Analytical Representations for the Basic Affine Jump Diffusion

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Abstract

The Basic Affine Jump Diffusion (BAJD) process is widely used in financial modeling. In this paper, we develop an exact analytical representation for its transition density in terms of a series expansion that is uniformly-absolutely convergent on compacts. Computationally, our formula can be evaluated to high level of accuracy by easily adding new terms which are given explicitly. Furthermore, it can be easily generalized to give an analytical expression for the transition density of the subordinate BAJD process which is more realistic than the BAJD process, while existing approaches cannot.

Keywords: Basic Affine Jump Diffusion, subordination, transition density.

1. Introduction

We consider the *Basic Affine Jump Diffusion (BAJD)* process introduced in Duffie and Gârleanu [1], which is the unique strong solution to the following stochastic differential equation:

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t + dJ_t, \quad X_0 = x \geq 0.$$

Here $\kappa, \theta, \sigma > 0$ are the rate of mean reversion, the long-run mean, and the volatility coefficient, respectively. $J := (J_t)_{t\geq 0}$ is a compound Poisson process with arrival rate $\varpi \geq 0$, and its jumps are exponentially distributed with mean $\mu > 0$. When the Feller condition $2\kappa\theta \geq \sigma^2$ is satisfied, zero is an unattainable boundary and the state space of this process, denoted by E, is given by $E = (0, \infty)$ (Cheridito et al. [2]). If $0 < 2\kappa\theta < \sigma^2$, the process is instantaneously reflected at zero and $E = [0, \infty)$. When $\varpi \equiv 0$ (i.e., $J \equiv 0$) the BAJD process reduces to the Cox, Ingersoll, and Ross [3] (CIR) process.

The BAJD process has found many applications in finance. For instance, it is used to model the default intensity in credit risk applications (see, e.g., Duffie and Gârleanu [1], Mortensen [4], Brigo and El-Bachir [5, 6], and Eckner [7]), the short-rate process in interest rate markets (see, e.g., Brigo and Mercurio [8]) and the volatility of an asset (see, e.g., Duffie et al. [9], Eraker et al. [10], and Eraker [11]). In energy markets, the BAJD process is used as the background process for modeling the spot price of electricity (Li et al. [12]).

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In these applications, one is interested in computing expectations of the form

$$\mathcal{P}_t^\alpha f(x) := \mathbb{E}_x\left[e^{-\alpha\int_0^t X_u du} f(X_t)\right] \ (\alpha \geq 0).$$

In financial terms, f is the payoff function and X is the default intensity factor in credit risk applications, or the short rate in interest rate models, or the spot price (in such case we are concerned with $\alpha = 0$). The collection of the operators $(\mathcal{P}_t^\alpha)_{t\geq 0}$ forms a Feynman-Kac (FK) semigroup of contractions on $\mathcal{B}_b(E)$, the space of Borel-measurable and bounded functions on E. The kernel of the BAJD semigroup is Sub-Markovian as $\mathcal{P}_t^\alpha 1 \leq 1$. It is absolutely continuous w.r.t. the Lebesgue measure and we denote its density by $p^\alpha(t,x,y)$, i.e.,

$$\mathcal{P}_t^\alpha f(x) = \int_E f(y) p^\alpha(t,x,y) dy. \tag{1.1}$$

Once $p^\alpha(t,x,y)$ is known, the integral in (1.1) can be obtained either analytically or numerically.

In spite of the extensive use of the BAJD process in applications, to our best knowledge, $p^\alpha(t,x,y)$ is unknown in any analytical form. In the literature, there exist two approaches for computing $p^\alpha(t,x,y)$. It is well known that the Laplace transform is given by (c.f. Duffie and Gârleanu [1])

$$\mathcal{P}_t^\alpha e^{-zx} = \mathbb{E}_x\left[e^{-\alpha\int_0^t X_u du} e^{-zX_t}\right] = C(\varpi,\alpha,z;t) D(\varpi,\alpha;t)\ A(\alpha,z;t)\exp\{-B(\alpha,z;t)x\},\ z,\alpha \geq 0. \tag{1.2}$$

where

$$\begin{aligned}
A(\alpha,z;t) &:= \Big(\frac{2\varepsilon e^{(\kappa+\varepsilon)t/2}}{2\varepsilon + (\varepsilon+\kappa+z\sigma^2)(e^{\varepsilon t}-1)}\Big)^b,\\
B(\alpha,z;t) &:= \frac{2\alpha(e^{\varepsilon t}-1) + z(\varepsilon-\kappa)e^{\varepsilon t} + z(\varepsilon+\kappa)}{2\varepsilon + (\varepsilon+\kappa+z\sigma^2)(e^{\varepsilon t}-1)},\\
C(\varpi,\alpha,z;t) &:= \Big(1 + \frac{(e^{\varepsilon t}-1)\big(\varepsilon+\kappa+z\sigma^2+\mu(2\alpha+z(\varepsilon-\kappa))\big)}{2\varepsilon(1+z\mu)}\Big)^{-\varpi\mathfrak{a}},\\
D(\varpi,\alpha;t) &:= \exp\Big\{-\varpi\Big(\frac{\kappa+\varepsilon}{2\varepsilon}\Big)\Big(\frac{\mathfrak{b}}{\mathfrak{b}-1}\Big)t\Big\},
\end{aligned} \tag{1.3}$$

with

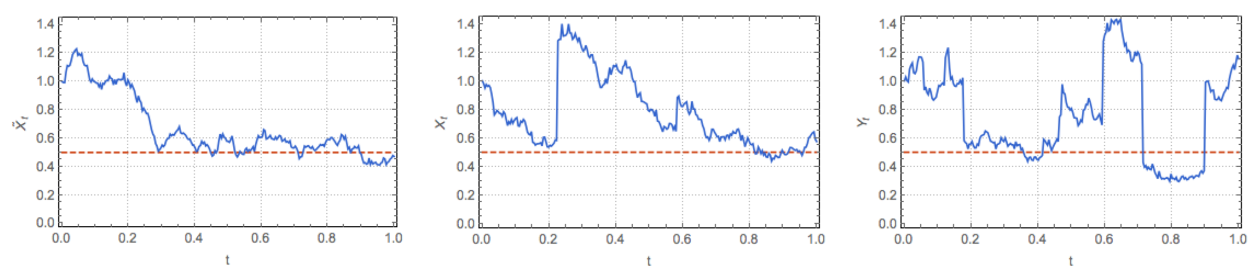
$$\mathfrak{a} := \frac{2\mu}{\sigma^2 - 2\mu\kappa - 2\alpha\mu^2}, \quad \mathfrak{b} := \frac{2\mu\varepsilon}{\sigma^2 + \mu(\varepsilon-\kappa)}, \quad \varepsilon := \sqrt{\kappa^2 + 2\alpha\sigma^2}, \quad \text{and} \quad b := \frac{2\kappa\theta}{\sigma^2}. \tag{1.4}$$

The formula can be obtained following the theory of affine processes (Duffie et al. [13]) to solve the corresponding generalized Riccati equation. Thus one approach to obtain $p^\alpha(t,x,y)$ is to invert the Laplace transform numerically. The other approach approximates the transition density either by polynomial approximations (Filipovic et al. [14]) or by approximations of the Kolmogorov forward/backward PIDE (Yu [15]).

In this paper, we derive an exact analytical expression for $p^\alpha(t,x,y)$ in terms of multiple infinite series which are uniformly-absolutely convergent on compacts. A series $\sum_{n=0}^\infty f_n(x)$ is said to converge *uniformly-absolutely convergent* if $\sum_{n=0}^\infty |f_n(x)|$ converges uniformly (a series of functions satisfying the Weierstrass's criterion for uniform convergence is uniformly-absolutely convergent, see, e.g., Itô [16], Definition 435.A, p.1647). As a by-product of our result for $p^\alpha(t,x,y)$, we also obtain the stationary density of the BAJD process.

In general, when the Laplace transform of a function is known, the function can be recovered from Laplace inversion via the Bromwich integral. In our case, we first derive an alternative representation for the Laplace transform $\mathcal{P}_t^\alpha e^{-zx}$ based on the spectral representation of the FK semigroup of the CIR process (Cox et al. [3]) and the binomial expansion. This representation allows us to calculate the Laplace inversion analytically.

To implement the existing closed-form approximations, one typically first fixes the number of terms to be used and then uses symbolic computational software to obtain the formula for these terms. Once the formula is obtained and stored, the subsequent evaluation at given parameter values can be done instantaneously. However, a potential drawback is that, one usually does not know a priori how many terms need to be used to achieve a certain level of accuracy, and adding a new term that has not been pre-computed can be costly. In contrast, in our expansion, every term is given explicitly and one can easily add a new term if it is needed to improve accuracy. In Section 4, we compare the approximation developed in Filipovic et al. [14] with our method, and it will be shown that the approximation formula which uses the first two to four terms can have quite significant error. Another nice feature of our method is that it can be easily generalized after subordination while the existing approaches cannot. The BAJD process is quite unrealistic in that it can only jump upward. Applying subordination to it allows us to develop more realistic models with two-sided jumps that are mean-reverting (see e.g., Boyarchenko and Levendorskii [17], Lim et al. [18], Mendoza-Arriaga and Linetsky [19] for applications of subordination to other processes in finance). Figure 1 illustrates typical sample paths for (a) the CIR process $\widetilde{X}$, (b) the BAJD process X, and (c) the Subordinate BAJD (SubBAJD) process Y. All three processes are mean reverting, the BAJD process exhibits only positive jumps, while the SubBAJD process exhibits mean reverting (positive and negative) jumps without leaving the state space E.



(a) *CIR process*, $\widetilde{X}$ (b) *BAJD process*, X (c) *SubBAJD process*, Y

Figure 1: Typical sample paths. All paths are started at $x = 1$, and the long-run mean is $\theta = 0.5$ (horizontal dashed line). Further details, including the values of the rest of the parameters, are provided in Section 4.

The rest of the paper is organized as follows. In Section 2, we obtain analytical representations for $p^\alpha(t, x, y)$ and $\mathcal{P}_t^\alpha f(x)$. In Section 3, we extend these results to the case with subordination. Section 4 presents numerical examples. All proofs are collected in the appendix.

2. Analytical formula for $p^\alpha(t, x, y)$

We make the following important observation: when $\varpi = 0$, since the BAJD process becomes the CIR process and $C(0, \alpha, z; t) = 1$, $D(0, \alpha; t) = 1$, the term $A(\alpha, z; t) \exp\{-B(\alpha, z; t)x\}$ is the Laplace transform of the CIR process. Hence we can rewrite Eq. (1.2) as

$$\mathcal{P}_t^\alpha e^{-zx} = C(\varpi, \alpha, z; t) D(\varpi, \alpha; t)\ \widetilde{\mathcal{P}}_t^\alpha e^{-zx}, \quad x \in E,\ z, \alpha, t \geq 0.$$

where $\widetilde{\mathcal{P}}^{\alpha} = (\widetilde{\mathcal{P}}^{\alpha}_t)_{t\geq 0}$ is the FK semigroup of the CIR process with killing rate αx. The FK semigroup of the CIR process can be represented by an eigenfunction expansion for functions that belong to $L^2(E,\mathfrak{m})$ where $\mathfrak{m}(dx) = \mathfrak{m}(x)dx$ is the CIR's speed measure with its density given by $\mathfrak{m}(x) = \frac{2x^{b-1}}{\sigma^2}e^{-2\kappa x/\sigma^2}$. Hence, from Proposition 9 in Davydov and Linetsky [20], for all $f \in L^2(E,\mathfrak{m})$ we have

$$\tilde{\mathcal{P}}^{\alpha}_t f(x) = \mathbb{E}_x[e^{-\alpha\int_0^t X_u du} f(X_t)] = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t}\varphi_n(x), \quad c_n = \int_E f(x)\varphi_n(x)\mathfrak{m}(x)dx, \tag{2.5}$$

where for $n = 0, 1, \cdots$,

$$\lambda_n = n\varepsilon + \frac{b}{2}(\varepsilon - \kappa), \quad \varphi_n(x) = N_n^{\alpha} e^{((\kappa-\varepsilon)x)/\sigma^2} L_n^{b-1}\left(\frac{2x\varepsilon}{\sigma^2}\right), \quad N_n^{\alpha} = \sqrt{\frac{\sigma^2 n!}{2\Gamma(b+n)}}\left(\frac{2\varepsilon}{\sigma^2}\right)^{b/2}, \tag{2.6}$$

with the variables b and ε defined in (1.4), and where $L_n^{\nu}(x)$ are the generalized Laguerre polynomials. It is straightforward to verify that for all $z \geq 0$, $e^{-zx} \in L^2(E,\mathfrak{m})$, hence we can calculate the Laplace transform of the CIR process using eigenfunction expansions. In particular, the expansion coefficients c_n entering into the expansion (2.5) for the function $f(x) = e^{-zx}$, $z \geq 0$, are available in close form (the calculation details are omitted), and they are given by

$$c_n(z) = \frac{1}{N_n^{\alpha}}\left(\frac{\kappa - \varepsilon + \sigma^2 z}{\kappa + \varepsilon + \sigma^2 z}\right)^n \left(\frac{2\varepsilon}{\kappa + \varepsilon + \sigma^2 z}\right)^b. \tag{2.7}$$

Lemma 1. *The spectral expansion (2.5) for the function $f(x) = e^{-zx}$, is uniformly-absolutely convergent on compacts for x, z and t.*

The function $C(\varpi, \alpha, z; t)$ can also be expanded in series such that time t enters the expression in an exponential form.

Lemma 2. *Define $Q(z) := \left(\frac{1}{\mathfrak{b}} - \frac{1}{\mathfrak{a}\varepsilon}\left(\frac{1}{\mu z+1}\right)\right)$ with $\mathfrak{a}$, $\mathfrak{b}$ and ε defined as in (1.4). Then, the function $C(\varpi, \alpha, z; t)$ of Eq. (1.3) accepts the following representation*

$$C(\varpi, \alpha, z; t) = \sum_{m=0}^{\infty} \frac{(\varpi\mathfrak{a})_m \left(\frac{Q(z)-1}{Q(z)+1}\right)^m}{m!(1+Q(z))^{\varpi\mathfrak{a}}} \left[\sum_{\ell=0}^{m} \binom{m}{\ell}(-1)^{m-\ell}2^{\ell+\varpi\mathfrak{a}}e^{-(\ell+\varpi\mathfrak{a})\varepsilon t}\right], \tag{2.8}$$

which converges uniformly-absolutely convergent for all $z, t \geq 0$. Here, $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

Next, for each $n = 0, 1, \ldots$ and $p = 0, 1, \cdots$; define

$$\vartheta_{m,n}(t) := \sum_{\ell=0}^{m} \binom{m}{\ell}(-1)^{m-\ell}2^{\ell+\varpi\mathfrak{a}}e^{-\beta_{\ell,n}t}, \quad \text{and} \quad \varrho_m(z) := \frac{(\varpi\mathfrak{a})_m}{m!}\frac{\left(\frac{Q(z)-1}{Q(z)+1}\right)^m}{(1+Q(z))^{\varpi\mathfrak{a}}}, \tag{2.9}$$

with $\beta_{\ell,n} = \left(\lambda_n + \varpi\left(\frac{\kappa+\varepsilon}{2\varepsilon}\right)\left(\frac{\mathfrak{b}}{\mathfrak{b}-1}\right) + (\ell+\varpi\mathfrak{a})\varepsilon\right)$. Hence, we arrive at the following lemma.

Lemma 3. *The Laplace transform (1.2) of the BAJD process can be written as,*

$$\mathcal{P}^{\alpha}_t e^{-zx} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \vartheta_{m,n}(t)\varrho_m(z)c_n(z)\varphi_n(x),$$

which is uniformly-absolutely convergent for x, z and t on compacts.

The following lemma allow us to rewrite the coefficients $c_n(z)$ and $\varrho_m(z)$ of (2.7) and (2.9), respectively, in a more convenient way.

Lemma 4. *The coefficients $c_n(z)$ and $\varrho_m(z)$ can be written as*

$$c_n(z) = \sum_{j=0}^{n} \widetilde{c}_{j,n}\Big(\Big(\frac{\kappa+\varepsilon}{\sigma^2}\Big)+z\Big)^{-(b+j)}, \quad \textit{with} \quad \widetilde{c}_{j,n} := \frac{1}{N_n^{\alpha}}\binom{n}{j}(-1)^j\Big(\frac{2\varepsilon}{\sigma^2}\Big)^{b+j}, \tag{2.10}$$

and

$$\varrho_m(z) = \sum_{k=0}^{\infty}\sum_{h=0}^{k} \widetilde{\varrho}_{h,k,m}\Big(z+\frac{1}{\mu}\Big)^{-h}, \quad \textit{with} \quad \widetilde{\varrho}_{h,k,m} := \binom{k}{h}\Big(-\frac{2}{\mu}\Big)^{h} U_m\, V_{k,m} \tag{2.11}$$

where

$$U_m := \frac{(\varpi\mathfrak{a})_m\Big(\frac{Q(1/\mu)-1}{Q(1/\mu)+1}\Big)^m}{(Q(1/\mu)+1)^{\varpi\mathfrak{a}}\, m!}, \quad \textit{and} \quad V_{k,m} := \frac{(\varpi\mathfrak{a})_k\ {}_2F_1\Big(\begin{matrix}-m,\ -k\\ \varpi\mathfrak{a}\end{matrix}; \frac{-2}{Q(1/\mu)-1}\Big)}{\big((-2\mathfrak{a}\varepsilon)(Q(1/\mu)+1)\big)^k\, k!}.$$

${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. (2.11) is uniformly-absolutely convergent for all $z \geq 0$.

Therefore, using Lemma 4, we arrive at the final series representation of the Laplace transform of the BAJD process,

$$\mathcal{P}_t^{\alpha} e^{-zx} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \vartheta_{m,n}(t)\varphi_n(x)\sum_{h=0}^{k}\sum_{j=0}^{n} \widetilde{c}_{j,n}\widetilde{\varrho}_{h,k,m}\Big(\Big(\frac{\kappa+\varepsilon}{\sigma^2}\Big)+z\Big)^{-(b+j)}\Big(z+\frac{1}{\mu}\Big)^{-h}. \tag{2.12}$$

We now obtain the transition density function $p^{\alpha}(t;x,y)$ of the BAJD process. For this purpose, we use Theorem 30.1 in Doetsch [21], which allows us to obtain $p^{\alpha}(t;x,y)$ as the term-by-term Laplace inversion of the series expansion (2.12). Indeed, observe that $\big(\big(\frac{\kappa+\varepsilon}{\sigma^2}\big)+z\big)^{-(b+j)}\big(z+\frac{1}{\mu}\big)^{-h} = \frac{\sigma^2}{2}\int_0^{\infty} e^{-zy}e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\frac{{}_1F_1(h;h+b+j;Ay)}{\Gamma(h+b+j)}y^{h+j}\mathfrak{m}(y)dy$, where $\mathfrak{m}(y) = \frac{2y^{b-1}}{\sigma^2}e^{-2\kappa y/\sigma^2}$ is the CIR's speed density, $A = \big(\frac{\kappa+\varepsilon}{\sigma^2}-\frac{1}{\mu}\big)$, and ${}_1F_1(a,b;x)$ is the confluent hypergeometric function (see Prudnikov et al. [22], Eq. 2.1.2.71). That is, $\big(\big(\frac{\kappa+\varepsilon}{\sigma^2}\big)+z\big)^{-(b+j)}\big(z+\frac{1}{\mu}\big)^{-h}$ is the Laplace transform of the function $f_{h,j}(y) = \frac{\sigma^2}{2}e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\frac{{}_1F_1(h;h+b+j;Ay)}{\Gamma(h+b+j)}y^{h+j}\mathfrak{m}(y)$. The analytical representation of $p^{\alpha}(t;x,y)$ is presented in the following theorem.

Theorem 1. *Let $\mathfrak{m}(y) = \frac{2y^{b-1}}{\sigma^2}e^{-2\kappa y/\sigma^2}$ be the CIR's speed density. Then, the transition density $p^{\alpha}(t;x,y)$ of the BAJD process X is given by*

$$p^{\alpha}(t;x,y) := p^{\alpha}_{\mathfrak{m}}(t;x,y)\mathfrak{m}(y), \quad \textit{with} \quad p^{\alpha}_{\mathfrak{m}}(t;x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\vartheta_{m,n}(t)\varphi_n(x)\psi_{m,n}(y), \tag{2.13}$$

and where

$$\psi_{m,n}(y) := \sum_{k=0}^{\infty}\widetilde{\psi}_{k,m,n}(y) \quad \textit{with} \quad \widetilde{\psi}_{k,m,n}(y) = \frac{\sigma^2}{2}\sum_{h=0}^{k}\sum_{j=0}^{n}\frac{\widetilde{c}_{j,n}\widetilde{\varrho}_{h,k,m}\ {}_1F_1\Big(\begin{matrix}h\\ h+b+j\end{matrix}; A\,y\Big)}{y^{-(h+j)}e^{\frac{\varepsilon-\kappa}{\sigma^2}y}\,\Gamma(h+b+j)}, \tag{2.14}$$

with $A := \big(\frac{\kappa+\varepsilon}{\sigma^2}-\frac{1}{\mu}\big)$. The expansion (2.13) of $p^{\alpha}_{\mathfrak{m}}(t;x,y)$ is uniformly-absolutely convergent on compacts for $x, y \geq 0$ and $t > 0$.

Theorem 2. *The probability that at time $t > 0$ the process X_t (has not yet been killed, $\alpha > 0$, and) is found at or below some level $R > 0$ is given by*

$$\mathbb{E}_x\left[e^{-\alpha\int_0^t X_u du}\mathbf{1}_{\{X_t \le R\}}\right] = \int_0^R p_{\mathfrak{m}}^{\alpha}(t;x,y)\mathfrak{m}(y)dy = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\vartheta_{m,n}(t)\varphi_n(x)f_{m,n}(R),$$

where $f_{m,n}(R) = \sum_{k=0}^{\infty}\widetilde{f}_{k,m,n}(R)$ *with*

$$\widetilde{f}_{k,m,n}(R) = \sum_{\ell=0}^{\infty}\sum_{h=0}^{k}\sum_{j=0}^{n}\frac{\widetilde{c}_{j,n}\widetilde{\varrho}_{h,k,m}(h)_\ell A^\ell}{\left(\frac{\kappa+\varepsilon}{\sigma^2}\right)^{h+b+j+\ell}\ell!}\widetilde{\gamma}\Big(h+b+j+\ell, \frac{\kappa+\varepsilon}{\sigma^2}R\Big). \tag{2.15}$$

$\widetilde{\gamma}(a,x) = \frac{\gamma(a,x)}{\Gamma(a)}$ *is the regularized lower incomplete gamma function.*

Our last result of this section is the stationary density $\pi(y)$ of the BAJD process X, which is the stationary density for the kernel $p^{\alpha}(t,x,y)$ with $\alpha = 0$ (a density function π is called stationary density for $p^{\alpha}(t,x,y)$ if $\pi(y) = \int_{(0,\infty)}\pi(x)p^{\alpha}(t,x,y)dx$). The Chapman-Kolmogorov equation says that $p^0(s+t,x,y) = \int_{(0,\infty)}p^0(s,x,z)p^0(t,z,y)dz$. Note that $\lim_{t\to\infty}p^0(t;x,y)$ exists and does not depend on x. Letting s go to ∞ in the C-K equation, we see that $\pi(y) = \lim_{t\to\infty}p^0(t;x,y)$. When $\alpha > 0$, $p^{\alpha}(t;x,y)$ can be interpreted as the transition density of a BAJD process that is killed by the additive functional $\int_0^t r(X_u)du$ with $r(x) = \alpha x$ (see, e.g., Applebaum [23], Sec.6.7.2). Since the process is killed a.s., its stationary density (and hence the stationary density for the kernel $p^{\alpha}(t;x,y)$) does not exist.

Theorem 3. *Let $\alpha = 0$, then the stationary density $\pi(y)$ of the BAJD process X is given by*

$$\pi(y) = 2^{\frac{2\varpi\mu}{\sigma^2-2\mu\kappa}-1}\sigma^2\,\mathfrak{m}(y)\Big(\frac{2\kappa}{\sigma^2}\Big)^b\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{k}\frac{(-1)^m\widetilde{\varrho}_{h,k,m}}{\Gamma(h+b)}\,{}_1F_1\Big(\begin{matrix}h\\h+b\end{matrix};\Big(\frac{2\kappa}{\sigma^2}-\frac{1}{\mu}\Big)y\Big)\,y^h, \tag{2.16}$$

with $\widetilde{\varrho}_{h,k,m}$ evaluated at $\alpha = 0$.

3. Subordination

We consider applying Bochner's subordination (see e.g., Schilling et al. [24] for detailed account) to the semigroup $(\mathcal{P}_t^{\alpha})_{t\ge0}$, that is, we define a new semigroup by

$$\mathcal{P}_t^{\alpha,\phi}f(x) := \int_{[0,\infty)}\mathcal{P}_s^{\alpha}f(x)q_t(ds), \tag{3.17}$$

where $(q_t)_{t\ge0}$ is the family of transition probabilities for a Lévy subordinator (a nonnegative Lévy process), and the Laplace transform of q_t is given by the Lévy-Kchinchine formula

$$\int_{[0,\infty)}e^{-\lambda s}q_t(ds) = e^{-\phi(\lambda)t}, \qquad \phi(\lambda) = \gamma\lambda + \int_{(0,\infty)}(1-e^{-\lambda s})\nu(ds),\ \lambda \ge 0, \tag{3.18}$$

where $\gamma \ge 0$ is the drift and ν is the Lévy measure satisfying the integrability condition $\int_{(0,\infty)}(s\wedge 1)\nu(ds) < \infty$. There is a Markov process Y associated with $(\mathcal{P}_t^{\alpha,\phi})_{t\ge0}$, i.e.,

$$\mathcal{P}_t^{\alpha,\phi}f(x) = \mathbb{E}_x\left[e^{-\int_0^t r_\alpha(Y_u)du}f(Y_t)\right].$$

We can find out the function $r_\alpha(x)$ and the infinitesimal generator of Y explicitly by extending the arguments in Lim et al. [18], and Mendoza-Arriaga and Linetsky [19]. The process Y now has a jump component that is two-sided and mean-reverting (see Figure 1). For brevity, we omit such discussions here. Below we derive an expression for $p^{\alpha,\phi}(s;x,y)$, the transition density of Y. As a direct consequence of (3.17), it is related to $p^\alpha(s;x,y)$ as

$$p^{\alpha,\phi}(t;x,y) = \int_{[0,\infty)} p^\alpha(s;x,y) q_t(ds). \tag{3.19}$$

Since time s enters the expression of $p^\alpha(s;x,y)$ in an exponential form (see (2.9) and (2.13)), assuming that we can interchange integration and summation, the calculation of the integral in (3.19) reduces to computing $\int_{[0,\infty)} e^{-\lambda s} q_t(ds)$ ($\lambda > 0$), which is the Laplace transform of the subordinator and it is known analytically. Thus we immediately obtain the expression for $p^{\alpha,\phi}(s;x,y)$. The following theorem provides the exact formula and gives out sufficient conditions under which the interchange is valid.

Theorem 4. *Let λ_n be given as in (2.6) and $\phi(\lambda)$ be the Lévy-Kintchine exponent (3.18). Assume that for all $t > 0$,*

$$\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}(1 + n1_{\{b\in(0,1)\}}) n^{\frac{2b-3}{4}} < \infty. \tag{3.20}$$

Then, the transition density of the subordinate BAJD process Y with $Y_0 = x \geq 0$ is given by $p^{\alpha,\phi}(t;x,y) := p^{\alpha,\phi}_{\mathfrak{m}}(t;x,y)\mathfrak{m}(dy)$, where

$$p^{\alpha,\phi}_{\mathfrak{m}}(t;x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \vartheta^{\phi}_{m,n}(t)\varphi_n(x)\psi_{m,n}(y), \tag{3.21}$$

with $\psi_{m,n}(y)$ defined as in Theorem 1, and the time-dependent coefficient $\vartheta^\phi_{m,n}$ is given by,

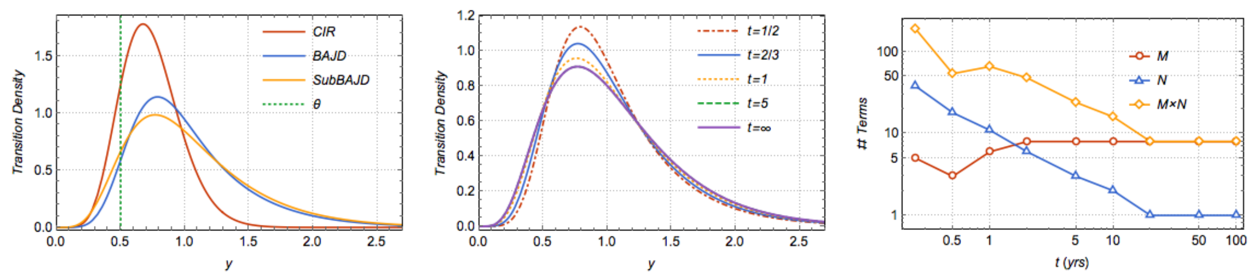
$$\vartheta^{\phi}_{m,n}(t) = \int_{[0,\infty)} \vartheta_{m,n}(s) q_t(ds) = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} 2^{\ell+\varpi\mathfrak{a}} e^{-\phi(\beta_{\ell,n})t},$$

with $\beta_{\ell,n} = \big(\lambda_n + \varpi\big(\frac{\kappa+\varepsilon}{2\varepsilon}\big)\big(\frac{\mathfrak{b}}{\mathfrak{b}-1}\big) + (\ell + \varpi\mathfrak{a})\varepsilon\big)$. (3.21) is uniformly-absolutely convergent on compacts for $x, y \geq 0$ and $t > 0$.

Observe that condition (3.20) on the Laplace transform of the Lévy subordinator is automatically satisfied if the drift of the subordinator is positive, i.e., $\gamma > 0$. On the other hand, when $\gamma = 0$, the previous assumption is satisfied for a large number of subordinators, including the temperate stable ones, for which $\phi(\lambda)$ is in the form of $-C\Gamma(-p)[(\lambda+\eta)^p - \eta^p]$ for some $p \in (0,1)$, $C, \eta > 0$ and $\Gamma(\cdot)$ is the Gamma function.

4. Numerical Examples

We implemented our formula (2.13) by approximating the double infinite series by a double finite sum which is accurate enough (how many terms to use is determined dynamically and we truncate when the specified error tolerance level is reached). To compute $\psi_{m,n}(y)$, we also truncate the infinite series in (2.14) when enough accuracy is obtained. We first provide the transition density function (with $\alpha = 0$) for the CIR process $\widetilde{X}$, the BAJD process X, and the SubBAJD process Y, whose sample paths are illustrated in Figure 1. All processes are started at $x = 1$ at time $t = 0$. The long-run mean is $\theta = 0.5$, the mean reversion level is $\kappa = 1.5$, and the volatility coefficient is $\sigma = 0.5$. The arrival rate of jumps for the compound Poisson component J is $\varpi = 2.5$, and the expected jump size is $\mu = 1/3$. For the SubBAJD process, we use the Inverse Gaussian subordinator, and its Laplace exponent is given by $\phi(\lambda) = \lambda\gamma + \frac{\delta^2}{\nu}\left(\sqrt{\frac{2\nu\lambda}{\delta}+1}-1\right)$, where δ is the mean rate and ν is the variance rate. We set $\gamma = 1$, $\delta = 3$, and $\nu = 24$. Figure 2a illustrates the 1/2 year transition density of these processes. Due to the presence of upward jumps, the transition density of the BAJD and SubBAJD processes exhibits a more pronounced right skew than the CIR transition density. Meanwhile, compared to the BAJD process, the transition density of the SubBAJD process exhibits a larger mass on the left side of the long-run mean θ, since it is possible for the SubBAJD process to return sooner to θ via downward mean-reverting jumps.



(a) *Transition densities for the CIR, BAJD and SubBAJD processes.*

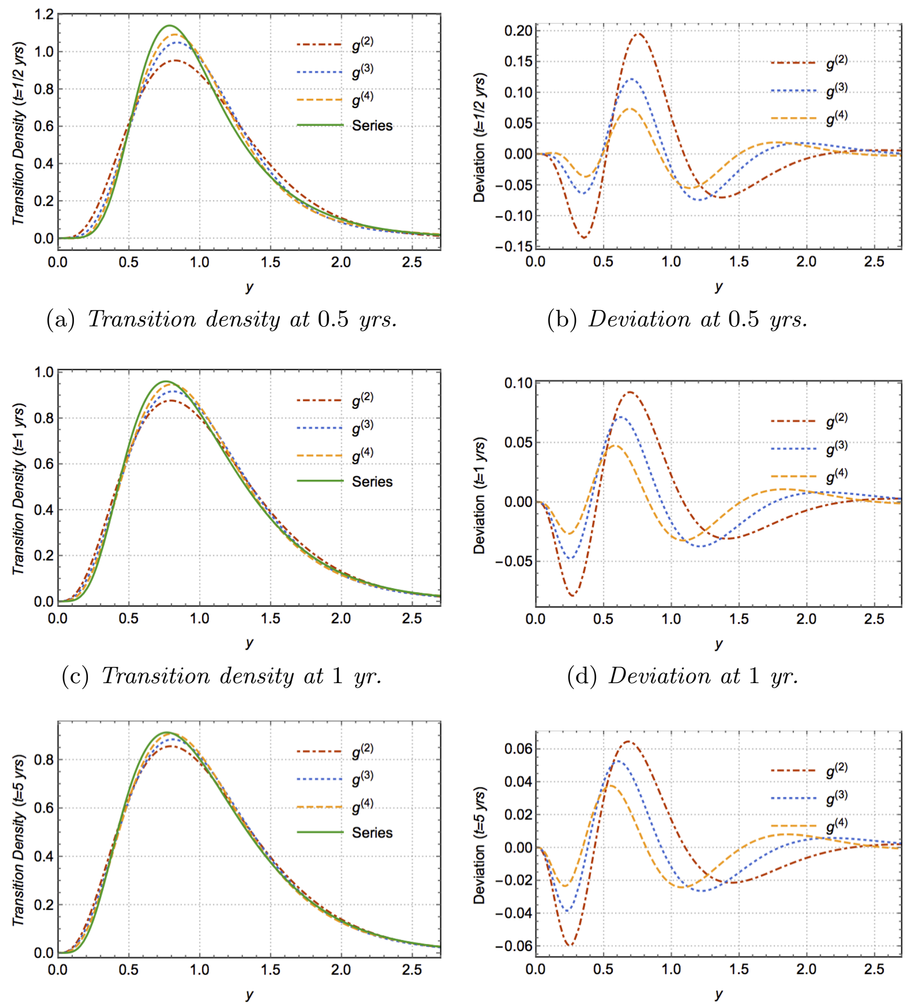
(b) *Stationary density, $\pi(y)$*

(c) *Maximum number of terms $M \times N$ as a function of time $t > 0$.*

Figure 2: Transition density analysis.

In Figure 2b we illustrate how the BAJD transition density converges to the stationary distribution $\pi(y)$, as we vary the time horizon t. We observe that for $t = 5$ yrs, the transition density has practically converged to the stationary density (and hence, it cannot be identified from the graph). In Figure 2c we illustrate the maximum number of terms required to compute the value of the transition density of the BAJD process from its initial level $x = 1$ to the end level $y = \theta = 0.5$ while varying t. The maximum number of terms is calculated as $M \times N$, where M and N correspond to the number of terms required in each sum of (2.13), so that each partial sum $S^P = \sum_{p=0}^{P} a_p$, $P \in \{M, N\}$, has an absolute relative error $ARE = \left|\frac{a_{P+1}}{S^P}\right| < 10^{-6}$. We observe that, the maximum number of terms N practically decays to $N = 1$ at an exponential rate with respect to t, while the number of terms M increase slightly but it is stabilized by $t = 2$ where it reaches $M = 8$. The maximum number of terms $M \times N$ required at $t = 1/4$ is 190 and it decays to $M \times N = 8$ by $t = 20$. This shows our formula converges faster as the time to maturity increases.

We compare our method to the closed-form approximation derived in Filipovic et al. [14] using the first two, three and four polynomial moments for different maturities. The results are plotted in Figure 3. As one can see, the approximation formula in Filipovic et al. [14] up to fourth order is less accurate than our formula (using $ARE < 10^{-6}$ as the stopping criterion). From Figure 3, we also observe that the deviation of Filipovic et al. [14] is more significant for shorter maturities even when four polynomial moments are used. Upon precomputing the moments and storing the formulas, subsequent calculations of Filipovic et al. [14] approximation can be done almost instantaneously. We implemented our formula in Mathematica. The time for evaluation of the series expansion (2.13) at each (x, y) pair varies, and it can take a few seconds. Nonetheless, we expect the computation time can be significantly reduced if we code the formula in more basic programming language like C. The real advantage of our method is that it allows high level of accuracy to be achieved by easily adding new terms, while it is not easy to do so in the existing approaches. In addition, our method can be extended to the subordination case effortlessly.



(a) *Transition density at* 0.5 *yrs.*

(b) *Deviation at* 0.5 *yrs.*

(c) *Transition density at* 1 *yr.*

(d) *Deviation at* 1 *yr.*

(e) *Transition density at* 5 *yrs.*

(f) *Deviation at* 5 *yrs.*

Figure 3: *Right:* BAJD's transition density $p^0(t;x,y)$ using the series expansion (2.13) and the approximation, $g(t;x,y)$, derived in Filipovic et al. [14], using 2, 3, and 4th order. *Left:* difference between the approximation and the series expansion, i.e., $p^0(t;x,y) - g(t;x,y)$.

Appendix A. Proofs

First, we need the following preliminary results (Lemmas 5-8).

Lemma 5. *Let $x \geq 0$ and $b > 0$, then for all $n \in \mathbb{N}_0$,* $|L_n^{b-1}(x)| \leq \frac{e^{x/2}}{n!}\Big[\mathbf{1}_{\{b\in(0,1)\}}\Big(\frac{(x+b)(b+1)_n}{b+n} + \frac{x(b+2)_n}{b+n}\Big) + \mathbf{1}_{\{b\geq 1\}}(b)_n\Big] = \frac{(b)_n e^{x/2}}{n!}\Big[1 + \mathbf{1}_{\{b\in(0,1)\}}\frac{2+2b+n}{b(b+1)}x\Big]$.

Proof. First observe that $L_n^{b-1}(x) = \frac{1}{b+n}((x+b)L_n^b(x) - xL_n^{b+1}(x))$, and that, for all $n \in \mathbb{N}_0$, and $\nu, x \geq 0$, we have $|L_n^\nu(x)| \leq e^{x/2}L_n^\nu(0) = e^{x/2}\frac{(\nu+1)_n}{n!}$ (see Olver et al. [25], Eq.18.14.8). Hence, the proof follows from an application of the triangular inequality. □

Lemma 6. *Let $a \in \mathbb{R}$ and $k \in \mathbb{N}_0$, then* $\lim_{k\to\infty}\left|\frac{(a)_{k+1}}{(k+1)!}\Big/\frac{(a)_k}{(k)!}\right| = \lim_{k\to\infty}\left|\frac{(a)_{k+1}}{k!}\Big/\frac{(a)_k}{(k-1)!}\right| = 1$.

Proof. It follows directly from substituting the asymptotic expansions $(a)_k \approx e^{-k}k^{k-\frac{1}{2}+a}$ and $k! \approx e^{-k}k^{k-\frac{1}{2}}$ as $k \to \infty$ and taking the limit. □

Lemma 7. *Let $x, b \in \mathbb{R}$ with b bounded, then* $\left|\frac{{}_1F_1(-a;b;x)}{{}_1F_1(1-a;b;x)}\right| \to 1$ *as $a \to \infty$. Similarly,* $\left|\frac{L_{n+1}^{b-1}(x)}{L_n^{b-1}(x)}\right| \to 1$ *as $n \to \infty$.*

Proof. For all $x, b \in \mathbb{R}$ and b bounded, we have ${}_1F_1(-a;b;x) \approx \frac{\Gamma(b)}{\sqrt{\pi}}e^{x/2}\big(\frac{b}{2}+a\big)^{\frac{1-2b}{4}}x^{\frac{1-2b}{4}}\cos\big(\frac{\pi}{4}(1-2b) + x^{1/2}\sqrt{2b+4a}\big)$ as $a \to \infty$ (see Abramowitz and Stegun [26], Eq.13.5.14). Assume that $x \geq 0$, in this case ${}_1F_1(-a;b;x) \approx \frac{\Gamma(b)}{\sqrt{\pi}}e^{x/2}\big(\frac{b}{2}+a\big)^{\frac{1-2b}{4}}x^{\frac{1-2b}{4}}$ (since $|\cos(z)| \leq 1$, $z \in \mathbb{R}$), and hence, $\left|\frac{{}_1F_1(-a;b;x)}{{}_1F_1(1-a;b;x)}\right| \approx \big(\frac{b+2a}{b+2a-2}\big)^{\frac{1-2b}{4}} \to 1$. Next assume $x < 0$, using the identities $|\cos(a+b\,i)| = \sqrt{\cos^2(a)\cosh^2(b) + \sin^2(a)\sinh^2(b)} = \sqrt{\frac{1}{2}(\cos(2a)+\cosh(2b))}$, we obtain $\left|\frac{{}_1F_1(-a;b;x)}{{}_1F_1(1-a;b;x)}\right| \approx \big(\frac{b+2a}{b+2a-2}\big)^{\frac{1-2b}{4}}\sqrt{\frac{\cos\left(\frac{\pi}{2}(1-2b)\right)+\cosh\left(2|x|^{1/2}\sqrt{2b+4a}\right)}{\cos\left(\frac{\pi}{2}(1-2b)\right)+\cosh\left(2|x|^{1/2}\sqrt{2b+4(a-1)}\right)}} \to 1$ (since $\cosh(0) = 1$, and for all $x \in \mathbb{R}\setminus\{0\}$, we have $\cosh(x\sqrt{a}) \to \infty$, and $\frac{\cosh(x\sqrt{a})}{\cosh(x\sqrt{a-1})} \to 1$ as $a \to \infty$). Since $L_n^{b-1}(x) = \frac{(b)_n}{n!}\,{}_1F_1(-n,b,x)$, then $\left|\frac{L_{n+1}^{b-1}(x)}{L_n^{b-1}(x)}\right| = \left|\frac{(b)_{n+1}n!}{(n+1)!(b)_n}\right|\left|\frac{{}_1F_1(-(n+1);b;x)}{{}_1F_1(-n;b;x)}\right| \to 1$, from the previous result and Lemma 6. □

Lemma 8. *Let $c, x \in \mathbb{R}$, and $m, k \in \mathbb{N}_0$, then* $\left|\frac{{}_2F_1(-m,-(k+1);c;x)}{{}_2F_1(-m,-k;c;x)}\right| \to 1$ *as $k \to \infty$.*

Proof. Using the identity $\frac{{}_2F_1(a,b;c;x)}{\Gamma(c)} = \frac{{}_2F_1(a,b+1;c;x)}{\Gamma(c)} - \frac{ax}{c}\frac{{}_2F_1(a+1,b+1;c+1;x)}{\Gamma(c)}$ (see Ramanathan [27], Eq.18) we obtain $\left|\frac{{}_2F_1(-m,-(k+1);c;x)}{{}_2F_1(-m,-k;c;x)}\right| = \left|1 + x\frac{m\,{}_2F_1(1-m,-k;c+1;x)/\Gamma(c+1)}{{}_2F_1(-m,-k;c;x)/\Gamma(c)}\right|$. Hence, we need to show that $\lim_{k\to\infty}\frac{x\,m\,{}_2F_1(1-m,-k;c+1;x)/\Gamma(c+1)}{{}_2F_1(-m,-k;c;x)/\Gamma(c)} = 0$. Using the definition of ${}_2F_1(-a,b,x)$ when $a \in \mathbb{N}$, the fact that $(a)_m\Gamma(a) = \Gamma(a+m)$, and the asymptotic $(a)_n \approx a^n$ as $a \to \infty$, we obtain $\frac{x\,m\,{}_2F_1 1-m,-k;c+1;x)/\Gamma(c+1)}{{}_2F_1(-m,-k;c;x)/\Gamma(c)} = \frac{x\sum_{p=0}^{m-1}\frac{m(1-m)_p x^p(-k)_p}{\Gamma(c+1+p)p!}}{\frac{(-m)_m x^m(-k)_m}{\Gamma(c+m)m!} + \sum_{p=0}^{m-1}\frac{(-m)_p x^p(-k)_p}{\Gamma(c+p)p!}} \approx \frac{x\,m\sum_{p=0}^{m-1}\frac{\Gamma(c+m)(1-m)_p x^p}{\Gamma(c+1+p)p!}(-k)^p}{k^m(-x)^m + \sum_{p=0}^{m-1}\frac{\Gamma(c+m)(-m)_p x^p}{\Gamma(c+p)p!}(-k)^p} \to 0$ as $k \to \infty$ and all $x \in \mathbb{R}$. □

Proof of Lemma 1: Let $K(t;x,z) = e^{-\frac{b}{2}(\varepsilon-\kappa)t}\big(\frac{2\varepsilon}{\kappa+\varepsilon+\sigma^2 z}\big)^b e^{\kappa x/\sigma^2}$ and $w = \frac{\kappa-\varepsilon+\sigma^2 z}{\kappa+\varepsilon+\sigma^2 z}$ (see that $|w| < 1$ for all $z < \infty$, and $|w| = 1$ in the limit as $z \to \infty$). From Lemma 5, we have $\sum_{n=0}^\infty e^{-\lambda_n t}|c_n(z)\varphi_n(x)| \leq K(t;x,z)\sum_{n=0}^\infty \frac{\Gamma(b+n)}{\Gamma(b+1)n!}|w|^n e^{-n\varepsilon t}\big((b+x) + x\frac{(b+n+1)}{(b+1)}\big)$, which is a continuous function of x, z, and t.

Moreover, the right-hand-side sum converges to $K(t;x,z)\big(1+\big(1+\frac{b|w|}{2(b+1)(e^{\varepsilon t}-|w|)}\big)\frac{2x}{b}\big)(1-|w|e^{-\varepsilon t})^{-b}$, which is finite for all x, z and t on compact sets. Hence the result follows. $\square$

Proof of Lemma 2: $C(\varpi,\alpha,z;t)$ can be written as

$$C(\varpi,\alpha,z;t)=e^{-\varpi\mathfrak{a}\varepsilon t}\Big(e^{-\varepsilon t}+(1-e^{-\varepsilon t})Q(z)\Big)^{-\varpi\mathfrak{a}}=e^{-\varpi\mathfrak{a}\varepsilon t}\Big(1+(Q(z)-1)(1-e^{-\varepsilon t})\Big)^{-\varpi\mathfrak{a}}. \quad \text{(A.1)}$$

We shall make use of the expansion valid for all $w\in\mathbb{R}$,

$$(a+bz)^w=(a+br)^w\sum_{k=0}^{\infty}\frac{(-1)^k(-w)_k}{k!}\Big(\frac{z-r}{(a/b)+r}\Big)^k=(a+br)^w\,{}_1F_0\Big(-w,;-\frac{z-r}{(a/b)+r}\Big), \quad \text{(A.2)}$$

where ${}_1F_0(a,;x)=\sum_{k=0}^{\infty}((a)_k/k!)x^k=(1-x)^{-a}$ is the generalized hypergeometric function, which converges absolutely provided $\big|\frac{z-r}{a/b+r}\big|<1$ (see Theorem 2.1.1 in Andrews et al. [28], p.62). Next, we obtain the expression (2.8). Applying (A.2) to (A.1) with $r=1-e^{-\varepsilon t}=\frac{1}{2}$ we obtain, $C(\varpi,\alpha,z;t)=2^{\varpi\mathfrak{a}}(1+Q(z))^{-\varpi\mathfrak{a}}e^{-\varpi\mathfrak{a}\varepsilon t}\sum_{m=0}^{\infty}\frac{(\varpi\mathfrak{a})_m}{m!}2^m\big(\frac{Q(z)-1}{Q(z)+1}\big)^m\big(e^{-\varepsilon t}-\frac{1}{2}\big)^m$. To show that we can actually apply (A.2), we need to show that $\big|2\big(\frac{Q(z)-1}{Q(z)+1}\big)\big(e^{-\varepsilon t}-\frac{1}{2}\big)\big|<1$. We already have $\big|2\big(e^{-\varepsilon t}-\frac{1}{2}\big)\big|\le 1$ (since $\varepsilon>0$). It is not difficult to show that $Q(z)>0$ for all $z\ge 0$. Indeed, it is easy to see that $Q(\infty)=1/\mathfrak{b}>0$, $Q(0)=(\kappa+\varepsilon+2\alpha\mu)/(2\varepsilon)>0$, and $Q(1/\mu)=(\sigma^2+2\alpha\mu^2+2\mu\varepsilon)/(4\mu\varepsilon)>0$. Applying the binomial expansion to $\big(e^{-\varepsilon t}-\frac{1}{2}\big)^m$ we obtain (2.8). We now show that the convergence of (2.8) is uniform for all $z\ge 0$ and all $t\ge 0$. Note that for $z\ge 0$, $Q(z)>0$ and $\max_{z\in[0,\infty]}Q(z)<\infty$. Hence, for all $z\ge 0$, there exists a $K\in(0,1)$ such that $\big|\frac{Q(z)-1}{Q(z)+1}\big|\le K<1$. Thus, for all $z\ge 0$ and all $t\ge 0$, $\big|2\big(\frac{Q(z)-1}{Q(z)+1}\big)\big(e^{-\varepsilon t}-\frac{1}{2}\big)\big|\le K<1$. Since $\sum_{m=0}^{\infty}\frac{(\varpi\mathfrak{a})_m}{m!}K^m$ is convergent (independently of t and z), then by virtue of the Weierstrass test, (2.8) is uniformly-absolutely convergent for all $z,t\ge 0$. $\square$

Proof of Lemma 3: Since $\vartheta_{m,n}(t)=e^{-(\lambda_n+\varpi(\frac{\kappa+\varepsilon}{2\varepsilon})(\frac{\mathfrak{b}}{\mathfrak{b}-1}))t}(2e^{-\varepsilon t})^{\varpi\mathfrak{a}}(2e^{-\varepsilon t}-1)^m$, it follows that $0\le|\vartheta_{m,n}(t)|\le e^{-\lambda_n t}2^{\varpi\mathfrak{a}}$ for all $t\ge 0$, which means that we can disentangle the double summation into a product of uniformly-absolutely convergent sums. Since the product of absolutely convergent series is absolute convergent, and each of them is also continuous uniformly convergent (on compacts), the result follows. $\square$

Proof of Lemma 4: Using the identity $\frac{b+az}{d+cz}=\frac{a}{c}+\frac{1}{c}\big(\frac{bc-ad}{d+cz}\big)$, $c_n(z)$ can be written as, $c_n(z)=\frac{(2\varepsilon/\sigma^2)^b}{N_n^{\alpha}}\big(1+\frac{-2\varepsilon/\sigma^2}{((\kappa+\varepsilon)/\sigma^2)+z}\big)^n\big(\frac{\kappa+\varepsilon}{\sigma^2}+z\big)^{-b}$. Applying the binomial formula we arrive at (2.10). Next we obtain the expansion of $\varrho_m(z)$. Applying again the identity $\frac{b+az}{d+cz}=\frac{a}{c}+\frac{1}{c}\big(\frac{bc-ad}{d+cz}\big)$ and the binomial formula we obtain: $\varrho_m(z)=\frac{(\varpi\mathfrak{a})_m}{m!}\sum_{\ell=0}^{m}\binom{m}{\ell}(-2)^{\ell}(Q(z)+1)^{-\varpi\mathfrak{a}-\ell}$. Expand $\big(Q(z)+1\big)^{-(\varpi\mathfrak{a}+\ell)}$ using (A.2) with $r=Q(1/\mu)$ to obtain: $(Q(z)+1)^{-\varpi\mathfrak{a}-\ell}=(1+Q(1/\mu))^{-(\varpi\mathfrak{a}+\ell)}\sum_{k=0}^{\infty}\big(2\mathfrak{a}\varepsilon(1+Q(1/\mu))\big)^{-k}\frac{(\varpi\mathfrak{a}+\ell)_k}{k!}2^k\big(\frac{1}{\mu z+1}-\frac{1}{2}\big)^k$. The latter converges uniformly-absolutely according to the Weierstrass test since, for all $z\in[0,\infty]$ and all $\sigma,\kappa,\mu>0$ and $\alpha\ge 0$, we have $\big|2\big(\frac{1}{\mu z+1}-\frac{1}{2}\big)\big|\le 1$, and $\big|\big(2\mathfrak{a}\varepsilon(1+Q(1/\mu))\big)^{-1}\big|=\Big|\frac{\sigma^2-2\mu(\kappa+\alpha\mu)}{\sigma^2+2\mu(3\varepsilon+\alpha\mu)}\Big|<1$. Thus, changing the summation order we arrive at $\varrho_m(z)=\sum_{k=0}^{\infty}\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-2)^{\ell}(\varpi\mathfrak{a})_m(\varpi\mathfrak{a}+\ell)_k}{(1+Q(1/\mu))^{\varpi\mathfrak{a}+\ell}m!\,k!}\Big(\frac{\frac{1}{\mu z+1}-\frac{1}{2}}{\mathfrak{a}\varepsilon(1+Q(1/\mu))}\Big)^k$. Using the binomial formula we obtain: $\varrho_m(z)=\sum_{k=0}^{\infty}\sum_{h=0}^{k}\widetilde{\varrho}_{h,k,m}\big(z+\frac{1}{\mu}\big)^{-h}$ with $\widetilde{\varrho}_{h,k,m}=\sum_{\ell=0}^{m}\binom{k}{h}\binom{m}{\ell}\frac{\mu^{-h}(-2)^{\ell+h-k}(\varpi\mathfrak{a}+\ell)_k(\varpi\mathfrak{a})_m}{(\mathfrak{a}\varepsilon)^k(1+Q(1/\mu))^{\varpi\mathfrak{a}+\ell+k}k!m!}$.

Next, observing that $\binom{m}{\ell} = \frac{m!}{\ell!\,(m-\ell)!}$, $(-m)_\ell = \frac{(-1)^\ell m!}{(m-\ell)!}$ for $m \in \mathbb{N}$ with $\ell \le m$, and $(\varpi\mathfrak{a}+\ell)_k = \frac{(\varpi\mathfrak{a})_k(\varpi\mathfrak{a}+k)_\ell}{(\varpi\mathfrak{a})_\ell}$, and using the definition of Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ with $-a \in \mathbb{N}$, we arrive at $\widetilde{\varrho}_{h,k,m} = \binom{k}{h}\left(\frac{1}{2}\right)^{k-h}\left(\frac{1}{\mu}\right)^h \frac{(\varpi\mathfrak{a})_m(\varpi\mathfrak{a})_k\ {}_2F_1(-m,\varpi\mathfrak{a}+k;\varpi\mathfrak{a};\frac{2}{1+Q(1/\mu)})}{(-1)^{h-k}(\mathfrak{a}\varepsilon)^k(1+Q(1/\mu))^{\varpi\mathfrak{a}+k}\,k!\,m!}$. Lastly, using the identity ${}_2F_1(a,b;c;x) = (1-x)^{-a}\,{}_2F_1(a,c-b;c;x/(x-1))$ we obtain (2.11). We now show that $\sum_{k=0}^\infty \big|\sum_{h=0}^k \widetilde{\varrho}_{h,k,m}(z+1/\mu)^{-h}\big|$ is uniformly convergent using the Weierstrass's criterion. First, observe that, $\big|\sum_{h=0}^k \widetilde{\varrho}_{h,k,m}(z+1/\mu)^{-h}\big| = |U_m\,V_{k,m}|\,\big|\frac{1-\mu z}{1+\mu z}\big|^k$, and that for all $z \in [0,\infty]$ we have $\big|\frac{1-\mu z}{1+\mu z}\big| \le 1$. Also, observe that from Lemmas 6 and 8, and since $\big|\frac{Q(1/\mu)-1}{Q(1/\mu)+1}\big| \le K < 1$ and $|(2\mathfrak{a}\varepsilon)(Q(1/\mu)+1)|^{-1} \le K < 1$, it follows that

$$\lim_{k\to\infty}\Big|\frac{V_{k+1,m}}{V_{k,m}}\Big| \le K < 1, \quad \text{and} \quad \lim_{m\to\infty}\Big|\frac{U_{m+1}V_{k,m+1}}{U_mV_{k,m}}\Big| \le K < 1. \tag{A.3}$$

The latter implies (ratio test) that $\sum_{k=0}^\infty \big|\sum_{h=0}^k \widetilde{\varrho}_{h,k,m}(z+1/\mu)^{-h}\big|$ is uniformly convergent for all $z \in [0,\infty]$. Hence, (2.11) is uniformly-absolutely convergent. □

Proof of Theorem 1: To show that the conditions of Theorem 30.1 in Doetsch [21] are satisfied, it suffices to show the convergence of the series $\sum_{k=0}^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty |\vartheta_{m,n}(t)|\,|\varphi_n(x)|\,\mathcal{L}^{|\widetilde{\psi}|}_{k,m,n}(z)$, where $\mathcal{L}^{|\widetilde{\psi}|}_{k,m,n}(z) = \int_0^\infty e^{-zy}|\widetilde{\psi}_{k,m,n}(y)|\mathfrak{m}(y)dy$ and where $\widetilde{\psi}_{k,m,n}(y)$ is defined in (2.14). Using the fact that for all $x, b \in \mathbb{R}$, ${}_1F_1(0,b,x) = 1$, and the integral representation $\frac{{}_1F_1(a;b;x)}{\Gamma(b)} = \frac{1}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}dt$, which is valid for $\Re(b) > \Re(a) > 0$, we obtain, $|\widetilde{\psi}_{k,m,n}(y)| = \Big|\frac{\sigma^2}{2}\, e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\frac{U_mV_{k,m}}{N_n^\alpha}\left(\frac{2\varepsilon}{\sigma^2}\right)^b\Big(\sum_{j=0}^n \frac{n!\,(\frac{2\varepsilon y}{\sigma^2})^j(-1)^j}{(n-j)!\,j!\,\Gamma(b+j)} + \int_0^1\Big(\sum_{h=1}^k\sum_{j=0}^n \frac{n!\,k!\,(\frac{2\varepsilon y}{\sigma^2})^j(-1)^{h+j}(\frac{2y}{\mu})^h}{(n-j)!\,j!\,(k-h)!\,h!}\frac{e^{Ayt}t^{h-1}(1-t)^{b+j-1}}{\Gamma(h)\Gamma(b+j)}\Big)dt\Big)\Big|$ $= \Big|\frac{\sigma^2}{2}\frac{U_mV_{k,m}}{N_n^\alpha\Gamma(b)}\left(\frac{2\varepsilon}{\sigma^2}\right)^b e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\Big({}_1F_1\big(-n,b,\frac{2\varepsilon y}{\sigma^2}\big) + \left(\frac{2y}{\mu}\right)\int_0^1(1-t)^{b-1}(-k)e^{Ayt}\,{}_1F_1\big(1-k;2;\frac{2y}{\mu}t\big)\,{}_1F_1\big(-n;b;\frac{2\varepsilon y}{\sigma^2}(1-t)\big)dt\Big)\Big|$. Using Eq. 7.11.1.24 in Prudnikov et al. [29], we obtain $|\widetilde{\psi}_{k,m,n}(y)| = \Big|\frac{\sigma^2}{2}\frac{U_mV_{k,m}}{N_n^\alpha\Gamma(b)}\left(\frac{2\varepsilon}{\sigma^2}\right)^b e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\Big({}_1F_1\big(-n,b,\frac{2\varepsilon y}{\sigma^2}\big) + \left(\frac{2y}{\mu}\right)\int_0^1(1-t)^{b-1}e^{Ayt}\,{}_1F_1\big(-n;b;\frac{2\varepsilon y}{\sigma^2}(1-t)\big)\big[{}_1F_1\big(-k;1;\frac{2y}{\mu}t\big) - (1+k)\,{}_1F_1\big(-k;2;\frac{2y}{\mu}t\big)\big]dt\Big)\Big| = \Big|\frac{\sigma^2}{2}\frac{U_mV_{k,m}}{N_n^\alpha\Gamma(b)}\left(\frac{2\varepsilon}{\sigma^2}\right)^b e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\frac{n!}{(b)_n}\Big(L_n^{b-1}\big(\frac{2\varepsilon y}{\sigma^2}\big) + \left(\frac{2y}{\mu}\right)\int_0^1(1-t)^{b-1}e^{Ayt}L_n^{b-1}\big(\frac{2\varepsilon y}{\sigma^2}(1-t)\big)\big[L_k^0\big(\frac{2y}{\mu}t\big) - L_k^1\big(\frac{2y}{\mu}t\big)\big]dt\Big)\Big| \le \frac{\sigma^2}{2}\frac{|U_mV_{k,m}|}{N_n^\alpha\Gamma(b)}\left(\frac{2\varepsilon}{\sigma^2}\right)^b e^{-\frac{\varepsilon-\kappa}{\sigma^2}y}\Big(e^{\frac{\varepsilon}{\sigma^2}y}\big[1 + \mathbf{1}_{\{b\in(0,1)\}}\frac{2+2b+n}{b(b+1)}\big(\frac{2\varepsilon y}{\sigma^2}\big)\big] + \left(\frac{2y}{\mu}\right)(2+k)\int_0^1 e^{\frac{\varepsilon y}{\sigma^2}(1-t)+\frac{y}{\mu}t+Ayt}(1-t)^{b-1}\big[1 + \mathbf{1}_{\{b\in(0,1)\}}\frac{2+2b+n}{b(b+1)}\big(\frac{2\varepsilon y}{\sigma^2}\big)(1-t)\big]dt\Big)$, where in the last inequality we used Lemma 5. Hence, after integrating and simplifying,

$$\begin{aligned}
|\widetilde{\psi}_{k,m,n}(y)| \le& \frac{\sigma^2}{2}\frac{|U_mV_{k,m}|}{N_n^\alpha}\left(\frac{2\varepsilon}{\sigma^2}\right)^b\Bigg(e^{\frac{\kappa}{\sigma^2}y}\Big[\frac{1}{\Gamma(b)} + \mathbf{1}_{\{b\in(0,1)\}}\Big(\frac{2+2b+n}{\Gamma(b+2)}\Big)\Big(\frac{2\varepsilon y}{\sigma^2}\Big)\Big] + (2+k)\Big(\frac{2y}{\mu}\Big)\times \\
& e^{\frac{2\kappa}{\sigma^2}y}\Bigg[\frac{{}_1F_1\big(b;b+1;-\frac{\kappa}{\sigma^2}y\big)}{\Gamma(b+1)} + \mathbf{1}_{\{b\in(0,1)\}}\Big(\frac{2+2b+n}{b+1}\Big)\Big(\frac{2\varepsilon y}{\sigma^2}\Big)\frac{{}_1F_1\big(b+1;b+2;-\frac{\kappa}{\sigma^2}y\big)}{\Gamma(b+2)}\Bigg]\Bigg) \\
=& \psi^*_{k,m,n}(y)
\end{aligned} \tag{A.4}$$

Then, multiplying both sides by $e^{-zy}\mathfrak{m}(y)$ and integrating (see Eq.3.35.1.3 in Prudnikov et al. [30]), we obtain the following inequality for the Laplace transform

$$\mathcal{L}^{|\widetilde{\psi}|}_{k,m,n}(z) \le \frac{|U_mV_{k,m}|}{N_n^\alpha}\left(\frac{2\varepsilon}{\sigma^2}\right)^b\left(1 + \frac{2(2+k)}{\mu z}\right)$$

$$\times\left[\left(\frac{\kappa}{\sigma^2}+z\right)^{-b}+\mathbf{1}_{\{b\in(0,1)\}}\left(\frac{2+2b+n}{b+1}\right)\left(\frac{2\varepsilon}{\sigma^2}\right)\left(\frac{\kappa}{\sigma^2}+z\right)^{-(b+1)}\right]$$
$$=\mathcal{L}^*_{k,m,n}(z). \tag{A.5}$$

Hence, from Eq. (A.3) and (A.5), it is clear that $\mathcal{L}^*_{k+1,m,n}(z)\Big/\mathcal{L}^*_{k,m,n}(z)\leq K<1$ as $k\to\infty$. Similarly, since $0\leq|\vartheta_{m,n}(t)|\leq e^{-\lambda_n t}2^{\varpi\mathfrak{a}}$ for all $t\geq 0$ (see the proof of Lemma 3), it follows that $|\vartheta_{m+1,n}(t)|\mathcal{L}^*_{k,m+1,n}(z)\Big/|\vartheta_{m,n}(t)|\mathcal{L}^*_{k,m,n}(z)\leq K<1$ as $m\to\infty$. Now, with respect to the index n, we have that $\sum_{n=0}^{\infty}|\vartheta_{m,n}(t)||\varphi_n(x)|\mathcal{L}^{|\widetilde{\psi}|}_{k,m,n}(z)\leq 2^{\varpi\mathfrak{a}}\sum_{n=0}^{\infty}e^{-\lambda_n t}|\varphi_n(x)|\mathcal{L}^*_{k,m,n}(z)$. From (A.3) and Lemma 7, and since $\lambda_n=n\varepsilon+\frac{b}{2}(\varepsilon-\kappa)$, it follows that for all $t>0$, $\left(e^{-\lambda_{n+1}t}|\varphi_{n+1}(x)|\mathcal{L}^*_{k,m,n+1}(z)\right)\Big/\left(e^{-\lambda_n t}|\varphi_n(x)|\mathcal{L}^*_{k,m,n}(z)\right)\leq K<1$ as $n\to\infty$. Hence, the series $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\vartheta_{m,n}(t)|\,|\varphi_n(x)|\,\mathcal{L}^{|\widetilde{\psi}|}_{k,m,n}(z)$ converges uniformly-absolutely on compacts for $z,x\geq 0$ and $t>0$. Therefore, the conditions of Theorem 30.1 in Doetsch [21] are satisfied. One can also prove that the expansion (2.13) of $p^{\alpha}_{\mathfrak{m}}(t;x,y)$ satisfies the Weierstrass criterion in exactly the same way as above (i.e., via the ratio test) by using the bound $\psi^*_{k,m,n}(y)$ in (A.4) instead of (A.5) (details are, therefore, omitted). Hence, we conclude that $p^{\alpha}_{\mathfrak{m}}(t;x,y)$ is uniformly-absolutely convergent on compacts for $x,y\geq 0$ and $t>0$. □

Proof of Theorem 2: Since $p^{\alpha}_{\mathfrak{m}}(t;x,y)$ defined as in (2.13) is uniformly-absolutely convergent on compacts for $x,y\geq 0$ and $t>0$, we observe that for all compact subsets I of $[0,\infty)$, the integral $\int_I p^{\alpha}(t;x,y)dy$ can be done term-by-term for all $t>0$. Hence, let $I=[0,R]$, then $f_{m,n}(R)=\int_I\psi_{m,n}(y)\mathfrak{m}(y)dy$. Applying the definition of ${}_1F_1(a;b;x)$, and since $A=\frac{\kappa+\varepsilon}{\sigma^2}-\frac{1}{\mu}$, we obtain

$$\widetilde{f}_{k,m,n}(R)=\sum_{h=0}^{k}\sum_{j=0}^{n}\frac{\widetilde{c}_{j,n}\widetilde{\varrho}_{h,k,m}}{\Gamma(h+b+j)}\sum_{\ell=0}^{\infty}\frac{(h)_\ell A^\ell}{(h+b+j)_\ell\ell!}\int_0^R e^{-\frac{\kappa+\varepsilon}{\sigma^2}y}y^{h+b+j+\ell-1}dy \tag{A.6}$$
$$=\sum_{h=0}^{k}\sum_{j=0}^{n}\frac{\widetilde{c}_{j,n}\widetilde{\varrho}_{h,k,m}}{\Gamma(h+b+j)}\Big[\sum_{\ell=0}^{\infty}\frac{(h)_\ell A^\ell}{(h+b+j)_\ell\ell!}\Big(\frac{\kappa+\varepsilon}{\sigma^2}\Big)^{-(h+b+j+\ell)}\gamma\Big(h+b+j+\ell,\frac{\kappa+\varepsilon}{\sigma^2}R\Big)\Big],$$

using the fact that $\frac{\gamma(h+b+j+\ell,\frac{\kappa+\varepsilon}{\sigma^2}R)}{\Gamma(h+b+j)(h+b+j)_\ell}=\frac{\gamma(h+b+j+\ell,\frac{\kappa+\varepsilon}{\sigma^2}R)}{\Gamma(h+b+j+\ell)}$ we arrive at (2.15). Now, we justify the interchange of the sum and integral in (A.6). Since for all $r>0$ and $x\geq 0$, and $\gamma(r,x)\leq\Gamma(r)$, then $0\leq\sum_{\ell=0}^{\infty}\frac{(h)_\ell|A|^\ell}{(h+b+j)_\ell\ell!}\big(\frac{\kappa+\varepsilon}{\sigma^2}\big)^{-(h+b+j+\ell)}\gamma\big(h+b+j+\ell,\frac{\kappa+\varepsilon}{\sigma^2}R\big)\leq\sum_{\ell=0}^{\infty}\frac{(h)_\ell|A|^\ell}{(h+b+j)_\ell\ell!}\big(\frac{\kappa+\varepsilon}{\sigma^2}\big)^{-(h+b+j+\ell)}$ $\Gamma(h+b+j+\ell)=\big(\frac{\kappa+\varepsilon}{\sigma^2}\big)^{-(h+b+j)}\big(1-|A|\frac{\sigma^2}{\kappa+\varepsilon}\big)^{-h}\Gamma(h+b+j)<\infty$ (observe that $|A|/((\kappa+\varepsilon)/\sigma^2)<1$, and hence, the identity ${}_1F_0(a;;x)=(1-x)^{-a}$ is valid). Therefore, the interchange is allowed by Fubini's theorem. □

Proof of Theorem 3: When $\alpha=0$, $\beta_{\ell,n}\geq 0$ is reduced to $\beta_{\ell,n}=\kappa(n+\ell)$. This implies that $\lim_{t\to\infty}\vartheta_{m,n}(t)=(-1)^m 2^{\varpi\mathfrak{a}}\mathbf{1}_{\{n=0\}}$. Hence $\lim_{t\to\infty}p^0(t;x,y)=2^{\varpi\mathfrak{a}}\sum_{m=0}^{\infty}(-1)^m\varphi_0(x)\psi_{m,0}(y)\mathfrak{m}(y)$, where $\psi_{m,0}(y)=\frac{\sigma^2}{2N_0^0}\big(\frac{2\kappa}{\sigma^2}\big)^b\sum_{k=0}^{\infty}\sum_{h=0}^{k}\frac{\widetilde{\varrho}_{h,k,m}\,{}_1F_1(h;h+b;Ay)}{\Gamma(h+b)}y^h$, and $\varphi_0(x)=N_0^0$. Using this limit in the transition density $p^{\alpha}(t;x,y)$ of Theorem 1 (with all the other parameters evaluated at $\alpha=0$) we arrive at (2.16). □

Proof of Theorem 4: To show that the integral $p^{\alpha,\phi}_{\mathfrak{m}}(s;x,y)=\int_{[0,\infty)}p^{\alpha}_{\mathfrak{m}}(s;x,y)q_t(ds)$ yields (3.21) we apply the dominated convergence theorem. Recall that $|\vartheta_{m,n}(t)|\leq e^{-\lambda_n t}2^{\varpi\mathfrak{a}}$. The dominating function for the integrand is given by $2^{\varpi\mathfrak{a}} \sum_{n=0}^{\infty} e^{-\lambda_n s} |\varphi_n(x)| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \psi^*_{k,m,n}(y)$ (recall that $|\widetilde{\psi}_{k,m,n}(y)| \leq \psi^*_{k,m,n}(y)$; see (A.4)). Hence if we can show that

$$\int_{[0,\infty)} 2^{\varpi\mathfrak{a}} \sum_{n=0}^{\infty} e^{-\lambda_n s} |\varphi_n(x)| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \psi^*_{k,m,n}(y) q_t(ds) = 2^{\varpi\mathfrak{a}} \sum_{n=0}^{\infty} e^{-\phi(\lambda_n)t} |\varphi_n(x)| \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \psi^*_{k,m,n}(y) < \infty,$$

then we can apply the dominated convergence theorem. Below we prove the convergence of the above series with respect to n (its convergence with respect to k and m can be easily proved using (A.3) and (A.4)). Indeed, using Lemma 5, we observe that $e^{-\phi(\lambda_n)t} |\varphi_n(x)| \psi^*_{k,m,n}(y) \propto K(y)(1 + n\mathbf{1}_{\{b\in(0,1)\}})e^{-\phi(\lambda_n)t} L_n^{b-1}\left(\frac{2x\varepsilon}{\sigma^2}\right)$ for some $0 < K(y) < \infty$ for $y \geq 0$ on compacts (the symbol "$\propto$" indicates "proportional to"). Using the asymptotic representation 4.22.19 in [31] we have $|L_n^{b-1}(x)| \propto n^{\frac{2b-3}{4}}$ as $n \to \infty$ for all $x \geq 0$ on compacts. Therefore, under our assumption, the sum converges uniformly-absolutely on compacts for $x, y \geq 0$ and $t > 0$. □

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