Additive Subordination and Its Applications in Finance[*]

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Abstract

This paper studies *additive subordination*, which we show is a useful technique for constructing time-inhomogeneous Markov processes with analytical tractability. This technique is a natural generalization of Bochner's subordination that has proven to be extremely useful in financial modelling. Probabilistically, Bochner's subordination corresponds to a stochastic time change with respect to an independent Lévy subordinator, while in additive subordination, the Lévy subordinator is replaced by an additive one. We generalize the classical Phillips Theorem for Bochner's subordination to the additive subordination case, based on which we provide Markov and semimartingale characterizations for a rich class of jump-diffusions and pure jump processes obtained from diffusions through additive subordination, and obtain spectral decomposition for them. To illustrate the usefulness of additive subordination, we develop an analytically tractable cross commodity model for spread option valuation that is able to calibrate the implied volatility surface of each commodity. Moreover, our model can generate implied correlation patterns that are consistent with market observations and economic intuitions.

Key Words: Bochner's subordination, additive subordination,
  time-inhomogeneous Markov processes, spread options.

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1   Introduction

Applications in finance often require the use of time-inhomogeneous Markov processes. This paper studies additive subordination, which we show is a useful technique for constructing time-inhomogeneous Markov processes with analytical tractability. In the first part of the paper (Section 2 to 5), we develop the theory of additive subordination. In the second part (Section 6), using additive subordination, we develop an analytically tractable cross commodity model for spread option valuation that is consistent with the implied volatility surface of each commodity. Since

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this technique is a natural generalization of Bochner's subordination, we start by providing a brief review of Bochner's subordination and its applications in financial modelling.

From the operator semigroup perspective, Bochner's subordination is a classical method for generating new semigroups from existing ones ([7, 8]). Given $(\mathcal{P}_t)_{t \geq 0}$, a strongly continuous semigroup of contractions on a Banach space $\mathfrak{B}$, and $(\mu_t)_{t \geq 0}$, a vaguely continuous convolution semigroup of probability measures on $[0, \infty)$, Bochner's subordination defines a new family of operators via the following Bochner integral

$$\mathcal{P}_t^\phi f := \int_{[0,\infty)} \mathcal{P}_u f \, \mu_t(du), \quad f \in \mathfrak{B}. \tag{1.1}$$

The superscript ϕ alludes to the Laplace exponent of $(\mu_t)_{t \geq 0}$, which is a Bernstein function with the following representation

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda\tau})\nu(d\tau), \quad \text{s.t.} \quad \int_{[0,\infty)} e^{-\lambda u}\mu_t(du) = e^{-\phi(\lambda)t}, \tag{1.2}$$

where $\gamma \geq 0$ is the drift, and $\nu(d\tau)$ is called the Lévy measure that satisfies $\int_{(0,\infty)} (\tau \wedge 1)\nu(d\tau) < \infty$. $(\mathcal{P}_t^\phi)_{t \geq 0}$ is again a strongly continuous semigroup of contractions on $\mathfrak{B}$. Let $\mathcal{G}$ and $\mathcal{G}^\phi$ be the infinitesimal generator of $(\mathcal{P}_t)_{t \geq 0}$ and $(\mathcal{P}_t^\phi)_{t \geq 0}$, respectively. A fundamental result in the theory of Bochner's subordination first derived in [66], and hence known as the Phillips Theorem, shows that $\mathfrak{D}(\mathcal{G}) \subseteq \mathfrak{D}(\mathcal{G}^\phi)$ (we use $\mathfrak{D}(A)$ to denote the domain of the operator A), and

$$\mathcal{G}^\phi f = \gamma\mathcal{G}f + \int_{[0,\infty)} (\mathcal{P}_\tau f - f)\nu(d\tau) \quad \text{for } f \in \mathfrak{D}(\mathcal{G}). \tag{1.3}$$

If $(\mathcal{P}_t)_{t \geq 0}$ is the transition semigroup of a time-homogeneous Markov process X, Bochner's subordination can be interpreted probabilistically (see [70], Remark 13.12). Note that $(\mu_t)_{t \geq 0}$ gives rise to a Lévy subordinator T. We can assume X and T are defined on a common probability space and they are independent. Define $X_t^\phi := X_{T_t}$. It follows that X^ϕ is a time-homogeneous Markov process w.r.t. its own filtration and its transition semigroup $(\mathcal{P}_t^\phi)_{t \geq 0}$ is given by (1.1). The corresponding Markov characterization of X^ϕ can be obtained from its infinitesimal generator $\mathcal{G}^\phi$, which is provided by the Phillips theorem.

Bochner's subordination is a powerful tool in financial modelling for two reasons. First, it provides a rich and flexible modelling framework. By appropriately choosing X and T, one can expect to construct a new Markov process X^ϕ with desirable features for applications. The case where X is a one-dimensional time-homogeneous diffusion is particularly interesting. Applying Bochner's subordination to X, the resulting process X^ϕ is a time-homogeneous Markov *jump-diffusion* if the subordinator's drift is strictly positive ($\gamma > 0$), or a *pure-jump process* otherwise ($\gamma = 0$). Moreover, the jump measure of X^ϕ is in general state-dependent except when X is a Brownian motion. Hereafter, such jump processes are referred to as Lévy subordinate diffusions. By appropriately choosing the diffusion process to subordinate, one can generate a variety of behaviors in the jumps. For instance, if X is a mean-reverting diffusion, then jumps exhibit mean-reversion as well. Second, analytical tractability can often be obtained for $(\mathcal{P}_t^\phi)_{t \geq 0}$. This is a big advantage in applications as it allows efficient model calibration and valuation of a portfolio of derivatives. The spectral method is particularly useful in deriving analytical characterization for Lévy subordinate diffusions ([53] provides a detailed study of the spectral decomposition method for diffusions). Indeed, going from the spectral decomposition of diffusions to that of Lévy subordinate diffusions only requires the knowledge of the Laplace transform of the Lévy subordinator, which is given by the Lévy-Khintchine Theorem (see [70], Remark 13.4). In general, to obtain explicit expressions for

the spectral decomposition, one can follow the resolvent operator approach, as was illustrated in [61]. When the spectrum is purely discrete, the spectral decomposition reduces to an eigenfunction expansion, which is more convenient to compute than the general decomposition. This is true for many diffusions used in finance such as the Ornstein-Uhlenbeck (OU), CIR ([25]) and CEV ([24]) process. The eigenfunction expansion method is developed by [27], [44, 45, 47] and [51] for pricing European options and exotic options with barrier and early exercise features. When X is a Brownian motion (BM), X^ϕ is a Lévy process, and hence, one can also use transform-based methods since the characteristic function of X^ϕ can be obtained in closed form. See, e.g., [16], [33, 34], [43], [9].

Applications of Bochner's subordination in finance can be traced back to [58] and [22]. Many popular Lévy processes used for equity modelling can be obtained by applying Bochner's subordination to the BM. Examples include the Variance Gamma process of [57], the Normal Inverse Gaussian process of [3], the CGMY process of [17], the generalized Hyperbolic process of [31] and the Meixner process of [71] ([56]). More recent applications of Bochner's subordination to general diffusions include [4] for equity modelling, [61] and [55] for credit-equity derivatives, [46] for commodity derivatives, [10] and [52] for interest rate modelling, and [60] for credit default modelling.

Although Bochner's subordination allows us to construct Markov processes with many appealing features, their local characteristics (i.e., drift, diffusion coefficient and jump measure) are time-homogeneous. This might be a serious limitation from the empirical standpoint. For instance, it is observed in electricity markets that spikes in the electricity spot price are concentrated in summer and/or winter. Moreover, it is well-documented in the literature that time-homogeneous models often have difficulty in calibrating the term structure of implied volatilities (see for example [18] for equity options, and [46] and [48] for commodity options).

Under Bochner's subordination, time-homogeneity of X results from the stationary increments property of T. If this property is dropped, then T becomes an additive subordinator, i.e., a nonnegative and nondecreasing additive process. So naturally we are led to consider time changing a time-homogeneous Markov process with an independent additive subordinator. We call this technique as *additive subordination*. It allows us to construct a large family of time-inhomogeneous Markov processes with time- and state-dependent local characteristics, which are better suited for reproducing empirical phenomena. Furthermore, analytical tractability of Bochner's subordination is retained under additive subordination, as the Laplace transform of an additive subordinator is also available analytically. *In short, additive subordination improves the realism of Bochner's subordination while retaining its advantage of being analytically tractable.*

From the operator semigroup perspective, additive subordination can be viewed as a technique for constructing two-parameter families of operators from given semigroups. More precisely, given $(\mathcal{P}_t)_{t\geq 0}$, a strongly continuous semigroup of contractions on a Banach space $\mathfrak{B}$, and $(q_{s,t})_{0\leq s\leq t}$, a two-parameter family of probability measures on $[0,\infty)$ with $q_{s,t}$ being the distribution of the increment $T_t - T_s$ of the additive subordinator T, additive subordination defines a two-parameter family of operators via the following Bochner's integral

$$\mathcal{P}^\psi_{s,t} f := \int_{[0,\infty)} \mathcal{P}_u f \; q_{s,t}(du), \quad f \in \mathfrak{B}. \tag{1.4}$$

where the superscript ψ alludes to the density of the Laplace exponent of T (see Section 2). We will show that $(\mathcal{P}^\psi_{s,t})_{0\leq s\leq t}$ is a strongly continuous propagator, as well as a backward propagator on $\mathfrak{B}$ (see Section 3 for the definition). To characterize $(\mathcal{P}^\psi_{s,t})_{0\leq s\leq t}$, one needs to study its family of infinitesimal generators, denoted by $(\mathcal{G}^\psi_t)_{t\geq 0}$, which is defined as $\mathcal{G}^\psi_t f := \lim_{h\to 0+}(\mathcal{P}^\psi_{t,t+h}f - f)/h$ for $f \in \mathfrak{B}$ such that the limit exists.

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An application of additive subordination to finance is given by [48] which develops a single commodity model, where the spot price is assumed to follow an exponential of an additive subordinate Ornstein-Uhlenbeck (ASubOU) process. This model is shown to calibrate the implied volatility surface for a single commodity very well. *However, the general theory of additive subordination is still lacking, and it is the purpose of the present paper to develop such theory as well as to explore further applications of additive subordination by developing a cross commodity model for spread option valuation.* In the rest of this section we present the organization of this paper together with a summary of our contributions in each part.

In Section 2, we start with general additive subordinators and obtain a representation for its Laplace transform. Then, we specify the class of additive subordinators we work with that are most relevant for financial applications. We also provide three approaches for constructing additive subordinators from Lévy subordinators.

In Section 3, we find the relation between $\mathcal{G}_t^\psi$ and $\mathcal{G}$ (the generator of $(\mathcal{P}_t)_{t\geq 0}$), which is the key to characterize additive subordinate Markov processes. For Bochner's subordination, the relation is given by the classical Phillips Theorem (see eq.(1.3)) and Theorem 3.1 generalizes it to the additive subordination case. The first generalization of the Phillips theorem is developed by [62]. They study the case where $(\mathcal{P}_t)_{t\geq 0}$ is the transition semigroup of a Feller-Dynkin process X and characterize the infinitesimal generator of the space-time process $(t_0 + t, X_{t_0+t}^\psi)_{t\geq 0}$ $(t_0 \geq 0, X_t^\psi = X_{T_t})$, which is again a Feller-Dynkin process (A Markov process X is a Feller-Dynkin process if their transition semigroup is strongly continuous and contracting on the space of bounded continuous functions vanishing at infinity; the terminology is not standardized in the literature and such processes are called Feller in [62]). The proof in [62] relies on some properties that pertain exclusively to Feller-Dynkin semigroups as well as on Sato's proof of the Phillips Theorem (see [69], Theorem 32.1). Our goal is to generalize the Phillips theorem in the most general setting by allowing $(\mathcal{P}_t)_{t\geq 0}$ to be a strongly continuous semigroup of contractions on an arbitrary Banach space. To this end, we deal directly with $(\mathcal{P}_{s,t}^\psi)_{0\leq s\leq t}$ and $(\mathcal{G}_t^\psi)_{t\geq 0}$ instead of their space-time counterpart. This approach is not only more natural, but also allows the differential characteristics of the additive subordinator to be "piecewise" continuous (see Theorem 3.1 condition (a) to (c)). We observe that [62] requires the differential characteristics of T to be continuous, which is an inherent assumption of their setting. However piecewise specifications are often found useful for calibrations in financial applications. We further remark that, to prove the claim in Theorem 3.1 requires a different approach compared to the proof of the classical Phillips Theorem since many arguments used there are only valid for semigroups. Our key observation is a commutativity property that allows us to use the polygonal approximation in [36]. The conditions in Theorem 3.1 are sufficient but not necessary for the results to hold. When $(\mathcal{P}_t)_{t\geq 0}$ is the semigroup associated with a Lévy process, we can weaken these conditions using the pseudo-differential operator representation of $(\mathcal{P}_{s,t}^\psi)_{0\leq s\leq t}$.

Section 4 presents a detailed study of one-dimensional additive subordinate diffusions, which are jump-diffusions and pure jump processes obtained by applying additive subordination to time-homogeneous diffusions. This is a rich family of jump processes that are useful in financial applications, for which we provide both Markov and semimartingale characterization based on the generalized Phillips Theorem.

Section 5 considers the case when $(\mathcal{P}_t)_{t\geq 0}$ is a strongly continuous semigroup of symmetric contractions on a Hilbert space generated by a self-adjoint dissipative operator. This case is very important in financial applications since the transition semigroup of one-dimensional diffusions fits this setting. We obtain the spectral decomposition of $\mathcal{P}_{s,t}^\psi$ and provide sufficient conditions under which the spectral representation reduces to an eigenfunction expansion that converges uniformly on compacts, which is important for computational purposes in applications.

In Section 6, we develop a cross commodity model for crack spread option valuation by applying additive subordination to the CIR diffusion. In general, spread options can be written on any two assets, but many spreads in the commodity market are based on commodities that have a production relationship. A very popular spread of this kind is the crack spread, which is the price difference between crude oil and its refined products (heating oil, or gasoline), where the reference price is typically the futures price. In this paper we focus our analysis on the crack spread but our modelling framework is applicable to any pairs of commodities that have a production relationship. An important issue for pricing spread options is to choose an underlying model that is consistent with the implied volatility surface of each asset. In the case of crack spread, single commodity options on both crude oil and its refined products are actively traded in the market. Hence, the implied volatilities of these options provide important information on the future evolution of these commodities, which should be incorporated for pricing crack spread options. However, this issue has been largely ignored in the literature. A further challenge in spread option pricing is that in general there are no analytical formulas. Computational methods based on the Fourier transform (e.g., [28], [38], [12]) and analytical approximations (e.g., [40], [13], [6], [74]) have been developed under various model assumptions. In this paper, we develop a two-factor model for crude oil and its refined product, where each factor is an additive subordinate CIR (ASubCIR) process. The reason why we choose the ASubCIR process as opposed to the exponential ASubOU process used for single commodity modelling in [48] is due to its analytical tractability in pricing spread options. Our model captures key empirical features of individual commodities, such as mean-reversion and jumps, as well as of their spread, and is analytically tractable for pricing both single commodity options and spread options. Furthermore, our model is able to calibrate the implied volatility surface of each commodity. To the best of our knowledge, it is the first analytically tractable cross commodity model for spread option valuation in the literature that is also able to match the implied volatility surface of each commodity. Moreover, it can produce implied correlation patterns that are consistent with empirical observations and economic intuitions.

Section 7 concludes the paper and discusses other applications of additive subordination. All proofs are collected in the appendix.

2 Additive Subordinators

A stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is an additive process if it has independent increments, $X_0 = 0$ almost surely (a.s.), it is stochastically continuous, and its sample paths are càdlàg a.s. ([69] Definition 1.6). An additive subordinator $(T_t)_{t\geq 0}$ is a nondecreasing additive process with values in $\mathbb{R}_+$. We denote the distribution of $T_t - T_s$ by $q_{s,t}$ $(0 \leq s \leq t < \infty)$, which is an *infinitely divisible* distribution on $\mathbb{R}_+$ and satisfies the following properties (c.f. [69], Theorem 9.7; $*$ denotes convolution):

$$q_{s,t} * q_{t,u} = q_{s,u}, \text{ for } 0 \leq s \leq t \leq u < \infty, \tag{2.1}$$

$$q_{s,s} = \delta_0, \text{ for } 0 \leq s < \infty, \tag{2.2}$$

$$q_{s,t} \to \delta_0, \text{ as } s \uparrow t \text{ and as } t \downarrow s. \tag{2.3}$$

Here δ_ξ is the Dirac measure concentrated at the point ξ, and the convergence in (2.3) is the weak convergence of probability measures. We first obtain an representation for the Laplace transform of a general additive subordinator.

Proposition 2.1. *T is a semimartingale and its Laplace transform ($\lambda > 0$) can be written as*

$$E\left[e^{-\lambda T_t}\right] = e^{-\lambda \int_0^t \gamma(s)F(ds) - \int_0^t \int_{(0,\infty)} (1-e^{-\lambda\tau})\nu(s,d\tau)F(ds)}$$

for some nonnegative continuous nondecreasing deterministic function $F(s)$, and $\gamma(s) \geq 0$ and $\int_{(0,\infty)} (\tau \wedge 1)\nu(s,d\tau) < \infty$ F-a.s. for s.

If the semimartingale characteristics of an additive subordinator are absolutely continuous with respect to the Lebesgue measure, then one may choose $F(s) = s$. In this case, $(\gamma, 0, \nu)$ are called the *differential characteristics* of T, which can be chosen to satisfy

$$\gamma(s) \geq 0, \quad \int_{(0,\infty)} (\tau \wedge 1)\nu(s,d\tau) < \infty, \text{ for all } s \geq 0. \tag{2.4}$$

For the remaining of this paper, we only consider *additive subordinators with differential characteristics satisfying* (2.4) (the case where $F(s)$ is a singular continuous function is not interesting from an application point of view). Its Laplace transform is given by:

$$E\left[e^{-\lambda T_t}\right] = e^{-\int_0^t \psi(\lambda,s)ds}, \ \psi(\lambda,s) = \lambda\gamma(s) + \int_{(0,\infty)} \left(1 - e^{-\lambda\tau}\right)\nu(s,d\tau), \ \lambda > 0, \tag{2.5}$$

where $\psi(\lambda, s)$ is called the density of the Laplace exponent.

In practice, which additive subordinator to use depends on the application at hand. Below we provide three approaches for constructing additive subordinators from Lévy subordinators, for which we already have many examples.

Approach 1: Let $L_t^1, L_t^2, \cdots$ be independent Lévy subordinators (they all start at 0). Consider $0 = t_0 < t_1 < \cdots$, and define $T_t := \sum_{i=1}^{n-1} L_{t_i-t_{i-1}}^i + L_{t-t_{n-1}}^n$ if $t \in [t_{n-1}, t_n)$. Then it is easy to see that $\psi(\lambda, s)$ is constant in s on each $[t_{n-1}, t_n)$. By appropriately choosing parameters, this piecewise specification assures excellent results in calibrating the implied volatility surface. However a potential drawback is that it requires many more parameters compared to using one Lévy subordinator.

Approach 2: Consider a Lévy subordinator L_t with drift $\gamma \geq 0$ and Lévy measure $\nu(d\tau)$. Its Laplace exponent $\phi(\lambda)$ is given by (1.2). Define $T_t := L_{A_t}$, where $A_t = \int_0^t a(s)ds$ and $a(s) \geq 0$, $s \geq 0$, is a deterministic function known as activity rate. In this case it is easy to see that

$$\gamma(s) = \gamma a(s), \ \nu(s,d\tau) = a(s)\nu(d\tau), \ E\left[e^{-\lambda T_t}\right] = e^{-\phi(\lambda)A_t}.$$

This type of additive subordinators is used by [50] to model seasonal spikes observed in electricity prices, where $a(s)$ is chosen as a continuous seasonal function.

Approach 3: This approach is based on [19] which constructs Sato processes from Lévy processes with self-decomposable laws. An additive process $(X_t)_{t\geq 0}$ is called a ρ-Sato process if it is ρ-self-similar, that is, X_{ct} has the same law as $c^\rho X_t$ for all $c > 0, t \geq 0$. The definition of self-decomposable distribution is given in [69], Definition 15.1. Given a self-decomposable distribution, for every $\rho > 0$, [19] shows how to construct a ρ-Sato process $(X_t)_{t\geq 0}$ such that the distribution of X_1 coincides with it. In our setting, we assume that the distribution of L_1 is self-decomposable. This is equivalent to assuming that its Laplace exponent $\phi(\lambda)$ can be expressed as

$$\phi(\lambda) = \gamma\lambda + \int_0^\infty (1 - e^{-\lambda\tau})\frac{h(\tau)}{\tau}d\tau,$$

where $h(\tau)$ is positive and decreasing on $(0, \infty)$ ([69], Corollary 15.11). Denote by $\mathcal{S}_t$ the ρ-Sato subordinator constructed from L_t. Then from [19],

$$\gamma_\mathcal{S}(s) = \gamma \rho s^{\rho-1}, \ \ \nu_\mathcal{S}(s, d\tau) = -\frac{\rho h'(\tau s^{-\rho})}{s^{\rho+1}} d\tau, \ \ E\left[e^{-\lambda \mathcal{S}_t}\right] = e^{-\phi(\lambda t^\rho)}. \tag{2.6}$$

An important family of Lévy subordinators with self-decomposable laws is the tempered stable family ([23], Section 4.4.2). Its Lévy measure has the following form:

$$\nu(d\tau) = C\tau^{-\alpha-1} e^{-\eta\tau} d\tau,$$

with $C > 0$, $0 < \alpha < 1$, and $\eta > 0$. The case $\alpha = 1/2$ corresponds to the *Inverse Gaussian* (IG) process ([3]), which is a popular choice in finance. The Laplace exponent is given by

$$\phi(\lambda) = \gamma\lambda - C\Gamma(-\alpha)\left[(\lambda+\eta)^\alpha - \eta^\alpha\right]. \tag{2.7}$$

From (2.6), the Lévy measure for Sato subordinators obtained from the tempered stable family is then given by,

$$\nu_\mathcal{S}(s, d\tau) = C\rho s^{\rho\alpha-1} \tau^{-\alpha-1} e^{-\eta\tau s^{-\rho}} (\alpha + \eta\tau s^{-\rho}) d\tau.$$

By (2.6), for $0 < \rho < 1$, $\gamma_\mathcal{S}(s)$ is singular at $s = 0$. From (2.5), (2.6) and (2.7), we obtain

$$\int_{(0,\infty)} (1 - e^{-\lambda\tau}) \nu_\mathcal{S}(s, d\tau) = \frac{d}{ds}\left\{-C\Gamma(-\alpha)\left[(\lambda s^\rho + \eta)^\alpha - \eta^\alpha\right]\right\} = -C\alpha\lambda\rho\Gamma(-\alpha)(\lambda s^\rho + \eta)^{\alpha-1} s^{\rho-1},$$

which is singular at $s = 0$ when $0 < \rho < 1$. So the same is true for $\int_{(0,\infty)} (\tau \wedge 1) \nu_\mathcal{S}(s, d\tau)$. To extend the Phillips Theorem to additive subordination (see Theorem 3.1), we need to avoid such kind of singular behavior, so we consider *regularized Sato-type tempered stable (RSTS) subordinators*. For $t \geq 0$, define $T_t = S_{t+t_0} - S_{t_0}$ for some t_0. If $0 < \rho < 1$, we choose $t_0 > 0$. If $\rho \geq 1$, we choose $t_0 \geq 0$. We can choose $t_0 = 0$ in this case since there is no singularity at $s = 0$ for S_t. It is easy to see that $(T_t)_{t\geq 0}$ is an additive subordinator with

$$\gamma(s) = \gamma\rho(s+t_0)^{\rho-1}, \ \ \nu(s, d\tau) = C\rho(s+t_0)^{\rho\alpha-1} \tau^{-\alpha-1} e^{-\eta\tau(s+t_0)^{-\rho}} (\alpha + \eta\tau(s+t_0)^{-\rho}) d\tau,$$

$$E[e^{-\lambda T_t}] = e^{-\left[\phi(\lambda(t+t_0)^\rho) - \phi(\lambda t_0^\rho)\right]}.$$

A big advantage of using RSTS subordinators is that the subordinate model remains parsimonious as the additive subordinator only requires one extra parameter, ρ, compared to its Lévy counterpart (the regularization parameter t_0 can be fixed in advance and do not need to be calibrated). In Section 6, the regularized Sato-type IG (RSIG) subordinator will be used in our cross commodity model for calibration.

3   Phillips Theorem Under Additive Subordination

We start with several definitions (see [37], Definition 2.1 and 2.2). A two-parameter family of operators $(Q_{s,t})_{0\leq s\leq t}$ $(s, t < \infty)$ on a Banach space $\mathfrak{B}$ is called a backward propagator if it satisfies (i) $Q(s,t) = Q(s,u)Q(u,t)$ for $0 \leq s \leq u \leq t$; (ii) $Q(t,t) = I$ for $t \geq 0$. Given a family of transition (sub)probability functions, the two-parameter family of operators associated with them (see [37], Eq.(2.7)) form a backward propagator on the space of Borel measurable bounded functions.

$(Q_{s,t})_{0\leq s\leq t}$ is called a propagator if in (i) we have $Q(s,t) = Q(u,t)Q(s,u)$. Our notation for a propagator is different from the standard notation in the literature. Here the smaller time variable

appears first whereas in the standard notation the larger time variable appears first. The reason for this deviation is that the two-parameter family $(\mathcal{P}^{\psi}_{s,t})_{0 \leq s \leq t}$ defined in (1.4) is a backward propagator as well as a propagator. Propagators and backward propagators generalize the notion of semigroup. A propagator is also called an evolution family or a generalized semigroup in the literature.

A propagator/backward propagator $(Q_{s,t})_{0 \leq s \leq t}$ is called strongly continuous if for every $f \in \mathfrak{B}$, $(s,t) \mapsto Q_{s,t}f$ is continuous $(0 \leq s \leq t < \infty)$. $Q_{s,t}$ is called a contraction if $\|Q_{s,t}f\| \leq \|f\|$ for every $f \in \mathfrak{B}$. Define $\mathcal{G}_t f = \lim_{h \to 0+} h^{-1}(Q_{t,t+h}f - f)$. The domain of $\mathcal{G}_t$ consists of $f \in \mathfrak{B}$ such that the limit exists. We call $(\mathcal{G}_t)_{t \geq 0}$ the family of infinitesimal generators of the propagator/backward propagator $(Q_{s,t})_{0 \leq s \leq t}$.

Theorem 3.1. *Let $(T_t)_{t \geq 0}$ be an additive subordinator satisfying (2.4) with its Laplace transform given by (2.5). Let $(\mathcal{P}_t)_{t \geq 0}$ be a strongly continuous contraction semigroup of linear operators on a Banach space $\mathfrak{B}$ with infinitesimal generator $\mathcal{G}$. Consider the two-parameter family of operators $(\mathcal{P}^{\psi}_{s,t})_{0 \leq s \leq t}$, with $\mathcal{P}^{\psi}_{s,t}$ defined as in (1.4).*

(i) $(\mathcal{P}^{\psi}_{s,t})_{0 \leq s \leq t}$ is a strongly continuous contraction propagator as well as backward propagator on $\mathfrak{B}$. We call it additive subordinate propagator/backward propagator *of $(\mathcal{P}_t)_{t \geq 0}$ w.r.t. $(T_t)_{t \geq 0}$.*

(ii) Define $\nu_F(t, A) = \int_A (\tau \wedge 1)\nu(t, d\tau)$ for any Borel set $A \subseteq (0, \infty)$, then $\nu_F(t, \cdot)$ is a finite measure on $(0, \infty)$. Suppose the following conditions are satisfied:

(a) For every $t \geq 0$, $\gamma(t-) := \lim_{s \to t-} \gamma(s)$ exists, and $\gamma(t+) := \lim_{s \to t+} \gamma(s)$ exists and $\gamma(t+) = \gamma(t)$ (we set $\gamma(0-) = \gamma(0)$; by saying that the limit exists we also mean that it is finite).

(b) For every $t \geq 0$, $\nu_F(t-, \cdot) := \lim_{s \to t-} \nu_F(s, \cdot)$ exists, and $\nu_F(t+, \cdot) := \lim_{s \to t+} \nu_F(s, \cdot)$ exists and $\nu_F(t+, \cdot) = \nu_F(t, \cdot)$ (we set $\nu_F(0-, \cdot) = \nu_F(0, \cdot)$; the mode of convergence is weak convergence; by saying that the limiting measure exists we also mean that it is a finite measure).

(c) For every $t > 0$, the set $\{s : \gamma(s-) \neq \gamma(s+)$ or $\nu_F(s-, \cdot) \neq \nu_F(s+, \cdot), 0 \leq s \leq t\}$ only has a finite number of points.

Denote the family of infinitesimal generators of $(\mathcal{P}^{\psi}_{s,t})_{0 \leq s \leq t < \infty}$ by $(\mathcal{G}^{\psi}_t)_{t \geq 0}$. Then $\mathfrak{D}(\mathcal{G}) \subseteq \mathfrak{D}(\mathcal{G}^{\psi}_t)$ for each $t \geq 0$ and

$$\mathcal{G}^{\psi}_t f = \gamma(t)\mathcal{G}f + \int_{(0,\infty)} (\mathcal{P}_\tau f - f)\nu(t, d\tau), \quad \text{for } f \in \mathfrak{D}(\mathcal{G}). \tag{3.1}$$

We also have for $0 \leq s < t$,

$$\lim_{h \to 0+} h^{-1}(\mathcal{P}^{\psi}_{s,t+h}f - \mathcal{P}^{\psi}_{s,t}f) = \mathcal{P}^{\psi}_{s,t}\mathcal{G}^{\psi}_t f = \mathcal{G}^{\psi}_t \mathcal{P}^{\psi}_{s,t}f, \quad \text{for } f \in \mathfrak{D}(\mathcal{G}). \tag{3.2}$$

That $(\mathcal{P}^{\psi}_{s,t})_{0 \leq s \leq t}$ is both a propagator and backward propagator is due to the independent increment property of the additive subordinator, which implies $\mathcal{P}^{\psi}_{s,u}\mathcal{P}^{\psi}_{u,t} = \mathcal{P}^{\psi}_{u,t}\mathcal{P}^{\psi}_{s,u}$ for $0 \leq s \leq u \leq t$. Below we briefly explain the idea to obtain (3.1). Note that for every $t \geq 0$, since $\gamma(t)$ and $\nu(t, \cdot)$ satisfy (2.4), there is a Lévy subordinator T^{ϕ_t} such that its drift and Lévy measure are given by $\gamma(t)$ and $\nu(t, \cdot)$ ($\phi_t(\cdot)$ is its Laplace exponent). We denote by $(\mathcal{P}^{\phi_t}_s)_{s \geq 0}$ the Lévy subordinate semigroup of $(\mathcal{P}_s)_{s \geq 0}$ w.r.t. T^{ϕ_t}. Its infinitesimal generator is denoted by $\mathcal{G}^{\phi_t}$. From the Phillips' Theorem,

$$\mathcal{G}^{\phi_t}f = \gamma(t)\mathcal{G}f + \int_{(0,\infty)} (\mathcal{P}_\tau f - f)\nu(t, d\tau), \quad \text{for } f \in \mathfrak{D}(\mathcal{G}). \tag{3.3}$$

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Now for any s, t such that $0 \le s < t$, consider partitions on $[s, t]$ of the form $\Pi : s = t_0 < t_1 < \cdots < t_n = t$, and let $|\Pi| := \max_{1 \le i \le n}(t_i - t_{i-1})$. We define

$$R_{s,t}^{\Pi} f := \prod_{i=0}^{n-1} \mathcal{P}_{t_{i+1}-t_i}^{\phi_{t_i}} f \quad \text{for } f \in \mathfrak{B}. \tag{3.4}$$

Intuitively, $R_{s,t}^{\Pi}$ is the transition operator of the additive subordinate process when a piecewise Lévy subordinator is used (see Approach 1 for constructing additive subordinators). One would expect that under some conditions, as $|\Pi| \to 0$, $R_{s,t}^{\Pi}$ converges and we denote its limit by $U_{s,t}$ (in the literature $R_{s,t}^{\Pi}$ is known as the polygonal approximation of $U_{s,t}$; see [36]). Our key observation is that the semigroup $(\mathcal{P}_u^{\phi_s})_{u \ge 0}$ and $(\mathcal{P}_u^{\phi_t})_{u \ge 0}$ commute for any $s, t \ge 0$. This fact, together with the assumed conditions (a) to (c) allow us to conclude that $U_{s,t}$ exists, and $(U_{s,t})_{0 \le s \le t}$ is a strongly continuous contraction propagator on $\mathfrak{B}$. Furthermore, for $f \in \mathfrak{D}(\mathcal{G})$,

$$\lim_{h \to 0+} h^{-1}(U_{t,t+h} f - f) = \mathcal{G}^{\phi_t} f. \tag{3.5}$$

We can further show that $U_{s,t}$ agrees with $\mathcal{P}_{s,t}^{\psi}$ on $\mathfrak{B}$. This result is expected, as the law of the piecewise Lévy subordinator converges to that of the additive subordinator under consideration when $|\Pi| \to 0$ under the assumed conditions. Therefore $\mathcal{G}_t^{\psi} f$ is given by (3.1) for $f \in \mathfrak{D}(\mathcal{G})$.

Condition (a) to (c) say that $\gamma(t)$ and $\nu_F(t, \cdot)$ are piecewise continuous in t, and at each discontinuity point, they are right continuous. In general, these conditions are just sufficient but not necessary for (3.1) to hold. As a simple example, consider the trivial transition semigroup on a Banach space $f \in \mathfrak{B}$, i.e., $\mathcal{P}_t f = f$ for all $f \in \mathfrak{B}$ and $t \ge 0$. In this case, (3.1) holds without any conditions on $\gamma(t)$ and $\nu_F(t, \cdot)$, as both sides of (3.1) equal to zero. However, from the application point of view, it is quite natural to use additive subordinators with piecewise continuous specifications for $\gamma(t)$ and $\nu_F(t, \cdot)$. That is, for every $t \ge 0$, the left and the right limit of $\gamma(t)$ and $\nu_F(t, \cdot)$ exist, and $\gamma(t-) = \gamma(t+) = \gamma(t)$ and $\nu_F(t-, \cdot) = \nu_F(t+, \cdot) = \nu_F(t, \cdot)$ for all t except for a finite number of t in any bounded intervals on $[0, \infty)$. Note that, for such subordinators, it is necessary to have $\gamma(t)$ and $\nu_F(t, \cdot)$ being right-continuous at the discontinuity points for (3.1) to hold at these points, which is essentially due to our definition of the generator as the right-derivative. To see this, suppose $\gamma(t)$ and $\nu_F(t, \cdot)$ are not right-continuous at the discontinuities. Then define a new additive subordinator $\overline{T}$ with $\overline{\gamma}(t)$ and $\overline{\nu}_F(t, \cdot)$ which take the same value as $\gamma(t)$ and $\nu_F(t, \cdot)$ when they are continuous at t, and take the value of $\gamma(t+)$ and $\nu_F(t+, \cdot)$ when t is a discontinuity point. Now $\overline{\gamma}(t)$ and $\overline{\nu}_F(t, \cdot)$ satisfy condition (a) to (c). It is easy to see that the Laplace transform of $\overline{T}$ and T are equal. Therefore the transition probability $\overline{q}_{s,t}(\cdot)$ and $q_{s,t}(\cdot)$ are identical, which implies $\mathcal{P}_{s,t}^{\overline{\psi}} = \mathcal{P}_{s,t}^{\psi}$ and $\mathcal{G}_t^{\overline{\psi}} = \mathcal{G}_t^{\psi}$. Applying Theorem 3.1, at each discontinuous t, for $f \in \mathfrak{D}(\mathcal{G})$,

$$\mathcal{G}_t^{\psi} f = \mathcal{G}_t^{\overline{\psi}} f = \gamma(t+)\mathcal{G}f + \int_{(0,\infty)} \frac{\mathcal{P}_\tau f - f}{\tau \wedge 1} \nu_F(t+, d\tau) \neq \gamma(t)\mathcal{G}f + \int_{(0,\infty)} (\mathcal{P}_\tau f - f)\nu(t, d\tau).$$

Therefore (3.1) is not valid at these points.

When $(\mathcal{P}_t)_{t \ge 0}$ has a special structure, it is sometimes possible to derive (3.1) under conditions that are weaker than those imposed in Theorem 3.1 (ii). Below we provide an example that is relevant for financial applications. Let X be a Lévy process in $\mathbb{R}^d$ with characteristic exponent η_X, i.e., $E[e^{i\theta' X_t}] = e^{\eta_X(\theta)t}$ for $\theta \in \mathbb{R}^d$ (θ' denotes the transpose of θ). Define $\mathcal{P}_t f(x) = E[f(x + X_t)]$. It is well-known that $(\mathcal{P}_t)_{t \ge 0}$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d, \mathbb{C}) :=$

$\{f(x) \in \mathbb{C} : \int_{\mathbb{R}^d} |f(x)|^2 dx < \infty\}$ ([2], Theorem 3.4.2), and $\mathcal{P}_t f(x)$ is given by the following pseudo-differential operator (PDO) representation ([2], Exercise 3.4.3):

$$\mathcal{P}_t f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\theta' x} e^{\eta_X(\theta)t} \hat{f}(\theta) d\theta, \ f \in L^2(\mathbb{R}^d, \mathbb{C}), \tag{3.6}$$

where $\hat{f}(\theta) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\theta' x} f(x) dx$ is the Fourier transform of f. Let $\mathcal{G}$ be the generator of $(\mathcal{P}_t)_{t \geq 0}$. It is given by ([2], Theorem 3.4.4)

$$\mathcal{G}f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\theta' x} \eta_X(\theta) \hat{f}(\theta) d\theta, \ \mathfrak{D}(\mathcal{G}) = \left\{ f \in L^2(\mathbb{R}^d, \mathbb{C}) : \int_{\mathbb{R}^d} |\eta_X(\theta)|^2 |\hat{f}(\theta)|^2 d\theta < \infty \right\}. \tag{3.7}$$

Now let T be an additive subordinator independent of X, satisfying (2.4) with its Laplace transform given by (2.5). The characteristic function of X_{T_t} is given by

$$E[e^{i\theta' X_{T_t}}] = E[e^{\eta_X(\theta)T_t}] = e^{-\int_0^t \psi(-\eta_X(\theta), u) du} \text{ and } E[e^{i\theta'(X_{T_t} - X_{T_s})}] = e^{-\int_s^t \psi(-\eta_X(\theta), u) du}.$$

Using the argument in [2], Exercise 3.4.3, it can be shown that $\mathcal{P}_{s,t}^\psi f(x)$ is represented as follows:

$$\mathcal{P}_{s,t}^\psi f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\theta' x} e^{-\int_s^t \psi(-\eta_X(\theta), u) du} \hat{f}(\theta) d\theta. \tag{3.8}$$

From (3.8), the Fourier transform of $h^{-1}(\mathcal{P}_{t,t+h}^\psi f - f)$ is given by $h^{-1}(e^{-\int_t^{t+h} \psi(-\eta_X(\theta), u) du} - 1)\hat{f}(\theta)$. If it has a limit in $L^2(\mathbb{R}^d, \mathbb{C})$ as h tends to 0, then $\lim_{h \to 0+} h^{-1}(\mathcal{P}_{t,t+h}^\psi f - f)$ also exists in $L^2(\mathbb{R}^d, \mathbb{C})$, as the Fourier transform is an isomorphism on $L^2(\mathbb{R}^d, \mathbb{C})$. Note that for almost all t on $[0, \infty)$, we have

$$\lim_{h \to 0+} h^{-1}(e^{-\int_t^{t+h} \psi(-\eta_X(\theta), u) du} - 1) = -\psi(-\eta_X(\theta), t). \tag{3.9}$$

It can be proved that $\psi(-\eta_X(\theta), t)\hat{f}(\theta) \in L^2(\mathbb{R}^d, \mathbb{C})$ (we will verify this in the proof of Proposition 3.1). Thus if (3.9) holds at t, then $\mathfrak{D}(\mathcal{G}) \subseteq \mathfrak{D}(\mathcal{G}_t^\psi)$, and

$$\mathcal{G}_t^\psi f(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\theta' x} \psi(-\eta_X(\theta), t)\hat{f}(\theta) d\theta, \ f \in \mathfrak{D}(\mathcal{G}).$$

Now by (2.5), (3.6), (3.7) and Fubini's Theorem,

$$\mathcal{G}_t^\psi f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\theta' x} \left(\gamma(t)\eta_X(\theta) + \int_{(0,\infty)} (e^{\eta_X(\theta)\tau} - 1)\nu(t, d\tau) \right) \hat{f}(\theta) d\theta$$

$$= \gamma(t)\mathcal{G}f(x) + \int_{(0,\infty)} (\mathcal{P}_\tau f(x) - f(x))\nu(t, d\tau), \ f \in \mathfrak{D}(\mathcal{G}).$$

The results are summarized in the next proposition.

Proposition 3.1. *Let $(\mathcal{P}_t)_{t \geq 0}$ be the semigroup associated with a Lévy process in $\mathbb{R}^d$, which is strongly continuous and contracting on $L^2(\mathbb{R}^d, \mathbb{C})$. Consider an additive subordinator T satisfying (2.4) with its Laplace transform given by (2.5). If (3.9) is valid at t, then $\mathfrak{D}(\mathcal{G}) \subseteq \mathfrak{D}(\mathcal{G}_t^\psi)$ and (3.1) holds.*

Note that (3.9) holds for almost all $t \geq 0$, so the conclusions are valid almost everywhere. It is easy to see that for additive subordinators which satisfy condition (a) to (c) in Theorem 1 (ii), they also satisfy (3.9) at every t. However, for those additive subordinators such that their $\gamma(t)$ and $\nu_F(t, \cdot)$ do not have left limits or have an infinite number of discontinuities, it is possible that (3.9) is still valid at every t. Therefore, this condition is weaker than those imposed in Theorem 1. In this specific case, we are able to weaken the conditions thanks to the PDO representation of the additive subordinate propagator.

We regularize the Sato-type tempered stable (STS) subordinator in Approach 3. Now we use an example to explain why regularization is important when $0 < \rho < 1$. In this case, both $\gamma(t, \cdot)$ and $\nu_F(t, \cdot)$ explode as t tends to 0, hence condition (b) is not satisfied at $t = 0$. Consider an additive subordinate Lévy process X^ψ on $\mathbb{R}^d$ using a STS subordinator with $0 < \rho < 1$ as the time change. The characteristic function of X_t^ψ is given by

$$E[e^{i\theta' X_t^\psi}] = e^{\gamma \eta_X(\theta) t^\rho + C\Gamma(-\alpha)[(-\eta_X(\theta) t^\rho + \eta)^\alpha - \eta^\alpha]}.$$

The limit of the Fourier transform of $h^{-1}(\mathcal{P}_{0,h}^\psi f - f)$ as h tends to 0 is equal to

$$\lim_{h \to 0+} h^{-1}(e^{\gamma \eta_X(\theta) h^\rho + C\Gamma(-\alpha)[(-\eta_X(\theta) h^\rho + \eta)^\alpha - \eta^\alpha]} - 1)\hat{f},$$

which is infinity when $0 < \rho < 1$. Therefore $\mathfrak{D}(\mathcal{G}_0^\psi) = \emptyset$, and the conclusions of Theorem 3.1 (ii) do not hold at $t = 0$.

4 One-Dimensional Additive Subordinate Diffusions

We call a non-homogeneous Markov process an additive subordinate diffusion, or ASubDiff for short, if it can be obtained by time changing a time-homogeneous diffusion with an independent additive subordinator. Equivalently, its transition operator can be represented in the form of (1.4), where $(\mathcal{P}_u)_{u \geq 0}$ is the transition semigroup of a time-homogeneous diffusion and $(q_{s,t}(\cdot))_{0 \leq s < t}$ is the family of transition functions of an additive subordinator. In this paper, we develop Markov and semimaringale characterization for ASubDiffs. Here we restrict our discussions to one-dimensional (1D) ASubDiffs due to their importance in financial modelling and analytical tractability for derivative pricing. However, the following results can be extended to the multidimensional case using similar arguments.

We first specify the class of 1D diffusions that we work with. A 1D diffusion X on an interval $I \subseteq \mathbb{R}$ is a normal time-homogeneous strong Markov process with state space $I_\Delta := I \cup \{\Delta\}$ (I_Δ is the one-point compactification of I and Δ is the point at infinity) and its sample paths are continuous in I for $0 \leq t < \zeta$, and equal to Δ for $t \geq \zeta$, where $\zeta = \inf\{t \geq 0 : X_t = \Delta\}$ is called the lifetime (see [41], Definition 4.3.1). Here Δ is an abstract point that represents the state of cemetery. For the diffusion X to be considered, we assume $I = (l, r)$ ($-\infty \leq l < r \leq \infty$), i.e., the endpoints are not included in the state space. This means that we kill X when it hits an accessible endpoint and send it to Δ. We also assume that X is regular, i.e., for any point $x, y \in (l, r)$ with $x \neq y$, there is a positive probability that X reaches y in finite time when starting from x.

We first consider a class of diffusions that are conservative on (l, r), i.e., killing cannot occur inside the interval ([41], Definition 7.1.2). The more general class of diffusions under our study will be constructed from these processes later. Consider two functions $\mu(x)$ and $\sigma(x)$ defined on I. We assume that they are both continuous and $\sigma(x) > 0$ on I. Let

$$s(x) := \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)} dy\right), \quad \mathfrak{s}(dx) := s(x)dx, \quad m(x) := \frac{2}{\sigma^2(x)s(x)}, \quad \mathfrak{m}(dx) := m(x)dx. \quad (4.1)$$

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Thus we have defined two absolutely continuous measures on I. From [41], Theorem 7.2.2 and Corollary 7.2.2, there is a unique (in law) regular diffusion conservative on (l, r) with $\mathfrak{s}$ as the scale measure and $\mathfrak{m}$ as the speed measure. We denote this process by X^0. Its lifetime $\zeta_0 = \inf\{t \geq 0 : X_t^0 = \Delta\}$. Note that many diffusions used in finance are in this setting, including e.g., BM, OU, CIR (with Feller condition) and CEV.

An endpoint $p \in \{l, r\}$ is called accessible if $\int_{x_0}^p \mathfrak{m}((z, x)) s(x) dx < \infty$ for some $z \in (l, r)$. Let $C(X^0)$ denote the collection of bounded continuous functions on (l, r) with finite limits at l and at r, and with limit 0 at each accessible endpoint and value 0 at Δ. Then by Theorem 7.2.2 in [41], the transition semigroup of X^0 is strongly continuous on $C(X^0)$, and its generator is given by

$$\mathcal{G}^0 f(x) = \tfrac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x), \ \mathfrak{D}(\mathcal{G}^0) = \{f \in C(X^0) : \tfrac{1}{2}\sigma^2(x) f''(x) + \mu(x) f'(x) \in C(X^0)\}.$$

In particular, $C_c^2(I)$ (twice continuously differentiable functions on $I = (l, r)$ with compact support) is a subset of $\mathfrak{D}(\mathcal{G}^0)$. Theorem 16.84 and Proposition 16.82 in [11] show that for every $\varepsilon > 0$, as h tends to 0, for $x \in I$ (note that the diffusion considered in [11] Chapter 16 Section 12 is slightly different from X^0 in that X^0 is killed at accessible endpoints while the diffusion in [11] is not; however it is easy to see that those results still apply to our case),

$$h^{-1} P_x \left[|X_h^0 - x| > \varepsilon, \ \zeta_0 > h \right] \xrightarrow{bp} 0, \tag{4.2}$$

$$h^{-1} E_x \left[(X_h^0 - x) 1_{\{|X_h^0 - x| \leq \varepsilon, \ \zeta_0 > h\}} \right] \xrightarrow{bp} \mu(x), \tag{4.3}$$

$$h^{-1} E_x \left[(X_h^0 - x)^2 1_{\{|X_h^0 - x| \leq \varepsilon, \ \zeta_0 > h\}} \right] \xrightarrow{bp} \sigma^2(x), \tag{4.4}$$

where the convergence is bounded pointwise on all compact intervals in (l, r) ($f_h(x)$ is said to converge boundedly pointwise to $f(x)$ on some subset A of their domains as $h \to 0$ if $\lim_{h\to 0} f_h(x) = f(x)$ for all $x \in A$ and $\sup_{x \in A} |f_h(x)| \leq M < \infty$ for all h sufficiently small; see [11], Definition 15.47). The functions $\mu(x)$ and $\sigma(x)$ are known as the drift and diffusion coefficient of X^0. From the proof of Theorem 16.84 in [11] (see its claim (iº)), we also have

$$h^{-1} P_x \left[0 < \zeta_0 \leq h \right] \xrightarrow{bp} 0. \tag{4.5}$$

We now construct a more general class of diffusions that can be killed inside (l, r) by killing X^0, the conservative diffusion on (l, r) with drift $\mu(x)$, diffusion coefficient $\sigma(x)$ and lifetime ζ_0, using continuous additive functionals. Let $k(x) \geq 0$ be a continuous function on (l, r) and $\mathcal{E}$ be an exponential r.v. with mean 1, independent of X^0. $\int_0^t k(X_s^0) ds$ is a continuous additive functional. Define $\zeta_k = \inf\{t \geq 0 : \int_0^t k(X_s^0) ds \geq \mathcal{E}\}$. A new diffusion X is constructed from X^0 as

$$X_t = \begin{cases} X_t^0, & \text{if } t < \zeta, \\ \Delta, & \text{if } t \geq \zeta, \end{cases} \text{ with lifetime } \zeta = \zeta_0 \wedge \zeta_k.$$

Now X can be killed inside (l, r). Killing by continuous additive functionals provides a natural tool for modelling jump to default in finance (see e.g., [15] for an application in unified credit-equity modelling). Let $C(X)$ denote the space of bounded continuous functions on (l, r) with finite limits at l and at r, with value 0 at Δ, and with limit 0 at each finite excluded endpoint and at each infinite endpoint p except those p which are entrance points with $\lim_{x\to p} P_x(\zeta < \varepsilon) \neq 1$ for some $\varepsilon > 0$ (see [41], Definition 7.3.1 for entrance boundaries). Theorem 7.4.2 in [41] shows that the transition semigroup of X is strongly continuous on $C(X)$ and Theorem 7.4.3 of the same reference

gives its generator, which is

$$\mathcal{G}f(x) = \tfrac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - k(x)f(x),$$

with $\mathfrak{D}(\mathcal{G}) = \{f \in C(X) : \tfrac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - k(x)f(x) \in C(X)\}$. In particular, $C_c^2(I) \subset \mathfrak{D}(\mathcal{G})$. The next proposition provides the limit relations for X, which are critical in the analysis of ASubDiffs. The function $k(x)$ is known as the killing rate of X.

Proposition 4.1. *For X, the following limit relations hold for every $\varepsilon > 0$ and $x \in I$:*

$$h^{-1}P_x\left[|X_h - x| > \varepsilon,\ \zeta > h\right] \xrightarrow{bp} 0, \tag{4.6}$$

$$h^{-1}E_x\left[(X_h - x)1_{\{|X_h - x| \le \varepsilon,\ \zeta > h\}}\right] \xrightarrow{bp} \mu(x), \tag{4.7}$$

$$h^{-1}E_x\left[(X_h - x)^2 1_{\{|X_h - x| \le \varepsilon,\ \zeta > h\}}\right] \xrightarrow{bp} \sigma^2(x), \tag{4.8}$$

$$h^{-1}P_x\left[0 < \zeta \le h\right] \xrightarrow{bp} k(x). \tag{4.9}$$

The convergence is bounded pointwise on all compact intervals in (l, r).

Now we construct an additive subordinate diffusion with the additive subordinator satisfying (2.4) and conditions (a) to (c) in Theorem 3.1 (ii). Suppose that the diffusion X is defined on a measurable space $(\Omega^1, \mathcal{F}^1)$, and $\{P_x^1\}_{x \in I_\Delta}$ is a family of probability measures on this space such that $P_x^1(X_0 = x) = 1$ and $P_x^1(X_t \in A) = \mathcal{P}_t 1_A(x)$ for any measurable set A on I_Δ. Assume that the additive subordinator T is defined on a measurable space $(\Omega^2, \mathcal{F}^2)$, and $\{P_{s,u}^2\}_{s \ge 0, u \ge 0}$ is a family of probability measures on this space such that $P_{s,u}^2(T_s = u) = 1$, $(T_t)_{t \ge s}$ is an additive subordinator (but starting at u) and $E_{s,u}^2[e^{-\lambda(T_t - T_s)}] = e^{-\int_s^t \psi(\lambda, v) dv}$ with $\psi(\lambda, v)$ given by (2.5). Let $\Omega := \Omega^1 \times \Omega^2$ and $\mathcal{F} := \mathcal{F}^1 \otimes \mathcal{F}^2$ (the product sigma-algebra). Consider the product space $(\Omega, \mathcal{F})$ and let $P_{s,x} := P_x^1 \otimes P_{s,0}^2$ be the product probability measure on $\mathcal{F}$. For all $\omega = (\omega_1, \omega_2) \in \Omega$, define $X_t^\psi(\omega) := X_{T_t(\omega_2)}(\omega_1)$. Let $\mathcal{F}_{s,t}^0 := \sigma(X_u^\psi : s \le u \le t)$ be the double filtration generated by X^ψ (see [73], Definition 2.14). Then it is not difficult to see that $P_{s,x}(X_s^\psi = x) = 1$, $P_{s,x}(X_t^\psi \in A) = \mathcal{P}_{s,t}^\psi 1_A(x)$ for any measurable set A on I_Δ, and $(X_t^\psi, \mathcal{F}_{s,t}^0, P_{s,x})$ is a time-inhomogeneous Markov process (see [37] Definition 1.4).

We now determine $(\mathcal{G}_t^\psi)_{t \ge 0}$ (the family of infinitesimal generators of X^ψ), which gives the Markov characterization of X^ψ. For the diffusion X, its transition probability measure restricted to I has a density w.r.t. the Lebesgue measure ([59]), which we denote by $p(\tau, x, y)$ $(x, y \in I)$. We extend $p(\tau, x, y)$ from $y \in I$ to $y \in \mathbb{R}$ by defining $p(\tau, x, y) = 0$ for $y \notin I$. $P(\tau, x, \{\Delta\})$ is the probability that X is killed by time τ when starting from x. Using Theorem 3.1 (ii), we obtain an integro-differential representation for $\mathcal{G}_t^\psi$ on $C_c^2(I)$, from which one can see that X^ψ is in general a jump-diffusion with state-dependent and time-dependent drift, diffusion coefficient, killing rate and jump intensity. It is a pure jump process if $\gamma(t) = 0$ for all $t \ge 0$.

Theorem 4.1. *For $f \in C_c^2(I)$,*

$$\mathcal{G}_t^\psi f(x) = \frac{1}{2}(\sigma^\psi(t, x))^2 f''(x) + \mu^\psi(t, x)f'(x) - k^\psi(t, x)f(x)$$

$$+ \int_{y \ne 0}\left(f(x + y) - f(x) - 1_{\{|y| \le 1\}}yf'(x)\right)\Pi^\psi(t, x, dy), \tag{4.10}$$

where for $t \ge 0$ and $x \in I$,

$$\sigma^\psi(t, x) = \sqrt{\gamma(t)}\sigma(x),$$

$$\mu^\psi(t,x) = \gamma(t)\mu(x) + \int_{(0,\infty)} \left(\int_{\{|y|\leq 1\}} y p(\tau, x, x+y) dy \right) \nu(t, d\tau),$$

$$k^\psi(t,x) = \gamma(t)k(x) + \int_{(0,\infty)} P(\tau, x, \{\Delta\}) \nu(t, d\tau),$$

$$\Pi^\psi(t,x,dy) = \pi^\psi(t,x,y) dy, \quad \pi^\psi(t,x,y) = \int_{(0,\infty)} p(\tau, x, x+y) \nu(t, d\tau) \quad \text{for } y \neq 0.$$

$\Pi^\psi(t,x,dy)$ is a Lévy-type measure, i.e., $\int_{y\neq 0}(1 \wedge y^2)\Pi^\psi(t,x,dy) < \infty$.

Remark 4.1. For each (s,x), $\int_{y\neq 0} \pi^\psi(s,x,y)dy < \infty$ if $\int_{(0,\infty)} \nu(s,d\tau) < \infty$. This is because

$$\int_{y\neq 0} \pi^\psi(s,x,y)dy = \int_{y\neq 0} \int_{(0,\infty)} p(\tau, x, x+y)\nu(s, d\tau) dy$$

$$= \int_{(0,\infty)} \int_{y\neq 0} p(\tau, x, x+y) dy\, \nu(s, d\tau) \leq \int_{(0,\infty)} \nu(s, d\tau).$$

The interchange of integration order is justified by Tonelli's Theorem.

Semimartingales play a fundamental role in finance. From the time change construction, it is easy to see that X^ψ has càdlàg sample paths. However, note that the cemetery state Δ is an abstract point which does not have to be real-valued, so X^ψ is not a semimartingale in general. However, it can be turned into a semimartingale. We follow [21] and first identify Δ with a real-valued point. If $I \neq \mathbb{R}$, we identify Δ with an arbitrary point in $\mathbb{R} \setminus I$. If $I = \mathbb{R}$, such a point does not exist. In this case, one can embed I into $\mathbb{R}^2$ by mapping $x \mapsto (x,0)$, and extend σ^ψ, μ^ψ, k^ψ and π^ψ to $\mathbb{R}^2$ as in [21], p.1719. Then Δ can be identified with a point in $\mathbb{R}^2$ outside $\{(x,0) : x \in I\}$. To simplify the discussion, without loss of generality, in the following we assume that Δ is identified with a point ∂ in $\mathbb{R} \setminus I$, and the distance between ∂ and any point in I is greater than 1, which is the truncation level for jumps.

Let ζ^ψ be the lifetime of X^ψ. Consider a new process $\hat{X}^\psi$ defined as

$$\hat{X}_t^\psi := X_t^\psi 1_{\{0 \leq t < \zeta^\psi\}} + \partial 1_{\{t \geq \zeta^\psi\}}.$$

We make the convention that $\sigma^\psi(t,\partial) = \mu^\psi(t,\partial) = k^\psi(t,\partial) = \Pi^\psi(t,\partial,\cdot) = 0$, and define for each $\omega \in \Omega$,

$$B_t^\psi(\omega) := \int_0^t \mu^\psi(s, \hat{X}_{s-}(\omega)) ds,$$

$$C_t^\psi(\omega) := \int_0^t (\sigma^\psi(s, \hat{X}_{s-}(\omega)))^2 ds,$$

$$\nu^\psi(\omega, dt, dy) := \left[\Pi^\psi(t, \hat{X}_{t-}(\omega), dy) + k^\psi(t, \hat{X}_{t-}(\omega)) \delta_{\partial - \hat{X}_{t-}(\omega)}(dy) \right] dt,$$

where $\delta_x(\cdot)$ is the Dirac measure concentrated at x. Based on the integro-differential representation (4.10), we characterize $\hat{X}^\psi$ as a semimartingale and obtain its semimartingale characteristics (see, e.g., [39], Section II.2 for the definitions). The sample path decomposition can then be given based on these characteristics (see [39], Theorem II.2.34) which shows the path behavior.

Theorem 4.2. *For each $s \geq 0$, $x \in I \cup \{\partial\}$, under the probability measure $P_{s,x}$, $(\hat{X}_t^\psi)_{t \geq s}$ is a semimartingale with respect to $(\mathcal{F}_{s,t+}^0)_{t \geq s}$ $(\mathcal{F}_{s,t+}^0 := \bigcap_{\tau > t} \mathcal{F}_{s,\tau}^0)$, with characteristics $(B^\psi, C^\psi, \nu^\psi)$ with respect to the truncation function $h(x) = x 1_{\{|x| \leq 1\}}$.*

The predictable finite variation process B_t^ψ gives the drift of $\hat{X}^\psi$ and C_t^ψ is the quadratic variation process of its continuous local martingale part. The compensator of the random jump measure of $\hat{X}^\psi$ has two parts. The first part is absolutely continuous which shows the intensity of jumps into another state in I. The second part reflects the possibility of "jump to default". It concentrates on the point ∂ and gives the intensity of jumps into the cemetery state, which equals the killing rate.

Remark 4.2. *Based on the semimartingale characterization, [49] develops a class of equivalent measure changes that transforms one ASubDiff into another ASubDiff of the same structure. Using such measure transformations, one can develop financial models that are tractable under both the physical and the pricing measure based on ASubDiffs.*

5 Spectral Representation for Symmetric Semigroups under Additive Subordination

We consider the case where $(\mathcal{P}_t)_{t\geq 0}$ is a semigroup of symmetric contractions defined on a Hilbert space $\mathcal{H}$, generated by a self-adjoint dissipative operator $\mathcal{G}$. Since $\mathcal{P}_t$ is bounded, it is also self-adjoint, and it admits a spectral decomposition. In Theorem 5.1, we obtain the spectral decomposition of $\mathcal{P}_{s,t}^\psi$ (defined in (1.4)).

This case is very important in financial applications. The class of 1D diffusions we consider are symmetric Markov processes ([35]), and their transition semigroups are strongly continuous semigroups of symmetric contractions on $L^2(I, \mathfrak{m}) := \{f : \int_I f^2(x)\mathfrak{m}(dx) < \infty\}$ (recall that $\mathfrak{m}$ is the speed measure defined in (4.1)), which are generated by self-adjoint dissipative second-order differential operators ([59]). Once the spectral decomposition of the underlying diffusion process is known, Theorem 5.1 gives us the spectral decomposition of the additive subordinate diffusion, thus allowing us to derive analytical formulas for financial derivatives with square-integrable payoffs under models based on additive subordinate diffusions.

In the following, we use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the inner product and the norm of the Hilbert space $\mathcal{H}$ respectively. We first recall some basic results. A linear operator A is said to be *self-adjoint* if its domain $\mathfrak{D}(A) \subseteq \mathcal{H}$ is dense and $(A, \mathfrak{D}(A)) = (A^*, \mathfrak{D}(A^*))$, where A^* is the adjoint operator of A. If $\langle Au, u \rangle \leq 0$ for all $u \in \mathfrak{D}(A)$, then it is also dissipative. For a self-adjoint operator A, Proposition 12.2 in [70] shows that it is dissipative if and only if its spectrum σ_A is contained in $(-\infty, 0]$. Theorem 12.4 in [70] gives the spectral representation of a self-adjoint dissipative operator $(A, \mathfrak{D}(A))$ on $\mathcal{H}$: there exists an orthogonal projection-valued measure E on the Borel sets of $\mathbb{R}$ with support σ_A such that for all Borel sets $I, J \subset \mathbb{R}$:

(i) $E(\emptyset) = 0$, $E(\mathbb{R}) = \mathrm{id}$;

(ii) $E(I \cap J) = E(I)E(J)$;

(iii) $E(I) : \mathfrak{D}(A) \to \mathfrak{D}(A)$ and $AE(I) = E(I)A$;

(iv) $Af = \int_{\sigma_A} \lambda E(d\lambda)f$ for $f \in \mathfrak{D}(A) = \{f \in \mathcal{H} : \|Af\|^2 = \int_{\sigma_A} \lambda^2 \langle E(d\lambda)f, f \rangle < \infty\}$.

Now let $(\mathcal{G}, \mathfrak{D}(\mathcal{G}))$ be a dissipative self-adjoint operator on $\mathcal{H}$ with spectral measure E and $(\mathcal{P}_t)_{t\geq 0}$ be the semigroup it generates (see Example 12.5 in [70]). We have

$$\mathcal{P}_t f = e^{t\mathcal{G}} f = \int_{(-\infty, 0]} e^{t\lambda} E(d\lambda)f, \quad f \in \mathcal{H}, \quad t \geq 0. \tag{5.1}$$

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and $(\mathcal{P}_t)_{t\geq 0}$ is a strongly continuous semigroup of symmetric contractions. Below we obtain the spectral decomposition for the additive subordinate propagator $(\mathcal{P}_{s,t}^\psi)_{0\leq s\leq t}$.

Theorem 5.1. *Suppose the additive subordinator T satisfy* (2.4) *with its Laplace transform given by* (2.5). $(\mathcal{P}_{s,t}^\psi)_{0\leq s\leq t}$ *is a strongly continuous propagator/backward propagator of symmetric contractions on $\mathcal{H}$, and*

$$\mathcal{P}_{s,t}^\psi f = \int_{(-\infty,0]} e^{-\int_s^t \psi(-\lambda,u)du} E(d\lambda)f, \quad \text{for all } f \in \mathcal{H}, \quad 0 \leq s < t. \tag{5.2}$$

We next fix $\mathcal{H} = L^2(I, \mathfrak{m})$ (I is a locally compact separable metric space and $\mathfrak{m}$ is a positive Radon measure with full support on I) and consider the case where the spectrum of $\mathcal{G}$ is purely discrete. In this case, (5.1) becomes an eigenfunction expansion, i.e.,

$$\mathcal{P}_t f(x) = \sum_{n=1}^\infty e^{-\lambda_n t} f_n \varphi_n(x), \ f_n = \langle f, \varphi_n \rangle, \ f \in L^2(I, \mathfrak{m}), \tag{5.3}$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, each $\varphi_n \in L^2(I, m)$ and satisfies $\mathcal{P}_t \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x)$. $\varphi_n(x)$ is called an eigenfunction and $(\varphi_n(x))_n$ form a complete orthonormal basis of $L^2(I, \mathfrak{m})$. Using Theorem 5.1, $\mathcal{P}_{s,t}^\psi$ is also represented by an eigenfunction expansion as follows:

$$\mathcal{P}_{s,t}^\psi f(x) = \sum_{n=1}^\infty e^{-\int_s^t \psi(\lambda_n,u)du} f_n \varphi_n(x), \ f_n = \langle f, \varphi_n \rangle, \ f \in L^2(I, \mathfrak{m}), \tag{5.4}$$

Eigenfunction expansions are easier to compute than the general spectral representation. Many diffusion processes used in finance have purely discrete spectrum and explicit expressions of λ_n and $\varphi_n(x)$ for many examples can be found in [53]. Below we are interested in sufficient conditions for the eigenfunction expansion to exist, and also to converge uniformly on compacts (u.o.c). In general, the eigenfunction expansion (5.4) converges under the $L^2(I, \mathfrak{m})$ norm. However, in financial applications, it is more desirable to have (5.4) converge u.o.c, since we are interested in derivative prices at particular values of the underlying variable in a compact domain, and L^2 convergence does not guarantee convergence at a given point. The next proposition provides sufficient conditions for $\mathcal{P}_{s,t}^\psi$ to be represented by an eigenfunction expansion that converges u.o.c.

Proposition 5.1. *We assume that for each $t > 0$, $\mathcal{P}_t$ is trace-class (see [68] p.206 for the definition).*
(i) For every $t \geq 0$ and (s,t) with $0 \leq s \leq t$, $\mathcal{P}_t f$ and $\mathcal{P}_{s,t}^\psi f$ are represented by (5.3) *and* (5.4) *respectively. In this case, $\mathcal{P}_t$ admits a symmetric kernel $p_t(x,y)$, i.e., $\mathcal{P}_t f(x) = \int_I p_t(x,y) f(y) \mathfrak{m}(dy)$ and $p_t(x,y) = p_t(y,x)$.*
(ii) We further assume $p_t(x,y)$ is continuous in (x,y) for each $t > 0$. Consider an additive subordinator T satisfying (2.4) *with its Laplace transform given by* (2.5). *Given (s,t) such that $0 \leq s < t$, assume one of the following two conditions is satisfied:*

(a) $\int_s^t \gamma(u)du > 0$.

(b) $\int_s^t \gamma(u)du = 0$, but for any compact set $J \subseteq I$ there exists some constant C_J such that for all n, $|\varphi_n(x)| \leq C_J$ for all $x \in J$, and $\sum_{n=1}^\infty e^{-\int_s^t \psi(\lambda_n,u)du} < \infty$.

Then the eigenfunction expansion (5.4) *converges u.o.c, and $\mathcal{P}_{s,t}^\psi f(x)$ is a continuous function in x for all $f \in L^2(I, \mathfrak{m})$.*

For additive subordinators with drift, condition (a) is satisfied, so the L^2-convergent expansion automatically converges u.o.c. For those without drift, condition (b) is also mild. For example, the regularized Sato-type tempered stable subordinators fulfil this condition.

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6   Crack Spread Option Valuation

In this section, we apply additive subordination to CIR diffusions to develop a cross commodity model for pricing crack spread options. We will sometimes call crude oil as the primary commodity, and refer to its refined product (heating oil or gasoline) as the daughter commodity. Before proceeding to the model, we discuss additive subordinate CIR processes first.

6.1   Additive Subordinate CIR Processes

Recall that a CIR diffusion X is the unique solution to the SDE $dX_t = \kappa(\theta - X_t) + \sigma\sqrt{X_t}dB_t$, where $\kappa, \theta, \sigma > 0$ and B is the standard 1D BM. An additive subordinate CIR (ASubCIR) process is obtained by time changing X with an independent additive subordinator T. For the rest of Section 6, we make the following assumption.

Assumption 6.1. *(1) The Feller condition is satisfied, i.e., $2\kappa\theta \geq \sigma^2$. Under this condition, X cannot hit zero and hence $I = \mathbb{R}_{++} := (0,\infty)$. (2) T is an additive subordinator with differential characteristics $(\gamma(t), \nu(t,\cdot))$ satisfying (2.4) and conditions (a) to (c) in Theorem 3.1 (ii), and its Laplace transform is given by (2.5). Furthermore, condition (a) or (b) in Proposition 5.1 holds.*

We denote the ASubCIR process by X^ψ and call $(\kappa, \theta, \sigma, \gamma(t), \nu(t,\cdot))$ as its generating tuple. Applying Theorem 4.1 and 4.2 gives us the Markov and semimartingale characterization of X^ψ. In particular, its jump measure is mean-reverting and time-dependent. If $\gamma(t) = 0$ for all $t \geq 0$, the continuous local martingale part vanishes and X^ψ is a pure jump process with mean-reversion realized only via jumps.

We next give the eigenfunction expansion for X^ψ. For the CIR diffusion X, its speed density is given by

$$m(x) = \frac{2}{\sigma^2}x^{\beta-1}e^{-\alpha x}, \text{ where } \alpha := \frac{2\kappa}{\sigma^2}, \ \beta := \frac{2\kappa\theta}{\sigma^2}, \tag{6.1}$$

and $\mathfrak{m}(dx) := m(x)dx$ is the speed measure. The CIR transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ is a strongly continuous semigroup of symmetric contractions on $L^2(\mathbb{R}_{++}, \mathfrak{m})$, with the following eigenfunction expansion for $f \in L^2(\mathbb{R}_{++}, \mathfrak{m})$ ([53]):

$$\mathcal{P}_t f(x) = \sum_{n=0}^{\infty} f_n e^{-\lambda_n t}\varphi_n(x), \ \lambda_n = \kappa n, \ \varphi_n(x) = \sqrt{\frac{n!\kappa}{\Gamma(\beta+n)}}\alpha^{\frac{\beta-1}{2}}L_n^{(\beta-1)}(\alpha x), \tag{6.2}$$

where $(L_n^{(v)}(\cdot))_{n\geq 0}$ are the generalized Laguerre polynomials (see, e.g., [5] p.113, Eq.(4.5.2)) and $\Gamma(\cdot)$ is the Gamma function. Here, we label the eigenvalues and eigenfunctions starting from 0 rather than 1 as in (5.4). This notation is more convenient when working with orthogonal polynomials. One can directly verify that the CIR transition semigroup is trace-class and from [59], its kernel is jointly continuous. Furthermore, $\varphi_n(x)$ are uniformly bounded in n for x on compacts due to the property of generalized Laguerre polynomials ([64] p.54, Eq.(27a)). Under Assumption 6.1 for T, for any $f \in L^2(\mathbb{R}_{++}, \mathfrak{m})$, $\mathcal{P}_{s,t}^\psi f$ can be represented by the eigenfunction expansion (5.4) that converges u.o.c, with $e^{-\lambda_n t}$ replaced by $e^{-\int_s^t \psi(\lambda_n, u)du}$.

6.2   The Model

Let S_t^1 and S_t^2 be the spot price at time t of crude oil and its output, respectively. $F_i(t, T)$ is the futures price at time t of the contract maturing at time T for commodity i ($i = 1, 2$, $0 \leq t \leq T$). Since we are mainly interested in option pricing, we specify our model under a pricing measure,

which will be determined by calibrating the model to the market price of liquid options. In the following all expectations are taken under this pricing measure chosen by the market. We model S_t^1 and S_t^2 as

$$S_t^1 = a_1(t)X_t^{\psi_1}, \ X_0^{\psi_1} = x_1, \tag{6.3}$$

$$S_t^2 = a_2(t)\big(X_t^{\psi_1} + X_t^{\psi_2}\big), \ X_0^{\psi_2} = x_2. \tag{6.4}$$

We next describe each ingredient in the model.

(1) For $i = 1, 2$, X^{ψ_i} is an ASubCIR processes satisfying Assumption 6.1, with generating tuple $(\kappa_i, \theta_i, \sigma_i, \gamma_i(t), \nu_i(t, \cdot))$. X^{ψ_1} and X^{ψ_2} are assumed to be independent. In the calibration examples, we will use the regularized Sato-type IG subordinator, which satisfies Assumption 6.1.

(2) Let $F_i(t, T)$ be the futures price at time t of the contract maturing at time T for commodity i ($i = 1, 2$, $0 \leq t \leq T$), which are computed by $F_1(t, T) = E[S_T^1 | X_t^{\psi_1}]$ and $F_2(t, T) = E[S_T^2 | X_t^{\psi_1}, X_t^{\psi_2}]$ due to the Markov property. We select two deterministic functions $a_1(t)$ and $a_2(t)$ to match the initial futures curve for both commodities, i.e., $a_1(t) = F_1(0, t)/E[X_t^{\psi_1}]$ and $a_2(t) = F_2(0, t)/E[X_t^{\psi_1} + X_t^{\psi_2}]$. Most options in commodity markets are written on futures contracts, therefore a spot model should match the initial futures curve which represents important information from the futures market.

(3) For tractability and parsimony reasons, in (6.3) we model the evolution of crude oil using only one factor, X^{ψ_1}. As it will be seen from the calibration example, this one-factor model is already good enough to capture the implied volatility surface for crude oil futures options. Due to the production relationship, changes in the crude oil price also affect its output, which is modeled by incorporating X^{ψ_1} in (6.4). From our model construction, the prices for both commodities can jump simultaneously, but the magnitude is in general different. Events that only affect the daughter commodity are modeled by X^{ψ_2}. As our calibration example shows, having one extra factor is enough to calibrate the implied volatility surface of the daughter commodity. A similar factor structure with multiple factors is employed by [26], which models the joint evolution of the futures price of crude oil and its refined products using different stochastic drivers. However general spread option pricing is not analytically tractable in [26] and no calibration performance is shown there. We also remark that without time change, the model for the daughter commodity becomes a two-factor CIR model, which has been used to model the short rate in, e.g., [54] and [20].

Remark 6.1. *In a factor structure, it is natural to consider adding weights in (6.4) in the form $S_t^2 = a_2(t)\big(w_1 X_t^{\psi_1} + w_2 X_t^{\psi_2}\big)$ with constant $w_1, w_2 > 0$. However this general formulation can be reduced to (6.4) by observing that, for an ASubCIR process X^ψ, cX^ψ ($c > 0$) is another ASubCIR process with starting point cx, mean-level $c\theta$ and volatility $\sqrt{c}\sigma$, where x, θ, σ are the starting point, mean-level and volatility of X^ψ (the other parameters remain unchanged). Thus if we define $\tilde{a}_2(t) = a_2(t)w_1$ and $\tilde{X}_t^{\psi_2} = w_2/w_1 \cdot X_t^{\psi_2}$, then the dynamics of S_t^2 can be written as (6.4). Using this scaling property, we can also normalize $x_1 = 1$, where x_1 is the starting point of X^{ψ_1}.*

In our model, under the pricing measure, both commodities, and hence the spot spread $S_t^2 - S_t^1$, exhibit mean-reversion. Using the class of equivalent measure changes developed in [49], X^{ψ_1} and X^{ψ_2} remain to be ASubCIR processes under the physical measure. Hence these features are also observed in the physical model, which are consistent with empirical studies ([72], [29]).

6.3 Derivatives Pricing

To price futures options for the daughter commodity and the spread option, we need to compute the expectation of a function of $X_t^{\psi_1}$ and $X_t^{\psi_2}$. Let $\mathfrak{m}_i$ be the speed measure of X^{ψ_i} and define

$\mathfrak{M}(dx_1, dx_2) := \mathfrak{m}_1(dx_1)\mathfrak{m}_2(dx_2)$. Put $\mathbf{x} := (x_1, x_2)$ and $\mathbb{R}_{++}^2 := (0, \infty) \times (0, \infty)$. Due to the independence of $X_t^{\psi_1}$ and $X_t^{\psi_2}$, for $f \in L^2(\mathbb{R}_{++}^2, \mathfrak{M})$,

$$E_{s,\mathbf{x}}\left[f(X_t^{\psi_1}, X_t^{\psi_2})\right] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{nm} e^{-\int_s^t (\psi_1(\kappa_1 n, u) + \psi_2(\kappa_2 m, u))du} \varphi_n^1(x_1)\varphi_m^2(x_2),$$

$$f_{nm} = \int_{\mathbb{R}_{++}^2} f(x_1, x_2)\varphi_n^1(x_1)\varphi_m^2(x_2)\mathfrak{m}_1(dx_1)\mathfrak{m}_2(dx_2).$$

Here $\varphi_n^i(x)$ denotes the n-th eigenfunction of X^{ψ_i}. Under Assumption 6.1 for the additive subordinators, it is easy to verify that the above series converges u.o.c using Proposition 5.1.

We first obtain the futures price for both commodities, which is affine in the state variables.

Proposition 6.1. *For* $0 \le t \le T$,

$$F_1(t, T) = a_1(T)e^{-\int_t^T \psi_1(\kappa_1, u)du} X_t^{\psi_1} + a_1(T)\theta_1\left(1 - e^{-\int_t^T \psi_1(\kappa_1, u)du}\right),$$

$$F_2(t, T) = a_2(T)e^{-\int_t^T \psi_1(\kappa_1, u)du} X_t^{\psi_1} + a_2(T)e^{-\int_t^T \psi_2(\kappa_2, u)du} X_t^{\psi_2}$$
$$+ a_2(T)\theta_1\left(1 - e^{-\int_t^T \psi_1(\kappa_1, u)du}\right) + a_2(T)\theta_2\left(1 - e^{-\int_t^T \psi_2(\kappa_2, u)du}\right)$$

where

$$a_1(T) = \frac{F_1(0, T)}{\theta_1 + (x_1 - \theta_1)e^{-\int_0^T \psi_1(\kappa_1, u)du}},$$

$$a_2(T) = \frac{F_2(0, T)}{\theta_1 + \theta_2 + (x_1 - \theta_1)e^{-\int_0^T \psi_1(\kappa_1, u)du} + (x_2 - \theta_2)e^{-\int_0^T \psi_2(\kappa_2, u)du}}.$$

Next we price options. We will need several special functions in the following: the scaled generalized Laguerre polynomial $l_n^{(\nu)}(x)$ ((B.1)), the scaled Kummer's confluent hypergeometric function $\mathbf{M}(a, c; z)$ ((B.4)), Tricomi's confluent hypergeometric function $U(a, c; x)$ ((B.9)) and the Gauss Hypergeometric function $_2F_1$ ((B.10)). Their definitions and some useful identities are collected in Appendix B.

We consider pricing European-style futures options for each commodity. In a put option for commodity i, the payoff at the option maturity time T is given by $(K - F_i(T, T_i))^+$ where $K > 0$ is the strike price and T_i is the expiration time of the underlying futures contract. In practice $T < T_i$. Below we only price the put. The call option price can be obtained by the put-call parity. Alternatively, it can also be priced by eigenfunction expansions. We assume deterministic risk-free rate and $B(0, T)$ denotes the discount factor from T to time zero. Note that in the following α_i and β_i ($i = 1, 2$) are defined as in (6.1) using the parameters for X^{ψ_i}.

Proposition 6.2. *(1) Suppose* $K > a_1(T_1)\theta_1\left(1 - e^{-\int_T^{T_1} \psi_1(\kappa_1, u)du}\right)$ *(otherwise the put price is zero). Let* $F_1 = F_1(0, T_1)$, $A = a_1(T_1)e^{-\int_T^{T_1} \psi_1(\kappa_1, u)du}$, $B = a_1(T_1)\theta_1\left(1 - e^{-\int_T^{T_1} \psi_1(\kappa_1, u)du}\right)$, *and* $K_0 = \frac{K - B}{A}$. *Then for the primary commodity, the put option price* $P_1(F_1, T, T_1, K)$ *has the following expansion*

$$P_1(F_1, T, T_1, K) = B(0, T)A\Bigg\{K_0 P(\beta_1, \alpha_1 K_0) + P(\beta_1 + 1, \alpha_1 K_0)\left((\theta_1 - x_1)e^{-\int_0^T \psi_1(\kappa_1, u)du} - \theta_1\right)$$

$$+ \alpha_1^{\beta_1} K_0^{\beta_1+1} e^{-\alpha_1 K_0} \sum_{n=2}^{\infty} \frac{e^{-\int_0^T \psi_1(\kappa_1 n, u)du}}{\sqrt{n(n-1)}} l_{n-2}^{(\beta_1+1)}(\alpha_1 K_0) l_n^{(\beta_1-1)}(\alpha_1 x_1)\Bigg\},$$

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where $P(c, x)$ is the regularized incomplete gamma function defined as $P(c, x) = \gamma(c, x)/\Gamma(c)$ with the lower incomplete gamma function $\gamma(c, x) = \int_0^x t^{c-1}e^{-t}dt$ for any $c > 0$, and $l_n^{(\nu)}(x)$ is the scaled generalized Laguerre polynomial defined in (B.1).

(2) Define $K_1 := K - a_2(T_2)\theta_1\left(1 - e^{-\int_T^{T_2}\psi_1(\kappa_1,u)du}\right) - a_2(T_2)\theta_2\left(1 - e^{-\int_T^{T_2}\psi_2(\kappa_2,u)du}\right)$ and

$$h(n, m) := e^{-\int_0^T \left(\psi_1(\kappa_1 n, u) + \psi_2(\kappa_2 m, u)\right)du}\sqrt{\frac{\Gamma(\beta_1 + n)\Gamma(\beta_2 + m)}{n!m!}}l_n^{(\beta_1 - 1)}(\alpha_1 x_1)l_m^{(\beta_2 - 1)}(\alpha_2 x_2), \quad (6.5)$$

Suppose $K_1 > 0$ (otherwise the put price is zero). Let $F_2 = F_2(0, T_2)$, $\omega_1 = a_2(T_2)e^{-\int_T^{T_2}\psi_1(\kappa_1,u)du}$ and $\omega_2 = a_2(T_2)e^{-\int_T^{T_2}\psi_2(\kappa_2,u)du}$ and $\gamma_1 = \frac{\alpha_1}{\omega_1}K_1$ and $\gamma_2 = \frac{\alpha_2}{\omega_2}K_1$. Then for the daughter commodity, the put option price $P_2(F_2, T, T_2, K)$ has the following expansion

$$P_2(F_2, T, T_2, K) = B(0, T)K_1\gamma_1^{\beta_1}\gamma_2^{\beta_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi_{n,m}(\gamma_1, \gamma_2)h(n, m),$$

where for $n, m \geq 0$,

$$\pi_{n,m}(\gamma_1, \gamma_2) := \int_0^1 y^{\beta_1 - 1}(1 - y)^{\beta_2 + 1}\mathbf{M}(n + \beta_1, \beta_1; -\gamma_1 y)\mathbf{M}\left(m + \beta_2, \beta_2 + 2; -\gamma_2(1 - y)\right)dy$$

$$= \sum_{p=0}^{\infty}\left[\frac{(m + \beta_2)_p}{p!}(-\gamma_2)^p\mathbf{M}\left(n + \beta_1, \beta_1 + \beta_2 + 2 + p; -\gamma_1\right)\right]. \quad (6.6)$$

Here $\mathbf{M}(a, c; z)$ is the scaled Kummer's confluent hypergeometric function defined in (B.4).

We now consider pricing the put type spread option. The price of a call type spread option can be obtained from the put-call parity. The put payoff at the option maturity time T is given by $(K - (F_2(T, T_2) - F_1(T, T_1)))^+$ where in practice $T < T_i$ and in general $T_1 \neq T_2$. We do not require the strike price K to be positive, although for the crack spread option, K is almost always positive in traded options as the crack spread rarely becomes negative. Note that

$$K - F_2(T, T_2) + F_1(T, T_1) = K_1 + \omega_0 X_T^{\psi_1} - \omega_2 X_T^{\psi_2}, \text{ with}$$

$$\omega_2 = a_2(T_2)e^{-\int_T^{T_2}\psi_2(\kappa_2,u)du}, \quad \omega_0 = a_1(T_1)e^{-\int_T^{T_1}\psi_1(\kappa_1,u)du} - a_2(T_2)e^{-\int_T^{T_2}\psi_1(\kappa_1,u)du},$$

$$K_1 = K + \theta_1\left(a_1(T_1) - a_2(T_2) - \omega_0\right) - \theta_2 a_2(T_2)\left(1 - e^{-\int_T^{T_2}\psi_2(\kappa_2,u)du}\right).$$

Here ω_2 is positive for all $T > 0$ and $T_2 \geq T$.

Remark 6.2. *Conversion Ratio: In the market, crude oil price is quoted per barrel while its refined products are quoted per gallon. To define the crack spread, both commodities need to be quoted on the same basis. Since there are 42 gallons in a barrel, the market price of heating oil or gasoline futures should be multiplied by 42. In our formulation, we assume $F_2(t, T)$ is the futures price after conversion, i.e., it is quoted per barrel.*

Proposition 6.3. *Let $\omega_1 = |\omega_0|$, $F_1 = F_1(0, T_1)$, $F_2 = F_2(0, T_2)$, $\gamma_1 = \frac{\alpha_1}{\omega_1}K_1$ and $\gamma_2 = \frac{\alpha_2}{\omega_2}K_1$.*
(a) If $\omega_0 < 0$ and suppose $K_1 > 0$ (otherwise the option price is zero), the spread put option price $SP(F_1, F_2, T, K)$ is given by

$$SP(F_1, F_2, T, K) = B(0, T)K_1\gamma_1^{\beta_1}\gamma_2^{\beta_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\pi_{n,m}(\gamma_1, \gamma_2)h(n, m),$$

where $h(n, m)$ and $\pi_{n,m}(\gamma_1, \gamma_2)$ are defined in (6.5) and (6.6) respectively.

(b) If $\omega_0 > 0$, the spread put option price $SP(F_1, F_2, T, K)$ is given by

$$
SP(F_1, F_2, T, K) = \begin{cases}
B(0,T)K_1\gamma_1^{\beta_1}\gamma_2^{\beta_2} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m}^1(\gamma_1, \gamma_2)h(n, m), & \text{if } K_1 > 0, \\[2em]
B(0,T)\omega_1\omega_2(\alpha_1\omega_2)^{\beta_1}(\alpha_2\omega_1)^{\beta_2} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m}^2(\omega_1, \omega_2)h(n, m), & \text{if } K_1 = 0, \\[2em]
B(0,T)|K_1||\gamma_1|^{\beta_1}|\gamma_2|^{\beta_2} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \pi_{n,m}^3(\gamma_1, \gamma_2)h(n, m), & \text{if } K_1 < 0,
\end{cases}
$$

where

$$
\pi_{n,m}^1(\gamma_1, \gamma_2) := \int_0^\infty x^{\beta_1 - 1}(1 + x)^{\beta_2 + 1}\mathbf{M}(n + \beta_1, \beta_1; -\gamma_1 x)\mathbf{M}(m + \beta_2, \beta_2 + 2; -\gamma_2(1 + x))\,dx,
$$

$$
\pi_{n,m}^2(\omega_1, \omega_2) := \int_0^\infty x^{\beta_1 + \beta_2}\mathbf{M}(n + \beta_1, \beta_1; -\alpha_1\omega_2 x)\mathbf{M}(m + \beta_2, \beta_2 + 2; -\alpha_2\omega_1 x)\,dx,
$$

$$
\pi_{n,m}^3(\gamma_1, \gamma_2) := \int_0^\infty x^{\beta_2 + 1}(1 + x)^{\beta_1 - 1}\mathbf{M}(n + \beta_1, \beta_1; \gamma_1(1 + x))\mathbf{M}(m + \beta_2, \beta_2 + 2; \gamma_2 x)\,dx.
$$

For $n, m \geq 0$,

$$
\pi_{n,m}^1(\gamma_1, \gamma_2) = \sum_{k=0}^{\infty}\left[\frac{(m + \beta_2)_k(-\gamma_2)^k}{k!\Gamma(k + \beta_2 + 2)} \sum_{l=0}^{n}\binom{n}{l}(-\gamma_1)^l U(l + \beta_1, k + l + \beta_1 + \beta_2 + 2; \gamma_1) \right],
$$

where $U(a, c; x)$ is Tricomi's confluent hypergeometric function defined in (B.9). For $n, m \geq 0$, if $\beta_2 + 2 + p = n$ for some non-negative integer p, then

$$
\pi_{n,m}^2(\omega_1, \omega_2) = \frac{\Gamma(\beta_1 + \beta_2 + 1)}{\Gamma(n + \beta_1)(\alpha_1\omega_2)^{\beta_1 + \beta_2 + 1}} \sum_{k=0}^{p} \frac{(m + \beta_2)_k(\beta_1 + \beta_2 + 1)_k}{k!\Gamma(\beta_2 + 2 + k)}\left(-\frac{\alpha_2\omega_1}{\alpha_1\omega_2}\right)^k(-\beta_2 - k - 1)_n,
$$

Otherwise ($_2F_1$ is the Gauss Hypergeometric function defined in (B.10))

$$
\pi_{n,m}^2(\omega_1, \omega_2) = \frac{(-1)^n\Gamma(\beta_1 + \beta_2 + 1)}{\Gamma(n + \beta_1)\Gamma(\beta_2 + 2 - n)(\alpha_1\omega_2)^{\beta_1 + \beta_2 + 1}} \,_2F_1\left(\begin{matrix} m + \beta_2, \beta_1 + \beta_2 + 1 \\ \beta_2 + 2 - n \end{matrix}; -\frac{\alpha_2\omega_1}{\alpha_1\omega_2} \right).
$$

Lastly, for $n, m \geq 0$,

$$
\pi_{n,m}^3(\gamma_1, \gamma_2) = e^{\gamma_1} \sum_{k=0}^{\infty}\left[\frac{(m + \beta_2)_k\gamma_2^k}{k!} \sum_{l=0}^{n}\binom{n}{l}\frac{\gamma_1^l}{\Gamma(l + \beta_1)}U(k + \beta_2 + 2, k + l + \beta_1 + \beta_2 + 2; -\gamma_1) \right].
$$

(c) If $\omega_0 = 0$ and suppose $K_1 > 0$ (otherwise the option price is zero), the spread put option price $SP(F_1, F_2, T, K)$ is given by ($K_0 = K_1/\omega_2$)

$$
SP(F_1, F_2, T, K) = B(0, t)w_2\left\{ K_0 P(\beta_2, \alpha_2 K_0) + P(\beta_2 + 1, \alpha_2 K_0)\left((\theta_2 - x_2)e^{-\int_0^T \psi_2(\kappa_2, u)du} - \theta_2 \right) \right.
$$

$$
\left. + \alpha_2^{\beta_1}K_0^{\beta_2 + 1}e^{-\alpha_2 K_0} \sum_{n=2}^{\infty} \frac{e^{-\int_0^T \psi_2(\kappa_2 n, u)du}}{\sqrt{n(n-1)}}l_{n-2}^{(\beta_2 + 1)}(\alpha_2 K_0)l_n^{(\beta_2 - 1)}(\alpha_2 x_2) \right\}.
$$

The pricing formulas in Proposition 6.1 to 6.3 can be implemented in C++. The scaled generalized Laguerre polynomial can be computed efficiently through the recursion in (B.2) and all other special functions can be computed by calling functions from the GNU Scientific Library. Alternatively, one can use the software *Mathematica*, which includes all the special functions in the pricing formulas as built-in functions. For each infinite expansion, one can truncate it when a given error tolerance is reached.

6.4   Calibration Examples and Model Implied Correlations

In the current market, single commodity options on crude oil and its refined products are actively traded. In contrast, the trading volume for crack spread options is substantially smaller and it is practically quite impossible to have reliable calibration using such data. For this reason, we will calibrate our model from the implied volatility surface of crude oil and its refined product, which can be done via the following two-step procedure.

- Step 1: Calibrate the parameters of $X_t^{\psi_1}$ from the implied volatility surface of crude oil.

- Step 2: Calibrate the parameters of $X_t^{\psi_2}$ from the implied volatility surface of the refined product.

In our calibration examples, we use the regularized Sato-type IG (RSIG) subordinator, which is the regularized Sato-type additive subordinator constructed from the Inverse Gaussian Lévy subordinator. This type of additive subordinators is a parsimonious extension of Lévy subordinators, hence it is promising in capturing the market in a parsimonious way.

In general our model is described by the following parameters: x_i (the starting point), κ_i (the CIR mean-reversion speed), θ_i (the CIR long-run level), σ_i (the CIR volatility), γ_i (the IG subordinator drift), μ_i (the mean rate of the IG subordinator without drift), v_i (the variance rate of the IG subordinator), ρ_i (the self-similarity index), t_0^i (the regularization parameter) for $i = 1, 2$. Note that for a Lévy subordinator L_t, $E[L_t] = E[L_1]t$ and $\text{Var}[L_t] = \text{Var}[L_1]t$ for any $t > 0$. We call $\mu := E[L_1] - \gamma$ as the mean rate without drift and $v := \text{Var}[L_1]$ as the variance rate. To be parsimonious, we fix the value of the following parameters for $i = 1, 2$.

- t_0^i: It is just a regularization parameter to rule out singularity at time zero for the IG-Sato subordinator. We fix it to be very close to zero.

- γ_0^i: We set it to be zero. In this case, X^{ψ_i} is an infinite activity pure jump process. When jumps have infinite activity, diffusion components seem to be unnecessary as frequent small movements can be captured by small jumps of infinite activity.

- μ_i: We fix it to be one. For a generic ASubCIR process obtained from the RSIG subordinator, we have the following scale invariance that can be directly verified from the eigenfunction expansion formula: changing κ, σ, μ, v to $c\kappa$, $\sqrt{c}\sigma$, $\frac{1}{c}\mu$, $\frac{1}{c^2}v$ does not change the ASubCIR transition density. Hence fixing $\mu_i = 1$ does not cause any loss of generality.

From Remark 6.1, we can also normalize $x_1 = 1$. Hence in total, we only have 11 parameters (x_2 and $\kappa_i, \theta_i, \sigma_i, v_i, \rho_i$ for $i = 1, 2$) to be calibrated from the data.

We calibrate our model for both the heating oil–crude oil pair, and the gasoline–crude oil pair. Market data of implied volatilities for these commodities are downloaded from Bloomberg for February 25, 2014. Due to liquidity, we consider twelve maturities (approximately 1 to 12 months) for crude oil, eight maturities (approximately 1 to 8 months) for heating oil, and five maturities (approximately 1 to 5 months) for gasoline. In each maturity, we look at implied volatility at

nine different levels of moneyness: 0.8, 0.9, 0.95, 0.975, 1.0, 1.025, 1.05, 1.1, 1.2, where moneyness is defined as the strike price divided by the current futures price. In total we have 108 implied volatilities for crude oil, 72 for heating oil and 45 for gasoline.

The calibration is performed by minimizing the sum of squared errors between model and market implied volatilities. The goodness of fit is measured by the average percentage error (APE) as suggested in [19], which is defined as the average absolute pricing error divided by the average option price (all options used are OTM except the ATM ones). This measure is commonly used by practitioners and, as pointed out in [19], market practice regards a particular model as having failed if its APE exceeds 5%. We calculate the APE for the futures options of each commodity. The results are summarized in Table 1. All of them are well below 2%, indicating excellent performance of our model.

	Crude Oil	Heating Oil	Gasoline
APE	1.11%	1.66%	1.47%

Table 1: Calibration errors using the IG-Sato subordinator for data on February 25, 2014

To investigate implications of the calibrated model on the spread option price, we first calculate the spread option price under our model using the calibrated parameters for different moneyness and maturities, and then calculate the implied correlation for each price. The results are shown in Figure 1. The implied correlation is defined as the value of correlation in the classical two asset-GBM model that makes the spread option price exactly equal to a given price. In this model, each asset is assumed to follow a geometric Brownian motion, and the correlation between two driving Brownian motions is constant. To find out the implied correlation, we need to know the volatility for each asset and how to price the spread option. There are different conventions on which volatility to use (see [1]). In this paper we follow [14] and set the volatility of a commodity to be the average of all implied volatilities for that commodity. Since there are no analytical formulas for the spread option under the two asset-GBM model, an approximate pricing formula is used. A standard choice is the Kirk's formula ([40]). We use the approximation developed by [6], which is more accurate than the Kirk's approximation. Given the volatility of each asset and a closed-form approximation formula, the implied volatility can be found out easily by a numerical root-finding algorithm such as bisection.

For a given maturity, implied correlation varies with moneyness, where moneyness is defined as the ratio of strike and the current price difference. In practice, a frown like shape is commonly observed ([32]), which implies the tails of the joint distribution of returns are heavier than the bivariate normal distribution ([1]). In Figure 1, this shape is clearly observed for each maturity and each commodity pair.

For given moneyness, implied correlation increases with time to maturity. This matches economic intuitions, as the price difference of longer-dated futures contracts more closely reflect the average refining cost, while the price difference of shorter-dated contracts also reflect additional short-term issues (see [26]). Finally, all the implied correlations are positive, which is expected due to the production relation between crude oil and its refined products.

7   Conclusions

The first part of this paper develops the theory of additive subordination. Starting with $(\mathcal{P}_t)_{t\geq 0}$, a strongly continuous semigroup of contractions on a Banach space $\mathfrak{B}$, additive subordination de-
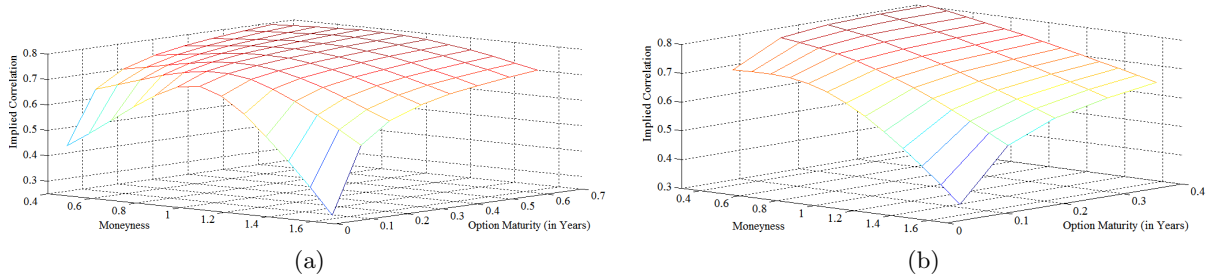
Figure 1: Implied correlation surface for heating oil–crude oil (a) and gasoline–crude oil (b) on February 25, 2014

fines $(\mathcal{P}_{s,t}^{\psi})_{0 \le s \le t}$, a strongly continuous propagator as well as backward propagator of contractions on $\mathfrak{B}$, through (1.4). Under some weak conditions on the differential characteristics of the additive subordinator, we obtain a relation between the infinitesimal generator of $(\mathcal{P}_{s,t}^{\psi})_{0 \le s \le t}$ and that of $(\mathcal{P}_t)_{t \ge 0}$, which generalizes the classical Phillips Theorem. Probabilistically, additive subordination can be viewed as a stochastic time change with respect to an independent additive subordinator. Given a time-homogeneous Markov process X and an additive subordinator T, $Y_t := X_{T_t}$ is a time-inhomogeneous Markov process. Motivated by financial applications, we are particularly interested in the case where X is a diffusion with killing in general. In this case Y is a Markov jump-diffusion or pure jump process with in general time- and state-dependent jumps and killing rate. We provide both Markov and semimartingale characterization for Y, and study a class of equivalent measure changes for it. When $(\mathcal{P}_t)_{t \ge 0}$ is a strongly continuous semigroup of symmetric contractions in a Hilbert space $\mathcal{H}$, we derive spectral decomposition of $(\mathcal{P}_{s,t}^{\psi})_{0 \le s \le t}$ based on the spectral decomposition of $(\mathcal{P}_t)_{t \ge 0}$. We also provide mild sufficient conditions under which the spectral representation becomes an eigenfunction expansion that converges uniformly on compacts. The spectral expansion method provides an analytical approach for derivatives pricing in models based on additive subordinate diffusions.

The second part of this paper illustrates the usefulness of additive subordination as a technique to construct time-inhomogeneous Markov processes with analytical tractability by developing a cross commodity model for crack spread option valuation. Our model captures the essential empirical features of each commodity as well as of their spread, and it admits a closed-form formula for the spread option. Furthermore it is consistent with the implied volatility surface of each commodity and generates implied correlation patterns that match empirical observations and economic intuitions. In a separate paper ([50]), we develop a tractable electricity model using additive subordination that successfully captures seasonal spikes observed in electricity spot prices. We also anticipate additive subordination to be used for modelling time-dependency in various other markets, such as equity, credit and fixed income, where Bochner's subordination has been used. For these potential applications, the theory developed in this paper can be readily applied.

A Proofs

Proposition 2.1: T is a semimartingale since it is a nondecreasing process which implies that it has finite variation over finite time intervals. Let (B_t, A_t, Π_t) be the generating triplet (see [69] Definition 8.2) of the infinitely divisible distribution $q_t = q_{0,t}$ (i.e., the distribution of T_t) for $t \ge 0$,

i.e.,

$$E[e^{i\lambda T_t}] = e^{i\lambda B_t - \frac{1}{2}\lambda^2 A_t + \int_{(0,\infty)}(e^{i\lambda\tau}-1-i\lambda\tau 1_{\{|\tau|\leq 1\}})\Pi_t(d\tau)}. \tag{A.1}$$

From Theorem 9.8 in [69], (i) $B_0 = 0$ and B_t is continuous in t. (ii) $\Pi_0 = 0$, and for all $B \in \mathcal{B}(\mathbb{R}_+)$, $\Pi_s(B) \leq \Pi_t(B)$ and $\Pi_s(B') \to \Pi_t(B')$ as $s \to t$, where $B' \subseteq (\varepsilon, \infty)$, $\varepsilon > 0$. Since T_t is nonnegative, Theorem 24.11 in [69] implies that $A_t = 0$, $\Pi_t((-\infty, 0)) = 0$ and $\int_{(0,\infty)}(\tau \wedge 1)\Pi_t(d\tau) < \infty$, and $E[e^{i\lambda T_t}]$ can also be written as

$$E[e^{i\lambda T_t}] = e^{i\lambda\Gamma_t + \int_{(0,\infty)}(e^{i\lambda\tau}-1)\Pi_t(d\tau)}. \tag{A.2}$$

where $\Gamma_t = B_t - \int_{(0,\infty)}(\tau \wedge 1)\Pi_t(d\tau)$ is nondecreasing in t and $\Gamma_t \geq 0$ for all $t \geq 0$.

Using [69] Remark 9.9, there exists a unique measure $\mathcal{V}$ on $[0,\infty) \times \mathbb{R}_+$ such that, $\mathcal{V}([0,t] \times B) = \Pi_t(B), t \geq 0$ and $B \in \mathcal{B}(\mathbb{R}_+)$, which satisfies $\mathcal{V}(\{t\} \times \mathbb{R}_+) = 0$ and $\int_{[0,t] \times (0,\infty)}(\tau \wedge 1)\mathcal{V}(dsd\tau) < \infty$. From (A.1) and Theorem II.4.15 in [39], the deterministic triplet $(B, 0, \mathcal{V})$ are the semimartingale characteristics for T. Furthermore since T is stochastically continuous, it has no fixed time of discontinuity, and hence it is quasi-left-continuous ([39], Corollary II.4.18). Applying Proposition II.2.9 of [39] to T, B and $\mathcal{V}$ can be written as

$$B_t = \int_0^t b(s)F(ds), \quad \mathcal{V}(dsd\tau) = \nu(s, d\tau)F(ds),$$

for some nonnegative continuous nondecreasing deterministic function $F(s)$. Define $\gamma(s) = b(s) - \int_{(0,\infty)}(\tau \wedge 1)\nu(s, d\tau)$, then $\Gamma_t = \int_0^t \gamma(s)F(ds)$. Since Γ_t is nondecreasing, we have $\gamma(s) \geq 0$ F-a.s. for s. $\int_0^t \int_{(0,\infty)}(\tau \wedge 1)\nu(s, d\tau)F(ds) < \infty$ for all $t \geq 0$ implies that $\int_{(0,\infty)}(\tau \wedge 1)\nu(s, d\tau) < \infty$ F-a.s. for s. The expression for the Laplace transform follows from (A.2). $\qquad\square$

Theorem 3.1: (i) Each $\mathcal{P}_{s,t}^{\psi}$ is a contraction because

$$\|\mathcal{P}_{s,t}^{\psi}f\| \leq \int_{[0,\infty)} \|\mathcal{P}_u f\| q_{s,t}(du) \leq \int_{[0,\infty)} \|f\| q_{s,t}(du) = \|f\|.$$

For $0 \leq s \leq t \leq r$,

$$\mathcal{P}_{s,t}^{\psi}\mathcal{P}_{t,r}^{\psi}f = \mathcal{P}_{s,t}^{\psi}\int_{[0,\infty)} \mathcal{P}_{u_1}f q_{t,r}(du_1) = \int_{[0,\infty)} \mathcal{P}_{s,t}^{\psi}\mathcal{P}_{u_1}f q_{t,r}(du_1)$$

$$= \int_{[0,\infty)}\int_{[0,\infty)} \mathcal{P}_{u_2}\mathcal{P}_{u_1}f q_{s,t}(du_2)q_{t,r}(du_1) = \int_{[0,\infty)}\int_{[0,\infty)} \mathcal{P}_{u_1+u_2}f q_{s,t}(du_2)q_{t,r}(du_1)$$

$$= \int_{[0,\infty)} \mathcal{P}_u f q_{s,r}(du) = \mathcal{P}_{s,r}^{\psi}f.$$

In the second to last equality we used the convolution property (2.1). This shows $\mathcal{P}_{s,r}^{\psi} = \mathcal{P}_{s,t}^{\psi}\mathcal{P}_{t,r}^{\psi}$. By (2.2), $q_{t,t} = \delta_0$, hence $\mathcal{P}_{t,t}^{\psi}f = f$. Together we have shown $(\mathcal{P}_{s,t}^{\psi})_{0\leq s\leq t}$ is a backward propagator. From Fubini's theorem for Bochner integrals (see, [30], Theorem E.8),

$$\int_{[0,\infty)}\int_{[0,\infty)} \mathcal{P}_{u_1+u_2}f q_{s,t}(du_2)q_{t,r}(du_1) = \int_{[0,\infty)}\int_{[0,\infty)} \mathcal{P}_{u_1+u_2}f q_{t,r}(du_1)q_{s,t}(du_2),$$

Hence we also have

$$\mathcal{P}_{s,r}^{\psi} = \mathcal{P}_{s,t}^{\psi}\mathcal{P}_{t,r}^{\psi} = \mathcal{P}_{t,r}^{\psi}\mathcal{P}_{s,t}^{\psi}. \tag{A.3}$$

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This implies $(\mathcal{P}^\psi_{s,t})_{0 \le s \le t}$ is also a propagator. Finally, we want to show that $(\mathcal{P}^\psi_{s,t})_{0 \le s \le t}$ is strongly continuous. From Theorem 2.1 in [37], the latter is equivalent to $(\mathcal{P}^\psi_{s,t})_{0 \le s \le t}$ being separately strongly continuous (i.e., for every fixed t and $f \in \mathfrak{B}$, $s \mapsto \mathcal{P}^\psi_{s,t}f$ is continuous on $[0,t]$, and for every fixed s and $f \in \mathfrak{B}$, $t \mapsto \mathcal{P}^\psi_{s,t}f$ is continuous on $[s,\infty)$) and locally uniformly bounded (i.e., for every compact set K of $\{(s,t) : 0 \le s \le t < \infty\}$, $\|\mathcal{P}^\psi_{s,t}\|$ is uniformly bounded for every $(s,t) \in K$). The locally uniformly boundedness is obvious, since every $\mathcal{P}^\psi_{s,t}$ is a contraction. Next, we consider the separate strong continuity. First, we prove that for every $f \in \mathfrak{B}$,

$$\mathcal{P}^\psi_{s,t}f \to f \text{ as } s \uparrow t, \quad \mathcal{P}^\psi_{s,t}f \to f \text{ as } t \downarrow s. \tag{A.4}$$

We have

$$\|\mathcal{P}^\psi_{s,t}f - f\| = \left\| \int_{(0,\infty)} (\mathcal{P}_u f - f) q_{s,t}(du) \right\| \le \int_{(0,\infty)} \|\mathcal{P}_u f - f\| q_{s,t}(du).$$

Since $\|\mathcal{P}_u f - f\|$ is a bounded continuous function in u, applying (2.3) shows (A.4). For fixed t and f, to show $s \mapsto \mathcal{P}^\psi_{s,t}f$ is continuous on $[0,t]$, we need to show for every $s < t$, $\mathcal{P}^\psi_{s-h,t}f \to \mathcal{P}^\psi_{s,t}f$ and $\mathcal{P}^\psi_{s+h,t}f \to \mathcal{P}^\psi_{s,t}f$ as $h \downarrow 0$. Note that by (A.3) and the contraction property,

$$\|\mathcal{P}^\psi_{s-h,t}f - \mathcal{P}^\psi_{s,t}f\| = \|\mathcal{P}^\psi_{s,t}(\mathcal{P}^\psi_{s-h,s} - I)f\| \le \|(\mathcal{P}^\psi_{s-h,s} - I)f\|,$$
$$\|\mathcal{P}^\psi_{s+h,t}f - \mathcal{P}^\psi_{s,t}f\| = \|\mathcal{P}^\psi_{s+h,t}(I - \mathcal{P}^\psi_{s,s+h})f\| \le \|(I - \mathcal{P}^\psi_{s,s+h})f\|,$$

Hence, (A.4) implies the continuity of $\mathcal{P}^\psi_{s,t}f$ in s. The continuity of $\mathcal{P}^\psi_{s,t}f$ in t can be proved similarly.

(ii) Recall that for every $t \ge 0$, T^{ϕ_t} is the Lévy subordinator whose drift and Lévy measure are given by $\gamma(t)$ and $\nu(t,\cdot)$. $(\mathcal{P}^{\phi_t}_u)_{u \ge 0}$ is the Lévy subordinate semigroup of $(\mathcal{P}_s)_{s \ge 0}$ w.r.t. T^{ϕ_t}. Its generator is denoted by $\mathcal{G}^{\phi_t}$ and we have (3.3) from the classical Phillips' Theorem. We want to show for any $s,t \ge 0$, $(\mathcal{P}^{\phi_s}_u)_{u \ge 0}$ and $(\mathcal{P}^{\phi_t}_u)_{u \ge 0}$ commute, that is, for any $u,v \ge 0$, $\mathcal{P}^{\phi_s}_u \mathcal{P}^{\phi_t}_v f = \mathcal{P}^{\phi_t}_v \mathcal{P}^{\phi_s}_u f$ for $f \in \mathfrak{B}$. Denote by $\pi^{\phi_s}_u(\cdot)$ and $\pi^{\phi_t}_v(\cdot)$ the distribution of $T^{\phi_s}_u$ and $T^{\phi_t}_v$. Then we have

$$\mathcal{P}^{\phi_s}_u \mathcal{P}^{\phi_t}_v f = \int_{[0,\infty)} \mathcal{P}_r \Big(\int_{[0,\infty)} \mathcal{P}_\tau f \pi^{\phi_t}_v(d\tau) \Big) \pi^{\phi_s}_u(dr) = \int_{[0,\infty)} \int_{[0,\infty)} (\mathcal{P}_r \mathcal{P}_\tau f) \pi^{\phi_t}_v(d\tau) \pi^{\phi_s}_u(dr)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} (\mathcal{P}_{r+\tau} f) \pi^{\phi_t}_v(d\tau) \pi^{\phi_s}_u(dr) = \int_{[0,\infty)} \int_{[0,\infty)} (\mathcal{P}_{\tau+r} f) \pi^{\phi_s}_u(dr) \pi^{\phi_t}_v(d\tau)$$

$$= \int_{[0,\infty)} \mathcal{P}_\tau \Big(\int_{[0,\infty)} \mathcal{P}_r f \pi^{\phi_s}_u(dr) \Big) \pi^{\phi_t}_v(d\tau) = \mathcal{P}^{\phi_t}_v \mathcal{P}^{\phi_s}_u f,$$

The interchange of the order of integration is justified by Fubini's theorem. We next verify the following statement: for $f \in \mathfrak{D}(\mathcal{G})$, $\mathcal{G}^{\phi_{t-}} f = \lim_{s \to t-} \mathcal{G}^{\phi_s} f$ and $\mathcal{G}^{\phi_{t+}} f = \lim_{s \to t+} \mathcal{G}^{\phi_s} f$ exists, and $\mathcal{G}^{\phi_{t+}} f = \mathcal{G}^{\phi_t} f$ for every $t \ge 0$. Condition (a) implies the statement for the first part on the RHS of (3.3). For the second part, we note that

$$\int_{(0,\infty)} (\mathcal{P}_\tau f - f) \nu(t, d\tau) = \int_{(0,\infty)} \frac{\mathcal{P}_\tau f - f}{\tau \wedge 1} \nu_F(t, d\tau)$$

It is easy to see that $\frac{\mathcal{P}_\tau f - f}{\tau \wedge 1}$ is a continuous function in τ, and bounded in the Banach space norm since $\|\mathcal{P}_\tau f - f\| \le \min\{\tau \|\mathcal{G}f\|, 2\|f\|\}$ ([70], eq.(13.3)). Therefore, by Theorem 2 of [63], the weak convergence of $\nu_F(t,\cdot)$ assumed in condition (b) implies the statement for the second part on the RHS of (3.3) ([63] deals with probability measures, but his result also applies to weakly convergent

finite measures.). From condition (c), we also have $\mathcal{G}^{\phi_t-}f = \mathcal{G}^{\phi_t+}f = \mathcal{G}^{\phi_t}f$ on $\mathfrak{D}(\mathcal{G})$ for all but a finite number of t in any bounded interval.

Recall $R_{s,t}^{\Pi}$ defined in (3.4). We have verified all the conditions of Theorem 3.1 in [36], which implies that $U_{s,t}f := \lim_{|\Pi|\to 0} R^{\Pi}f$ for $f \in \mathfrak{B}$ exists and $(U_{s,t})_{0\leq s\leq t}$ is a strongly continuous contraction propagator on $\mathfrak{B}$. Furthermore, for $f \in \mathfrak{D}(\mathcal{G})$, the family of generators of $(U_{s,t})_{0\leq s\leq t}$ is given by (3.5).

We now prove $U_{s,t} = \mathcal{P}_{s,t}^{\psi}$ on $\mathfrak{B}$ for $0 \leq s < t$. For $\Pi : s = t_0 < t_1 < \cdots < t_n = t$, define $q_{s,t}^{\Pi} := \pi_{t_1-t_0}^{\phi_{t_0}} * \pi_{t_2-t_1}^{\phi_{t_1}} * \cdots * \pi_{t_n-t_{n-1}}^{\phi_{t_{n-1}}}$, where $*$ denotes convolution. From the property of convolution, we have $R_{s,t}^{\Pi}f = \int_{[0,\infty)} \mathcal{P}_u f q_{s,t}^{\Pi}(du)$. The Laplace transform of $q_{s,t}^{\Pi}$ is

$$\int_{(0,\infty)} e^{-\lambda u} q_{s,t}^{\Pi}(du) = e^{-\sum_{i=0}^{n-1} \psi(\lambda,t_i)(t_{i+1}-t_i)}, \tag{A.5}$$

where $\psi(\lambda,\cdot)$ is defined in (2.5). Under the assumed conditions (a) to (c), $\psi(\lambda,t)$ is piecewise continuous in t. Hence as $|\Pi| \to 0$, (A.5) converges to the Laplace transform of $q_{s,t}$. Therefore, $q_{s,t}^{\Pi}$ converges to $q_{s,t}$ weakly, which implies that for any continuous linear functional l on $\mathfrak{B}$,

$$l(R^{\Pi}f) = \int_{(0,\infty)} l(\mathcal{P}_u f) q_{s,t}^{\Pi}(du) \to \int_{(0,\infty)} l(\mathcal{P}_u f) q_{s,t}(du) = l(\mathcal{P}_{s,t}^{\psi}f), \quad \text{for any } f \in \mathfrak{B},$$

since $l(\mathcal{P}_u f)$ is a continuous bounded function in u. Recall that $U_{s,t}f$ is the strong limit of $R^{\Pi}f$, hence $U_{s,t}f = \mathcal{P}_{s,t}^{\psi}f$. This allows us to conclude from (3.5) that

$$\lim_{h\to 0+} h^{-1}(\mathcal{P}_{t,t+h}^{\psi}f - f) = \mathcal{G}^{\phi_t}f, \quad \text{for } f \in \mathfrak{D}(\mathcal{G}).$$

Hence $\mathfrak{D}(\mathcal{G}) \subseteq \mathfrak{D}(\mathcal{G}_t^{\psi})$, and (3.3) gives (3.1). (3.2) follows from Theorem 3.1 in [36]. $\qquad \square$

Proposition 3.1: We only need to verify the finiteness of $\int_{\mathbb{R}^d} |\psi(-\eta_X(\theta),t)|^2 |\hat{f}(\theta)|^2 d\theta$ when $\int_{\mathbb{R}^d} |\eta_X(\theta)|^2 |\hat{f}(\theta)|^2 d\theta$ is finite. Notice that for $\psi(\lambda,t)$ ($\lambda \in \mathbb{C}$ with its real part $\Re(\lambda) \geq 0$):

$$|\psi(\lambda,t)| \leq \gamma(t)|\lambda| + 2(1+|\lambda|) \int_{(0,\infty)} (1 \wedge \tau)\nu(t,d\tau) \leq c(t)(1+|\lambda|)$$

for some suitable $c(t) > 0$ ($c(t)$ is a constant that only depends on t), which follows from the inequality $1 \wedge (|\lambda|\tau) \leq (1+|\lambda|)(1 \wedge \tau)$. Hence $|\psi(\lambda,t)|^2 \leq 2c^2(t)(1+|\lambda|^2)$, and the claim is implied by $\int_{\mathbb{R}^d} |\hat{f}(\theta)|^2 d\theta < \infty$ and $\int_{\mathbb{R}^d} |\eta_X(\theta)|^2 |\hat{f}(\theta)|^2 d\theta < \infty$. $\qquad \square$

Proposition 4.1: By conditioning on the sample path of X^0 between 0 and h and using the independence of the exponential r.v. and X^0, we obtain that for any measurable and bounded function f on I,

$$E_x \left[f(X_h) 1_{\{\zeta > h\}} \right] = E_x \left[e^{-\int_0^h k(X_u^0) du} f(X_h^0) 1_{\{\zeta_0 > h\}} \right].$$

We want to show that

$$h^{-1} E_x \left[f(X_h^0) 1_{\{\zeta_0 > h\}} \right] - h^{-1} E_x \left[f(X_h) 1_{\{\zeta > h\}} \right] = E_x \left[h^{-1}(1 - e^{-\int_0^h k(X_u^0) du}) f(X_h^0) 1_{\{\zeta_0 > h\}} \right]$$

$$\xrightarrow{bp} k(x) f(x),$$

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where the convergence is bounded pointwise on compact intervals of I. Let J be such an interval. Pick δ small enough such that for all $x \in J$, $[x-\delta, x+\delta] \subseteq \hat{J} \subset (l, r)$, where $\hat{J}$ is a compact interval. Let $\tau_x^\delta := \inf\{t \geq 0 : |X_t^0 - x| \geq \delta\}$. We have

$$E_x\left[h^{-1}(1 - e^{-\int_0^h k(X_u^0)du})f(X_h^0)1_{\{\zeta_0 > h\}}\right]$$

$$= E_x\left[h^{-1}(1 - e^{-\int_0^h k(X_u^0)du})f(X_h^0)1_{\{\tau_x^\delta > h\}}\right] + E_x\left[h^{-1}(1 - e^{-\int_0^h k(X_u^0)du})f(X_h^0)1_{\{\tau_x^\delta \leq h, \zeta_0 > h\}}\right].$$

The second term is bounded by $\|f\|_\infty E_x\left[h^{-1}1_{\{\tau_x^\delta \leq h, \zeta_0 > h\}}\right]$ ($\|f\|_\infty$ is the L^∞-norm of f), which converges to 0 boundedly pointwise on J as shown in the proof of Theorem 16.84 of [11] (see its claim (iº)). For the first term, notice that on $\{\tau_x^\delta > h\}$, $k(X_u^0)$ is bounded (say by M) for all $0 \leq u \leq h$ as $k(x)$ is continuous. Thus $|h^{-1}(1 - e^{-\int_0^h k(X_u^0)du})| \leq h^{-1}(1 - e^{-Mh})$, which is bounded for h sufficiently small. It follows that the first term is also bounded for h sufficiently small. Applying the dominated convergence theorem shows that it converges to $k(x)f(x)$ boundedly pointwise on J.

Now setting $f(y) = 1_{\{|y-x|>\varepsilon\}}$, $(y-x)1_{\{|y-x|\leq\varepsilon\}}$, $(y-x)^2 1_{\{|y-x|\leq\varepsilon\}}$, 1, respectively and applying (4.2), (4.3), (4.4) and (4.5) give us (4.6), (4.7), (4.8) and (4.9). $\qquad\square$

Theorem 4.1: Theorem 3.1 (ii) implies that, for $f \in C_c^2(I)$,

$$\mathcal{G}_t^\psi f(x) = \gamma(t)\frac{1}{2}\sigma^2(x)f''(x) + \gamma(t)\mu(x)f'(x) - \gamma(t)k(x)f(x) + \int_{(0,\infty)}(\mathcal{P}_\tau f(x) - f(x))\nu(t, d\tau).$$

where $(\mathcal{P}_t)_{t\geq 0}$ is the transition semigroup of the underlying diffusion. We write the last term as follows.

$$\int_{(0,\infty)}(\mathcal{P}_\tau f(x) - f(x))\nu(t, d\tau) = \int_{(0,\infty)}\left(\int_{\mathbb{R}} p(\tau, x, x+y)f(x+y)dy - f(x)\right)\nu(t, d\tau)$$

$$= \int_{(0,\infty)}\left\{\int_{\mathbb{R}} p(\tau, x, x+y)\left[(f(x+y) - f(x) - 1_{\{|y|\leq 1\}}yf'(x)) + f(x) + 1_{\{|y|\leq 1\}}yf'(x)\right]dy \right.$$

$$\left. - f(x)\right\}\nu(t, d\tau)$$

$$= \int_{\mathbb{R}}\left(f(x+y) - f(x) - 1_{\{|y|\leq 1\}}yf'(x)\right)\left(\int_{(0,\infty)} p(\tau, x, x+y)\nu(t, d\tau)\right)dy$$

$$+ f(x)\int_{(0,\infty)}\left(1 - \int_{\mathbb{R}} p(\tau, x, x+y)dy\right)\nu(t, d\tau)$$

$$+ f'(x)\int_{(0,\infty)}\left(\int_{\{|y|\leq 1\}} yp(\tau, x, x+y)dy\right)\nu(t, d\tau)$$

Combining this with other terms yields (4.10). We now justify the interchange of order of integration in the above derivation is valid. Notice that for $f \in C_c^2(I)$, $|f(x+y) - f(x) - 1_{\{|y|\leq 1\}}yf'(x)| \leq C_x(1 \wedge y^2)$ for some positive constant C_x which only depends on x. Thus, if we can show

$$\int_{(0,\infty)}\int_{\mathbb{R}}(1 \wedge y^2)p(\tau, x, x+y)dy\nu(t, d\tau) < \infty, \tag{A.6}$$

then we can apply the dominated convergence theorem to justify the interchange. This also implies that $\Pi^\psi(t, x, dy)$ is a Lévy-type measure. To prove (A.6), we notice that

$$\int_{(0,\infty)}\int_{\mathbb{R}}(1 \wedge y^2)p(\tau, x, x+y)dy\nu(t, d\tau)$$

$$= \int_{(0,\infty)} \int_{|y|\leq 1} y^2 p(\tau, x, x+y) dy \nu(t, d\tau) + \int_{(0,\infty)} \int_{|y|>1} p(\tau, x, x+y) dy \nu(t, d\tau)$$

Note that $\int_{|y|\leq 1} y^2 p(\tau, x, x+y) dy$ and $\int_{|y|>1} p(\tau, x, x+y) dy$ are bounded by 1 for all $\tau > 0$. (4.8) implies that $\int_{|y|\leq 1} y^2 p(\tau, x, x+y) dy \sim \sigma^2(x)\tau$ as $\tau \to 0$. From (4.6), $\int_{|y|>1} p(\tau, x, x+y) dy = o(\tau)$ as $\tau \to 0$. These facts together with $\int_{(0,\infty)} (\tau \wedge 1) \nu(t, d\tau) < \infty$ shows (A.6). Similar arguments also imply that the term $\int_{(0,\infty)} \left(\int_{\{|y|\leq 1\}} yp(\tau, x, x+y) dy \right) \nu(t, d\tau)$ and $\int_{(0,\infty)} P(\tau, x, \{\Delta\}) \nu(t, d\tau)$ are well-defined by noticing that (4.7) implies $\int_{\{|y|\leq 1\}} yp(\tau, x, x+y) dy \sim \mu(x)\tau$ and (4.9) implies $P(\tau, x, \{\Delta\}) \sim k(x)\tau$ as $\tau \to 0$. $\qquad \square$

Theorem 4.2: If we can show for $f \in C_b^2(I)$ (bounded and twice continuously differentiable functions on I),

$$M^f := f(\hat{X}^\psi) - f(x) - f'(\hat{X}_-^\psi) \cdot B^\psi - \frac{1}{2} f''(\hat{X}_-^\psi) \cdot C^\psi - (f(\hat{X}_-^\psi + y) - f(\hat{X}_-^\psi) - f'(\hat{X}_-^\psi)y 1_{\{|y|\leq 1\}}) * \nu^\psi$$

is a local martingale ("$\cdot$" and "$*$" denote stochastic integration w.r.t. a semimartingale and a random measure, respectively; see [39]), then Theorem II.2.42 in [39] implies that $\hat{X}^\psi$ is a semimartingale with $(B^\psi, C^\psi, \nu^\psi)$ as the characteristics. To show this, notice the following two things. 1. Eq.(3.2) and Theorem 4.1 imply that for $f \in C_c^2(I)$, $\mathcal{P}_{s,t}^\psi f - f = \int_s^t \mathcal{P}_{s,u}^\psi \mathcal{G}_u^\psi f du$. Thus under $P_{s,x}$, $f(X_t) - f(X_s) - \int_s^t \mathcal{G}_u^\psi f(X_u) du$ is a martingale w.r.t. $(\mathcal{F}_{s,t}^0)_{t\geq s}$. From [21], Remark 2.3, No.2, it is also a martingale w.r.t. $(\mathcal{F}_{s,t+}^0)_{t\geq s}$. 2. From the bounded pointwise convergence on compacts for (4.6), (4.7), (4.8) and (4.9) and using arguments from the proof of Theorem 4.1, we can show that on any compact interval in I for x,

$$\left| \int_{(0,\infty)} \left(\int_{\{|y|\leq 1\}} yp(\tau, x, x+y) dy \right) \nu(t, d\tau) \right| \leq C_1 \int_{(0,\infty)} (\tau \wedge 1) \nu(t, d\tau),$$

$$\left| \int_{(0,\infty)} P(\tau, x, \{\Delta\}) \nu(t, d\tau) \right| \leq C_2 \int_{(0,\infty)} (\tau \wedge 1) \nu(t, d\tau),$$

$$\left| \int_{y\neq 0} (y^2 \wedge 1) \Pi^\psi(t, x, dy) \right| \leq C_3 \int_{(0,\infty)} (\tau \wedge 1) \nu(t, d\tau).$$

for some positive constant C_1, C_2 and C_3 which do not depend on t and x. Furthermore, from conditions (a) to (c) of Theorem 3.1 (ii), on any compact interval for t, $\int_{(0,\infty)} (\tau \wedge 1) \nu(t, d\tau)$ and $\gamma(t)$ are continuous in t except for a finite number of points, hence bounded. Also note that $\mu(x)$, $\sigma(x)$ and $k(x)$ are continuous. These imply that

$$\mu^\psi(t, x), \sigma^\psi(t, x), k^\psi(t, x) \text{ and } \int_{y\neq 0} (y^2 \wedge 1) \Pi^\psi(t, x, dy)$$

are bounded on every compact set for t and x.

To prove the claim, based on the conclusions in 1 and 2, one can use the arguments in the proof of Proposition 3.2 in [21]. The details are omitted here. $\qquad \square$

Theorem 5.1: Theorem 3.1 (i) already implies that $(\mathcal{P}_{s,t}^\psi)_{0\leq s\leq t}$ is a strongly continuous propagator/backward propagator of contractions on $\mathcal{H}$. We next prove each $\mathcal{P}_{s,t}^\psi$ is symmetric. For $f, g \in \mathcal{H}$,

$$\left\langle \mathcal{P}_{s,t}^\psi f, g \right\rangle = \left\langle \int_{[0,\infty)} \mathcal{P}_u f q_{s,t}(du), g \right\rangle = \int_{[0,\infty)} \left\langle \mathcal{P}_u f, g \right\rangle q_{s,t}(du)$$

$$= \int_{[0,\infty)} \langle f, \mathcal{P}_u g \rangle \, q_{s,t}(du) = \left\langle f, \int_{[0,\infty)} \mathcal{P}_u g q_{s,t}(du) \right\rangle = \left\langle f, \mathcal{P}_{s,t}^\psi g \right\rangle.$$

This shows the symmetry. From Fubini's theorem, we observe that for all $f \in \mathcal{H}$ and $0 \le s \le t$,

$$\int_{[0,\infty)} \mathcal{P}_u f q_{s,t}(du) = \int_{[0,\infty)} \int_{(-\infty,0]} e^{\lambda u} E(d\lambda) f q_{s,t}(du) = \int_{(-\infty,0]} \int_{[0,\infty)} e^{\lambda u} q_{s,t}(du) E(d\lambda) f,$$

which yields the spectral decomposition (5.2). $\qquad\square$

Proposition 5.1: The claim can be proved using arguments similar to those used in the proof of Proposition 1 in [45]. We omit the details here. $\qquad\square$

Proposition 6.1: The futures price is the conditional expectation of the spot price under the pricing measure. In our model, $F_1(t,T) = a_1(T)E[X_T^{\psi_1}|X_t^{\psi_1}]$ and $F_2(t,T) = a_2(T)(E[X_T^{\psi_1}|X_t^{\psi_1}] + E[X_T^{\psi_2}|X_t^{\psi_2}])$ for any $0 \le t \le T$. Therefore we just need to calculate $\mathcal{P}_{t,T}^\psi f(x)$ with $f(x) = x$ for a generic ASubCIR process. It is easy to verify $x \in L^2(\mathbb{R}_{++}, \mathfrak{m})$. From [5], p.115, for a function $g(x)$ such that its derivatives up to order n are bounded as $x \to 0$ and of at most polynomial growth as $x \to \infty$ ($L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial),

$$\int_0^\infty g(x) L_n^{(\alpha)}(x) x^\alpha \, e^{-x} \, dx = \frac{(-1)^n}{n!} \int_0^\infty e^{-x} \, g^{(n)}(x) x^{n+\alpha} dx.$$

Hence for $n > 1$, $f_n = 0$. It is straightforward to find out f_0 and f_1 using the explicit expression of $\varphi_0(x)$ and $\varphi_1(x)$ (see (6.2)), as well as some elementary integration. They are given by

$$f_0 = \alpha^{-\frac{\beta+1}{2}} \frac{\Gamma(1+\beta)}{\sqrt{\kappa \Gamma(\beta)}}, \quad f_1 = -\alpha^{-\frac{\beta+1}{2}} \frac{\Gamma(1+\beta)}{\sqrt{\kappa \Gamma(\beta+1)}}.$$

Thus, for any $T > t$,

$$\begin{aligned}
E[X_T^\psi | X_t^\psi = x] &= e^{-\int_t^T \psi(0,u)du} \varphi_0(x) f_0 + e^{-\int_t^T \psi(\kappa,u)du} \varphi_1(x) f_1 \\
&= \frac{\beta}{\alpha} + e^{-\int_t^T \psi(\kappa,u)du} \left(x - \frac{\beta}{\alpha} \right) = \theta + e^{-\int_t^T \psi(\kappa,u)du} (x - \theta)
\end{aligned}$$

where we have used the definitions of α and β in (6.1). Some further simple calculations give us the claim. $\qquad\square$

Proposition 6.2: (1) Recall that $\mathbf{M}(a,c;z)$ is the scaled Kummer's confluent hypergeometric function defined in (B.4). We compute expansion coefficients for the payoff $(K-x)^+$ which is in $L^2(\mathbb{R}_{++}, \mathfrak{m})$.

$$\begin{aligned}
f_n(K) &= \int_0^\infty (K-x)^+ \varphi_n(x) \mathfrak{m}(dx) \\
&= \sqrt{\kappa} \alpha^{\frac{\beta-1}{2}} \sqrt{\frac{\Gamma(n+\beta)}{n!}} \frac{\alpha^{2-\beta}}{\kappa} \int_0^\infty (K-x)^+ (\alpha x)^{\beta-1} e^{-\alpha x} \mathbf{M}(-n, \beta; \alpha x) dx.
\end{aligned}$$

Using the Kummer transformation identity (B.7) and change of variable $x = Ky$, we have

$$f_n(K) = \alpha^{-\frac{\beta+1}{2}} \sqrt{\frac{\Gamma(n+\beta)}{n!\kappa}} (\alpha K)^{\beta+1} \int_0^1 \mathbf{M}(n+\beta, \beta; -\alpha Ky) y^{\beta-1} (1-y) dy$$

$$= \alpha^{-\frac{\beta+1}{2}} \sqrt{\frac{\Gamma(n+\beta)}{n!\kappa}} (\alpha K)^{\beta+1} \mathbf{M}(n+\beta, \beta+2; -\alpha K) \tag{A.7}$$

$$= \alpha^{-\frac{\beta+1}{2}} \sqrt{\frac{\Gamma(n+\beta)}{n!\kappa}} \, \mathrm{e}^{-\alpha K} (\alpha K)^{\beta+1} \mathbf{M}(2-n, \beta+2; \alpha K).$$

Here, we have used (B.8), the integral representation of $\mathbf{M}(a,b,z)$. Moreover, using (B.5), we can write $f_n(K)$ as

$$f_n(K) = \frac{1}{\sqrt{n(n-1)\kappa}} \alpha^{-\frac{\beta+1}{2}} \, \mathrm{e}^{-\alpha K} (\alpha K)^{\beta+1} l_{n-2}^{(\beta+1)}(\alpha K), \quad \text{for any} \ \ n \geq 2.$$

To express $f_0(K)$ and $f_1(K)$ in terms of incomplete gamma functions, we use $M(1, a+1; z) = \mathrm{e}^z \, az^{-a}\gamma(a,z)$ ([65], p.328, Eq.(13.6.5)), and $aM(a+1,b;z) = (a-b+1)M(a,b;z) + (b-1)M(a,b-1;z)$ ([65], p.325, Eq.(13.3.3)), where M is Kummer's confluent hypergeometric function defined in (B.3). We have

$$f_0(K) = \frac{\alpha^{-\frac{\beta+1}{2}}}{\sqrt{\kappa\Gamma(\beta)}} \Big[\alpha K \gamma(\beta, \alpha K) - \gamma(\beta+1, \alpha K) \Big], \ \ f_1(K) = \frac{1}{\sqrt{\kappa\Gamma(\beta+1)}} \alpha^{-\frac{\beta+1}{2}} \gamma(\beta+1, \alpha K).$$

The claim can be proved by substituting the expression for $f_n(K)$ back into the eigenfunction expansion and simplify.

(2) It is easy to verify $f(x_1, x_2) := (K - \omega_1 x_1 - \omega_2 x_2)^+ \in L^2(\mathbb{R}_{++}^2, \mathfrak{M})$. Let $k_1(x) = \frac{K}{\omega_2} - \frac{\omega_1}{\omega_2} x$.

$$f_{n,m} = \int_0^\infty \int_0^\infty (K - \omega_1 x_1 - \omega_2 x_2)^+ \varphi_n^1(x_1)\varphi_m^2(x_2)\mathfrak{m}_1(dx_1)\mathfrak{m}_2(dx_2)$$

$$= \omega_2 \int_0^\infty \mathbf{1}_{\{k_1(x_1)>0\}} \left[\int_0^{k_1(x_1)} \big(k_1(x_1) - x_2\big)\varphi_m^2(x_2)\mathfrak{m}_2(dx_2) \right] \varphi_n^1(x_1)\mathfrak{m}_1(dx_1)$$

$$= \frac{K^{\beta_1+\beta_2+1}}{\sqrt{\kappa_1\kappa_2}\omega_1^{\beta_1}\omega_2^{\beta_2}} \alpha_1^{\frac{\beta_1+1}{2}} \alpha_2^{\frac{\beta_2+1}{2}} \sqrt{\frac{\Gamma(n+\beta_1)\Gamma(m+\beta_2)}{n!m!}}$$

$$\times \int_0^1 y^{\beta_1-1}(1-y)^{\beta_2+1}\mathbf{M}(n+\beta_1, \beta_1; -\gamma_1 y)\mathbf{M}\big(m+\beta_2, \beta_2+2; -\gamma_2(1-y)\big)dy,$$

where we used (A.7). Some simplification gives us the put option formula for the daughter commodity. Using (B.3) and (B.8), we get

$$\pi_{n,m}(\gamma_1, \gamma_2) = \sum_{p=0}^\infty \left[\frac{(m+\beta_2)_p(-\gamma_2)^p}{p!\Gamma(\beta_2+2+p)} \int_0^1 y^{\beta_1-1}(1-y)^{p+\beta_2+1}\mathbf{M}(n+\beta_1, \beta_1; -\gamma_1 y)dy \right],$$

$$= \sum_{p=0}^\infty \frac{(m+\beta_2)_p(-\gamma_2)^p}{p!}\mathbf{M}\big(n+\beta_1, \beta_1+\beta_2+2+p; -\gamma_1\big).$$

This proves the claim. $\qquad\square$

Proposition 6.3: We only prove case (b). The other cases are similar to Proposition 6.2. We will compute expansion coefficients for the payoff $g(x_1, x_2) := (K + \omega_1 x_1 - \omega_2 x_2)^+ \in L^2(\mathbb{R}_{++}^2, \mathfrak{M})$ for $\omega_1 > 0$ and $\omega_2 > 0$. Let $k_1(x) = \frac{K}{\omega_2} + \frac{\omega_1}{\omega_2} x$. Using (A.7), (B.6) and (B.7),

$$g_{n,m} = \int_0^\infty \int_0^\infty (K + \omega_1 x_1 - \omega_2 x_2)^+ \varphi_n^1(x_1)\varphi_m^2(x_2)\mathfrak{m}_1(dx_1)\mathfrak{m}_2(dx_2)$$

$$= \omega_2 \int_0^\infty \mathbf{1}_{\{k_1(x_1)>0\}} \left[\int_0^{k_1(x_1)} \big(k_1(x_1) - x_2\big) \varphi_m^2(x_2) \mathfrak{m}_2(dx_2) \right] \varphi_n^1(x_1) \mathfrak{m}_1(dx_1)$$

$$= \frac{\omega_2}{\sqrt{\kappa_1 \kappa_2}} \alpha_1^{\frac{\beta_1+1}{2}} \alpha_2^{\frac{\beta_2+1}{2}} \sqrt{\frac{\Gamma(n+\beta_1)\Gamma(m+\beta_2)}{n!m!}}$$

$$\times \int_0^\infty y^{\beta_1-1} \big(\tfrac{K}{\omega_2} + \tfrac{\omega_1}{\omega_2}y\big)^{\beta_2+1} \mathbf{M}(n+\beta_1, \beta_1; -\alpha_1 y) \mathbf{M}\big(m+\beta_2, \beta_2+2; -\alpha_2\big(\tfrac{K}{\omega_2} + \tfrac{\omega_1}{\omega_2}y\big)\big) dy,$$

which can be expressed in terms of $\pi_{n,m}^1(\gamma_1, \gamma_2)$, $\pi_{n,m}^2(\omega_1, \omega_2)$ and $\pi_{n,m}^3(\gamma_1, \gamma_2)$ by change of variable. From (B.3), we have

$$\pi_{n,m}^1(\gamma_1, \gamma_2) = \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\gamma_2)^k}{k!\Gamma(\beta_2+2+k)} \int_0^\infty x^{\beta_1-1}(1+x)^{k+\beta_2+1} \mathbf{M}(n+\beta_1, \beta_1; -\gamma_1 x) dx$$

$$= \frac{n!}{\Gamma(n+\beta_1)} \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\gamma_2)^k}{k!\Gamma(\beta_2+2+k)} \int_0^\infty x^{\beta_1-1}(1+x)^{k+\beta_2+1} e^{-\gamma_1 x} L_n^{(\beta_1-1)}(\gamma_1 x) dx$$

$$= \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\gamma_2)^k}{k!\Gamma(\beta_2+2+k)} \left[\sum_{l=0}^n \frac{n!(-\gamma_1)^l}{l!(n-l)!\Gamma(l+\beta_1)} \int_0^\infty x^{l+\beta_1-1}(1+x)^{k+\beta_2+1} e^{-\gamma_1 x} dx \right].$$

Here, we have used the relation $\mathbf{M}(n+\beta_1, \beta_1; -\gamma_1 x) = e^{-\gamma_1 x} \frac{n!}{\Gamma(n+\beta_1)} L_n^{(\beta_1-1)}(\gamma_1 x)$ and the series representation of generalized Laguerre polynomials (e.g., [5], p.113, Eq.(4.5.3)). Using (B.9), we can obtain the formula for $\pi_{n,m}^1(\gamma_1, \gamma_2)$ after rearranging terms. Similarly, we can compute $\pi_{n,m}^3(\gamma_1, \gamma_2)$. Finally, we compute $\pi_{n,m}^2(\omega_1, \omega_2)$.

$$\pi_{n,m}^2(\omega_1, \omega_2) = \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\alpha_2\omega_1)^k}{k!\Gamma(\beta_2+2+k)} \int_0^\infty x^{\beta_1+\beta_2+k} \mathbf{M}(n+\beta_1, \beta_1; -\alpha_1\omega_2 x) dx$$

$$= \frac{n!}{\Gamma(n+\beta_1)} \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\alpha_2\omega_1)^k}{k!\Gamma(\beta_2+2+k)} \int_0^\infty x^{\beta_1+\beta_2+k} e^{-\alpha_1\omega_2 x} L_n^{(\beta_1-1)}(\alpha_1\omega_2 x) dx$$

$$= \frac{n!}{\Gamma(n+\beta_1)} \sum_{k=0}^\infty \frac{(m+\beta_2)_k(-\alpha_2\omega_1)^k}{k!\Gamma(\beta_2+2+k)} \frac{(-\beta_2-k-1)_n}{n!(\alpha_1\omega_2)^{\beta_1+\beta_2+k+1}} \Gamma(\beta_1+\beta_2+k+1)$$

$$= \frac{\Gamma(\beta_1+\beta_2+1)}{\Gamma(n+\beta_1)(\alpha_1\omega_2)^{\beta_1+\beta_2+1}} \sum_{k=0}^\infty \frac{(m+\beta_2)_k(\beta_1+\beta_2+1)_k}{k!\Gamma(\beta_2+2+k)} \Big(-\frac{\alpha_2\omega_1}{\alpha_1\omega_2}\Big)^k (-\beta_2-k-1)_n,$$

where the first equation comes from (B.3) and the second equation is from (B.5), and the third one follows from the integral identity ([67], p.463, Eq.(2.19.3.5))

$$\int_0^\infty x^{\alpha-1} e^{-cx} L_n^{(\lambda)}(cx) dx = \frac{(1-\alpha+\lambda)_n}{n!c^\alpha} \Gamma(\alpha).$$

When $\beta_2 + 2 - n \neq 0, -1, \cdots$, it follows from the identity $(a)_n = (-1)^n \Gamma(1-a)/\Gamma(1-a-n)$ that

$$\pi_{n,m}^2(\omega_1, \omega_2) = \frac{(-1)^n \Gamma(\beta_1+\beta_2+1)}{\Gamma(n+\beta_1)\Gamma(\beta_2+2-n)(\alpha_1\omega_2)^{\beta_1+\beta_2+1}} \sum_{k=0}^\infty \frac{(m+\beta_2)_k(\beta_1+\beta_2+1)_k}{k!(\beta_2+2-n)_k} \Big(-\frac{\alpha_2\omega_1}{\alpha_1\omega_2}\Big)^k$$

$$= \frac{(-1)^n \Gamma(\beta_1+\beta_2+1)}{\Gamma(n+\beta_1)\Gamma(\beta_2+2-n)(\alpha_1\omega_2)^{\beta_1+\beta_2+1}} {}_2F_1\left(\begin{array}{c} m+\beta_2, \beta_1+\beta_2+1 \\ \beta_2+2-n \end{array}; -\frac{\alpha_2\omega_1}{\alpha_1\omega_2} \right),$$

where the last equation is from the definition of the Gauss hypergeometric function $_2F_1$ (see (B.10)). If $\beta_2 + 2 + p = n$ for some non-negative integer p, the formula for $\pi_{n,m}^2(\omega_1, \omega_2)$ follows from the fact that $(-\beta_2 - k - 1)_n = 0$ for any positive integer $k > p$. $\qquad\square$

B Some Special Functions

We define the scaled generalized Laguerre polynomial as

$$l_n^{(\nu)}(x) := \sqrt{\tfrac{n!}{\Gamma(\nu+n+1)}} L_n^{(\nu)}(x), n = 0, 1, \cdots, \tag{B.1}$$

where $L_n^{(\nu)}(x)$ is the generalized Laguerre polynomial. We compute $l_n^{(\nu)}(x)$ in our implementation instead of $L_n^{(\nu)}(x)$. Based on the classical recursion for $L_n^{(\nu)}(x)$ (see, e.g., [42], p.241, Eq.(9.12.3)), $l_n^{(\nu)}(x)$ can be computed recursively as follows,

$$l_0^{(\nu)}(x) = \frac{1}{\sqrt{\Gamma(\nu+1)}}, \quad l_1^{(\nu)}(x) = \frac{1+\nu-x}{\sqrt{\Gamma(\nu+2)}},$$

$$l_n^{(\nu)}(x) = \frac{\nu+2n-1-x}{\sqrt{n(\nu+n)}} l_{n-1}^{(\nu)}(x) - \sqrt{\tfrac{(\nu+n-1)(n-1)}{(\nu+n)n}} l_{n-2}^{(\nu)}(x), \quad n \geq 2. \tag{B.2}$$

Let $M(a, c; x)$ denote Kummer's confluent hypergeometric function, which is defined as ([5], p.190, Eq(6.1.2))

$$M(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n = 1 + \frac{a}{c} z + \frac{a(a+1)}{c(c+1)2!} z^2 + \cdots. \tag{B.3}$$

for $c \neq 0, -1, -2, \cdots$ and $(a)_n$ is the Pochhammer symbol defined as $(a)_n = a(a+1)\cdots(a+n-1)$. We define the scaled Kummer's confluent hypergeometric function as

$$\mathbf{M}(a, c; z) := M(a, c; z)/\Gamma(c), c > 0. \tag{B.4}$$

Below we give several useful identities: (1) The scaled Kummar's confluent hypergeometric function and the generalized Laguerre polynomials are related as ([65] p.328, Eq.(13.6.19)):

$$\mathbf{M}(-n, a+1; z) = \frac{n!}{\Gamma(n+a+1)} L_n^{(a)}(z), \tag{B.5}$$

Using this relation, the eigenfunction of the ASubCIR process (see (6.2)) can be rewritten as

$$\varphi_n(x) = \sqrt{\kappa} \alpha^{\frac{\beta-1}{2}} \sqrt{\frac{\Gamma(n+\beta)}{n!}} \mathbf{M}(-n, \beta; \alpha x), \tag{B.6}$$

(2) Kummer's transformation identity ([65] p.325, Eq.(13.2.39)):

$$\mathbf{M}(a, b; z) = e^z \mathbf{M}(b-a, b, -z). \tag{B.7}$$

(3) Integral representation for $\mathbf{M}(a, b, z)$ ([65] p.326, Eq.(13.4.2)):

$$\mathbf{M}(a, b, z) = \frac{1}{\Gamma(b-c)} \int_0^1 \mathbf{M}(a, c, zt) t^{c-1} (1-t)^{b-c-1} dt, \quad \Re(b) > \Re(c) > 0, \tag{B.8}$$

where $\Re(x)$ denotes the real part of a complex number x.

Finally, Tricomi's confluent hypergeometric function $U(a, c; x)$ and the Gauss hypergeometric function $_2F_1$ are defined as ([5], p.194, Eq.(6.2.1) and p.268, Eq.(8.2.2))

$$U(a, c; x) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1} dt, \quad \Re(a) > 0, \tag{B.9}$$

$$_2F_1\binom{a, b}{c}; x := \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} x^n. \tag{B.10}$$

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