Optimal Stopping in Infinite Horizon: an Eigenfunction Expansion Approach

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Abstract

We develop an eigenfunction expansion based value iteration algorithm to solve discrete time infinite horizon optimal stopping problems for a rich class of Markov processes that are important in applications. We provide convergence analysis for the value function and the exercise boundary, and derive easily computable error bounds for value iterations. As an application we develop a fast and accurate algorithm for pricing callable perpetual bonds under the CIR short rate model.

Keywords: optimal stopping; symmetric Hunt processes; eigenfunction expansions; value iterations; callable perpetual bonds.

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1. Introduction

In this paper we consider infinite horizon optimal stopping problems with stopping allowed at a discrete set of times. Some applications in practice are naturally casted in this framework. A prominent example in finance is the callable perpetual bonds (a.k.a. *callable consol bonds* or *consols*). Consols are bonds that pay the periodic stated coupon in perpetuity (without stated maturity) unless they are called by the issuer. The call option embedded in the bond allows the issuer to call (buy back) the bond from the bond holders at the pre-specified call price, on a discrete set of dates. This gives the issuer an opportunity to refinance the bond should the interest rates fall. The first issuance of consols dates back to the middle of the eighteen century by the United Kingdom, and consols are still issued by various governments and large corporations today.

The class of processes we consider is a rich family of Markov processes, namely *symmetric Hunt processes*[1] taking values in Borel subsets of the real line, such that the corresponding Feynman-Kac (FK) semigroup, with the discount rate being a function of the Markov state, is represented by an eigenfunction expansion when defined on $L^2(E, m)$, where E is the state space of the Markov process, and m is the symmetrizing measure. Examples of Markov processes in this class include a number of one-dimensional diffusions important in applications, such as Ornstein-Uhlenbeck (OU) and Cox-Ingersoll-Ross (CIR), birth-and-death (BD) processes, as well as jump-diffusions and pure jump processes and continuous-time Markov chains obtained by time changing these diffusions and BD processes with Lévy subordinators. Applications of these processes have been found in a variety of financial markets and areas outside finance (see Li and Linetsky (2013) for a survey).

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[1]We refer the readers to Fukushima et al. (1994) for symmetric Markov processes and Hunt processes.

For the finite horizon optimal stopping problem with stopping at a discrete set of dates, Li and Linetsky (2013) have recently developed an eigenfunction expansion approach to solve it. The main idea of Li and Linetsky (2013) is to represent the FK operator (see Eq.(1) for the definition) by its eigenfunction expansion in $L^2(E, m)$, prove that when the payoff is in $L^2(E, m)$, the value function of the optimal stopping problem is also in $L^2(E, m)$, and construct an explicit recursion for the expansion coefficients of the value function. Li and Linetsky (2013) applies the method to valuing commodity futures options and real options in time-changed OU models. Lim et al. (2012) applies the method to valuing finite maturity callable bonds in diffusion and time-changed diffusion interest rate models. Li and Linetsky (2013) and Lim et al. (2012) provide computational experiments illustrating the superior computational performance of the eigenfunction expansion method for finite-horizon problems compared to alternative methods.

Infinite horizon optimal stopping problems in a continuous time setting for one-dimensional Markov processes can often be solved (semi-)analytically. See Dayanik and Karatzas (2003) and Dayanik (2008) for diffusions, and Christensen et al. (2013) for general Hunt processes. In a discrete time setting, in general, when solutions to finite-horizon problems are available, infinite-horizon problems can be solved by value iterations, which is an iterative procedure based on the convergence from finite horizon problems to the infinite horizon one as the maturity goes to infinity (see e.g. Bertsekas (1995) and Peskir and Shiryaev (2006)). However, in order to implement value iterations, issues of convergence, rate of convergence, and error bounds have to be addressed. From a practical point of view, error bounds are particularly important as they tell us when to stop the iteration with the error being properly controlled. Peskir and Shiryaev (2006); Shiryaev (1978) provides conditions for convergence of value iterations in optimal stopping problems for general Markov processes. However, issues of convergence rates and error bounds are not considered there. Bertsekas (1995) studies convergence issues in general control problems for Markov chains when payoffs are either bounded or positive or negative, and provides estimates for convergence rates and error bounds under bounded payoffs. In contrast, for the class of symmetric Hunt processes we consider, it is more natural to consider payoffs in $L^2(E, m)$, which do not necessarily satisfy the assumptions in Bertsekas (1995). Furthermore, the previous literature on value iterations has been primarily restricted to discounting at the constant interest rate. In contrast, in financial applications with stochastic interest rates one is lead to consider random discounting.

The main methodological contributions of the present paper are Theorems 1 and 2 that establish the following key results in our symmetric Hunt process framework, subject to appropriate regularity assumptions. (1) The sequence of value functions of the finite-horizon problems converges to the value function of the infinite-horizon problem both under the $L^2(E, m)$ norm and uniformly on compacts. This result supplements and strengthens the general pointwise convergence result in Peskir and Shiryaev (2006). (2) The convergence rate for value functions is *Q-linear* under the L^2-norm and *R-linear* pointwise. The convergence rate for optimal stopping boundaries is R-linear (cf. Nocedal and Wright (2006) for the definition of Q- and R-linear). (3) Theorems 1 and 2 give explicit and easy to compute error bounds when approximating the value function and optimal stopping boundary of the infinite-horizon problem with those of a finite-horizon problem. These bounds provide an explicit termination algorithm for value iterations. To the best of our knowledge, convergence rates and error bounds for optimal stopping boundaries have not been considered previously in the literature.

We note that in a related work Tsitsiklis and Van Roy (1999) studies value iterations for ergodic Markov processes under L^2 payoffs. However, our setting and assumptions are substantially different from theirs. We do not require ergodicity, but merely the symmetry of the Markov process. The symmetrizing measure m does not have to be a finite measure, thus including symmetric Markov processes without stationary distributions in our set-up. For ergodic Markov processes, Tsitsiklis and Van Roy (1999) proves convergence of values iterations under the L^2 norm. Under our assumptions, we obtain stronger results and show that convergence is, in fact, uniform on compacts. This allows us to easily obtain the continuity of the value function in our set-up. We also derive easily computable pointwise error bounds for the value function, which are more relevant for financial applications than error bounds under the L^2 norm, since in financial applications we are typically interested in the value of a security for a given value of the underlying financial variable. Furthermore, analytically and computationally, our expansion in the eigenfunctions of the Feynman-Kac operator is entirely different from the regression-based Monte Carlo method of Tsitsiklis and Van Roy (1999).

To illustrate the computational performance of our method, we apply our theory and develop a fast and accurate computational algorithm for pricing consols under the popular Cox-Ingersoll-Ross (CIR) interest rate model and show that under the error tolerance of one basis point for both the value function and the optimal stopping boundary, the average computation time is 0.037 seconds per bond, across a wide range of parameters. Consols have been widely studied in the finance literature (e.g., Brennan and Schwartz (1979), Brennan and Schwartz (1982), Delbaen (1993), Duffie et al. (1995), Dybvig et al. (1996)), where simplifying assumptions of continuous coupon payments and calls have often been made. Our method allows us to determine the optimal call policy and value callable consols in a more realistic setting with discrete periodic coupon payments and discrete call dates, including the so-called notice periods. Further applications of our method include infinite-horizon real option problems for mean-reverting assets studied in economics (Dixit and Pindyck (1994)).

2. The Markovian Set-Up and Assumptions

Let E be a Borel subset of $\mathbb{R}$, $\mathfrak{B}(E)$ denote the Borel σ-algebra on E, and $(X_t)_{t\geq 0}$ an E-valued conservative time-homogeneous Hunt process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ (roughly speaking, a Hunt process is a right-continuous strong Markov process that is also assumed to be quasi left-continuous; see Li and Linetsky (2013) for definitions and references). To simplify notation here we assume that the process is conservative to avoid dealing with killing, which can be done following the discussion in Li and Linetsky (2013). Let $r(x)$ be a nonnegative $\mathfrak{B}(E)$-measurable function. The corresponding Feynman-Kac operator that includes discounting at the (stochastic) interest rate $r_t = r(X_t)$ and taking the expectation is defined by

$$\mathcal{P}_t^r f(x) := \mathbb{E}_x\left[\exp\left(-\int_0^t r(X_u)du\right) f(X_t)\right]. \tag{1}$$

Discounting at a constant rate r is a special case where $\mathcal{P}_t^r = e^{-rt}\mathcal{P}_t$, where $(\mathcal{P}_t)_{t\geq 0}$ with $\mathcal{P}_t f(x) = \mathbb{E}_x\left[f(X_t)\right]$ is the transition semigroup of X. We assume that there is a measure $\mathfrak{m}$ with full support on E such that the FK semigroup $(\mathcal{P}_t^r)_{t\geq 0}$ defined on $L^2(E, \mathfrak{m})$ with the inner product $(f, g) := \int_E f(x)g(x)\mathfrak{m}(dx)$ and norm $\|f\| := \sqrt{(f, f)}$ is a strongly continuous semigroup of symmetric contractions. The symmetry property means that $(f, \mathcal{P}_t^r g) = (\mathcal{P}_t^r f, g)$ for all $f, g \in L^2(E, \mathfrak{m})$ and all t. The following assumption ensures the existence of an eigenfunction expansion with desirable convergence properties for financial applications.

Assumption 1. *The FK semigroup* $(\mathcal{P}_t^r)_{t\geq 0}$ *is trace class. This ensures that* $\mathcal{P}_t^r$ *admits a symmetric kernel* $p_t(x,y) = p_t(y,x)$ *with respect to* $\mathfrak{m}$*, i.e.* $\mathcal{P}_t^r f(x) = \int_E p_t(x,y)f(y)\mathfrak{m}(dy)$ *for any* $f \in L^2(E,\mathfrak{m})$*. We further assume that* $p_t(x,y)$ *is jointly continuous in* x *and* y*.*

Under Assumption 1, Proposition 1 and 2 in Li and Linetsky (2013) hold. In particular we have the eigenfunction expansion for any $f \in L^2(E,\mathfrak{m})$,

$$\mathcal{P}_t^r f(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} f_n \varphi_n(x), \quad f_n = (f, \varphi_n) \text{ for any } t > 0,$$

with convergence both in the L^2-norm and uniformly on compacts in x. $\varphi_n(x)$ are eigenfunctions of the operators $\mathcal{P}_t^r$ with the eigenvalues $e^{-\lambda_n t}$, $\mathcal{P}_t^r \varphi_n(x) = e^{-\lambda_n t}\varphi_n(x)$, and are eigenfunctions of the infinitesimal generator $\mathcal{G}^r$ of the FK semigroup with eigenvalues $-\lambda_n$, $\mathcal{G}^r\varphi_n(x) = -\lambda_n \varphi_n(x)$. Under our assumptions, λ_n are non-negative and increasing and satisfy $\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty$ for all $t > 0$ (the *trace-class condition*). The eigenfunctions $(\varphi_n)_{n\geq 1}$ form a complete orthonormal basis in $L^2(E,\mathfrak{m})$. Moreover, under Assumption 1 each φ_n is continuous, and $|\varphi_n(x)| \leq e^{\lambda_n t/2}\sqrt{p_t(x,x)}$ for all $t > 0$. The function $\mathcal{P}_t^r f(x)$ is also continuous in x. Assumption 1 is the same as in Li and Linetsky (2013) for finite-horizon problems.

To ensure convergence of value iterations for infinite-horizon problems in this paper, we impose an additional assumption that does not appear in Li and Linetsky (2013).

Assumption 2. $\lambda_1 > 0$*.*

When the discount rate is constant and positive $r > 0$, this condition is satisfied automatically since in this case $\lambda_1 \geq r$. This condition is also satisfied in all popular positive stochastic interest rate models which satisfy Assumption 1, including the CIR model.

3. Optimal Stopping in Infinite Horizon by Value Iterations

The decision maker receives the scheduled payments $g(X_{ih})$ at each payment date in the discrete set $\{0, h, 2h, \cdots\}$, $h > 0$. If the decision maker stops the game at time kh, he receives the terminal payoff $f(X_{kh})$, along with the final scheduled payment $g(X_{hk})$. After the game is stopped, no more scheduled payments are made. Functions f and g are assumed to be $\mathfrak{B}(E)$-measurable. Let $\mathcal{T}_h$ be the collection of all $(\mathcal{F}_t)_{t\geq 0}$-stopping times that take values in the discrete set $\{0, h, 2h, \cdots\}$. The decision maker maximizes the present value of his cash flow stream. The value function can then be written as:

$$V(x) = \sup_{\tau\in\mathcal{T}_h} \mathbb{E}_x \left[\sum_{i=0}^{\tau/h} \exp\left(-\int_0^{ih} r(X_u)du\right) g(X_{ih}) + \exp\left(-\int_0^{\tau} r(X_u)du\right) f(X_\tau)\right], \quad x \in E.$$

Assumption 3. $g, f \in L^2(E,\mathfrak{m})$*.*

Li and Linetsky (2013) considers this problem in finite horizon with maturity Nh, terminal payoff f, and without the intermediate scheduled payments g prior to maturity. To carry out value iterations, we are interested in computing a sequence of solutions to the finite horizon problems with $N = 1, 2, \cdots$. Let $C^N(x)$ and $V^N(x)$ be the continuation value and the value function at time zero for a problem with finite horizon Nh. Extending the arguments in Li and Linetsky (2013) to include g, we can show that the functions C^N and V^N are in $L^2(E,\mathfrak{m})$ and that $C^N(x)$ has the eigenfunction expansion $C^N(x)=\sum_{n=1}^{\infty}c_n^N e^{-\lambda_n h}\varphi_n(x)$ with the coefficients c_n^N satisfying the following recursion in N for each $n\geq 1$:

$$c_n^1=g_n+f_n,\ c_n^N=g_n+f_n(\mathcal{S}^{N-1})+\sum_{m=1}^{\infty}c_m^{N-1}e^{-\lambda_m h}\pi_{m,n}\left(\mathcal{C}^{N-1}\right),\quad N\geq 2,$$

where $\mathcal{C}^N:=\{x\in E: C^N(x)>f(x)\}$ is the continuation region at time zero for the problem with horizon Nh, $\mathcal{S}^N:=\{x\in E: C^N(x)\leq f(x)\}$ is the stopping region at time zero for the same problem, and $\pi_{m,n}(A):=(1_A\varphi_m,\varphi_n)$, $f_n(A):=(1_A f,\varphi_n)$ for $m,n\geq 1$ for $A\subseteq E$ ($1_A(x)$ is the indicator of the set A). The computational implementation of this recursion can be accomplished similarly to the recursion in Li and Linetsky (2013). The next theorem establishes convergence of value iterations and explicitly gives rates of convergence and error bounds for approximating the value function of the infinite horizon problem by the value function of the corresponding finite horizon problem.

Theorem 1. *Let $V(x)$ be the value function of the infinite-horizon problem and define $C(x):=\mathcal{P}_h^r V(x)$ (interpreted as the continuation value of the infinite-horizon problem), $\alpha_h:=\frac{e^{-\lambda_1 h}}{1-e^{-\lambda_1 h}}$, $\beta_h:=\frac{1}{1-e^{-\lambda_1 h}}$ and $m_h(x):=\inf_{0<u<h}\left(\sqrt{p_{2u}(x,x)}\sum_{n=1}^{\infty}e^{-\lambda_n(h-u)}\right)$. Under Assumptions 1, 2 and 3, the following results hold (recall that $\|\cdot\|$ denotes the L^2-norm w.r.t. $\mathfrak{m}$):*

(i) $V^N\to V$ and $C^N\to C$ in the L^2-norm with the Q-linear convergence rate, and

$$\|V^N-V\|\leq e^{-\lambda_1 h}\|V^{N-1}-V\|,\quad \|C^N-C\|\leq e^{-\lambda_1 h}\|C^{N-1}-C\|, \tag{2}$$
$$\|V^N-V\|\leq \alpha_h\|V^{N-1}-V^N\|,\quad \|C^N-C\|\leq \alpha_h\|C^{N-1}-C^N\|. \tag{3}$$

(ii) $V^N\to V$ and $C^N\to C$ uniformly on compacts with the R-linear convergence rate, and

$$|V^N(x)-V(x)|\leq |C^N(x)-C(x)|\leq \beta_h\|C^{N-1}-C^N\|m_h(x). \tag{4}$$

(iii) $C(x)=\sum_{n=1}^{\infty}c_n e^{-\lambda_n h}\varphi_n(x)$ with $c_n=\lim_{N\to\infty}c_n^N$ uniformly in n.

(iv) $C(x)$ is continuous. $V(x)$ is continuous if $g(x)$ and $f(x)$ are continuous.

To prove Theorem 1, we need the following lemma.

Lemma 1. *Under Assumption 1, for any $f\in L^2(E,\mathfrak{m})$, $|\mathcal{P}_h^r f(x)|\leq\|f\|m_h(x)$, $\|\mathcal{P}_h^r f\|\leq e^{-\lambda_1 h}\|f\|$.*

Proof. By Cauchy-Schwartz inequality we have $|f_n|\leq ||f||\cdot||\varphi_n||=||f||$. Assumption 1 implies for any $u\in(0,h)$, $|\varphi_n(x)|\leq e^{\lambda_n u}\sqrt{p_{2u}(x,x)}$. Hence we have for all $u\in(0,h)$,

$$|\mathcal{P}_h^r f(x)|=\left|\sum_{n=1}^{\infty}f_n e^{-\lambda_n h}\varphi_n(x)\right|\leq\sum_{n=1}^{\infty}\|f\|e^{-\lambda_n h}|\varphi_n(x)|\leq\|f\|\sqrt{p_{2u}(x,x)}\sum_{n=1}^{\infty}e^{-\lambda_n(h-u)}.$$

Thus, $|\mathcal{P}_h^r f(x)|\leq\|f\|m_h(x)$. The trace-class condition guarantees $m_h(x)$ is finite. To prove the second inequality, we note that $\|\mathcal{P}_t^r f\|^2=\sum_{n=1}^{\infty}e^{-2\lambda_n t}f_n^2\leq e^{-2\lambda_1 t}\sum_{n=1}^{\infty}f_n^2=e^{-2\lambda_1 t}\|f\|^2$. $\square$

Proof of Theorem 1. We first verify that under our assumptions the following holds for all $x \in E$

$$\mathbb{E}_x \left[\sum_{i=1}^{\infty} \exp\left(-\int_0^{ih} r(X_u) du \right) (|g(X_{ih})| + |f(X_{ih})|) \right] < \infty. \tag{5}$$

As in the proof of Lemma 1, this expectation is bounded by

$$\sum_{i=1}^{\infty} (\|g\| + \|f\|) \sqrt{p_h(x,x)} \sum_{n=1}^{\infty} e^{-\lambda_n (i-1/2)h} = (\|g\| + \|f\|) \sqrt{p_h(x,x)} \sum_{n=1}^{\infty} e^{\lambda_n h/2} \sum_{i=1}^{\infty} e^{-\lambda_n ih}$$

$$= (\|g\| + \|f\|) \sqrt{p_h(x,x)} \sum_{n=1}^{\infty} e^{-\lambda_n h/2} / (1 - e^{-\lambda_n h}) \le \frac{(\|g\| + \|f\|)}{1 - e^{-\lambda_1 h}} \sqrt{p_h(x,x)} \sum_{n=1}^{\infty} e^{-\lambda_n h/2} < \infty.$$

Above we used Assumption 2, $\lambda_n \le \lambda_{n+1}$, and the trace-class condition. The result (5) allows us to apply Peskir and Shiryaev (2006) Theorem 1.6 which shows that $V(x) = g(x) + \max\{f(x), C(x)\}$.
Part (i): We first note the inequality $|\max\{a,b\} - \max\{a,c\}| \le |b - c|$. This gives

$$|V^N(x) - V(x)| = |\max\{f(x), C^N(x)\} - \max\{f(x), C(x)\}| \le |C^N(x) - C(x)|.$$

Hence $\|V^N - V\| \le \|C^N - C\|$. By Lemma 1,

$$\|V^N - V\| \le \|C^N - C\| = \|\mathcal{P}_h^r V^{N-1} - \mathcal{P}_h^r V\| \le e^{-\lambda_1 h} \|V^{N-1} - V\| \le e^{-\lambda_1 h} \|C^{N-1} - C\|. \tag{6}$$

Repeating this procedure gives $\|V^N - V\| \le \|C^N - C\| \le (e^{-\lambda_1 h})^N \|g + f - V\|$. Letting $N \to \infty$ we see that $V^N \to V$ and $C^N \to C$ under the L^2-norm. To prove (3), we note that for any $M > 0$,

$$\begin{aligned}
\|V^N - V\| &= \|V^N - V^{N+1} + \sum_{k=1}^{M} (V^{N+k} - V^{N+k+1}) + V^{N+M+1} - V\| \\
&\le \|V^N - V^{N+1}\| + \sum_{k=1}^{M} \|V^{N+k} - V^{N+k+1}\| + \|V^{N+M+1} - V\| \\
&\le \|V^N - V^{N+1}\| + \sum_{k=1}^{M} (e^{-\lambda_1 h})^k \|V^N - V^{N+1}\| + \|V^{N+M+1} - V\|
\end{aligned}$$

Letting $M \to \infty$, we have $\|V^N - V\| \le \beta_h \|V^N - V^{N+1}\| \le \alpha_h \|V^{N-1} - V^N\|$. The last inequality is obtained from (6) by replacing V with V^{N+1}. Similarly we can prove the result for C^N and C.
Part (ii): The first inequality is already shown in (i). From Lemma 1 and its proof,

$$|C^N(x) - C(x)| = |\mathcal{P}_h^r V^{N-1}(x) - \mathcal{P}_h^r V(x)| \le \|V^{N-1} - V\| m_h(x) \le \|V^{N-1} - V\| \sqrt{p_h(x,x)} \sum_{n=1}^{\infty} e^{-\lambda_n h/2}.$$

Since $p_h(x,x)$ is continuous and $\|V^{N-1} - V\| \to 0$ as $N \to \infty$, $V^N \to V$ and $C^N \to C$ uniformly on compacts. Since $\|V^{N-1} - V\| \le (e^{-\lambda_1 h})^{N-1} \|g + f - V\|$, the convergence rate is R-linear. The last inequality in (4) follows from $\|V^{N-1} - V\| \le \beta_h \|C^{N-1} - C^N\|$ shown in (i).
Part (iii): Part (i) shows that $V \in L^2(E, \mathfrak{m})$. Since $C(x) = \mathcal{P}_h^r V(x)$, we can write $C(x) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n h} \varphi_n(x)$, where $c_n = (C, \varphi_n)$. Since $|c_n^N - c_n| = |(C^N, \varphi_n) - (C, \varphi_n)| = |(C^N - C, \varphi_n)| \le \|C^N - C\|$ and $C^N \to C$ in L^2, we have $\lim_{N \to \infty} c_n^N = c_n$ uniformly in n.
Part (iv): Each $C^N(x)$ is continuous as shown in Li and Linetsky (2013). The continuity of $C(x)$ follows from uniform convergence on compacts. When $f(x)$ and $g(x)$ are continuous, the continuity of $V(x)$ is then immediate from the continuity of $C(x)$. $\square$

Theorem 1 shows that, under our assumptions, the sequence of value functions of finite-horizon problems with increasing maturities converges both in L^2 and uniformly on compacts to the value function of the infinite-horizon problem. We note that when the discount rate r is constant, the constant in (2) is e^{-rh}. These types of results have appeared in the literature on infinite-horizon problems with constant discounting. Since $\lambda_1 \geq r$, our result improves the standard result for constant discounting, and also extends it to stochastic interest rates. Using the error bounds in (3), we can determine the termination level in the value iteration algorithm to achieve target error tolerance for the value function.

We next turn to the optimal stopping region. Let $\mathcal{S}$ and $\mathcal{C}$ denote the stopping and continuation region of the infinite-horizon problem respectively ($\mathcal{S} = \{x \in E : f(x) \geq C(x)\}$ and $\mathcal{C} = E \backslash \mathcal{S}$). $\mathcal{S}^N$ is the stopping region for the corresponding finite-horizon problems. Following the arguments in Bertsekas (1995) section 3.4, it can be shown that $\mathcal{S}^N \supseteq \mathcal{S}^{N+1}$ and $\lim_{N\to\infty} \mathcal{S}^N = \mathcal{S}$. A point $x \in E$ is a boundary point of $\mathcal{S}$ if $f(x) = C(x)$, and it is isolated if there exists a neighborhood of x^* such that there are no other boundary points in that neighborhood. The next theorem shows convergence rate and error bounds for isolated boundary points. In applications $\mathcal{S}$ and $\mathcal{S}^N$ are typically finite unions of disjoint intervals and, hence, the boundary points are isolated.

Theorem 2. *Suppose $E \subseteq \mathbb{R}$ is a finite or infinite interval, and $\mathcal{S}$ is nonempty. Let x^* be an isolated boundary point of $\mathcal{S}$, and x_N^* be the boundary point of $\mathcal{S}^N$ converging to x^*. Suppose Assumption 1, 2 and 3 hold, and further assume both f and C are continuously differentiable in a neighborhood of x^*, and $f'(x^*) \neq 0$. For large enough N, if $x_N^* \neq x^*$ then*

$$|x_N^* - x^*| = |C^N(x_N^*) - C(x_N^*)|/|f'(\xi_N) - C'(\eta_N)| \leq \beta_h \|C^{N-1} - C^N\| m_h(x_N^*)/|f'(\xi_N) - C'(\eta_N)| \quad (7)$$

for some ξ_N and η_N between x_N^* and x^*. If $f'(x^*) \neq C'(x^*)$, the convergence rate is R-linear.

Proof. Since x^* is isolated and f' is continuous, we can find $\delta > 0$ small enough such that for any $x \in (x^* - \delta, x^* + \delta)$, $f'(x) \neq 0$ and $f(x) \neq C(x)$ for $x \neq x^*$. Since $\lim_{N\to\infty} x_N^* = x^*$, for N sufficiently large, $x_N^* \in (x^* - \delta, x^* + \delta)$. We will only consider such x_N^* below. By the Mean Value Theorem, we have $f(x_N^*) - f(x^*) = f'(\xi_N)(x_N^* - x^*)$ for some ξ_N between x_N^* and x^*. Similarly, $C(x_N^*) - C(x^*) = C'(\eta_N)(x_N^* - x^*)$ for some η_N between x_N^* and x^*. Since $f'(\xi_N) \neq 0$ we have

$$\begin{aligned} x_N^* - x^* &= \frac{f(x_N^*) - f(x^*)}{f'(\xi_N)} = \frac{C^N(x_N^*) - C(x^*)}{f'(\xi_N)} = \frac{C^N(x_N^*) - C(x_N^*) + C(x_N^*) - C(x^*)}{f'(\xi_N)} \\ &= \frac{C^N(x_N^*) - C(x_N^*)}{f'(\xi_N)} + \frac{C'(\eta_N)}{f'(\xi_N)}(x_N^* - x^*) \end{aligned} \quad (8)$$

We note that $f'(\xi_N) \neq C'(\eta_N)$. If this was the case, then (8) implies $C^N(x_N^*) = C(x_N^*)$ and hence $f(x_N^*) = C(x_N^*)$. Since $x_N^* \neq x^*$, this contradicts that x^* is an isolated boundary point in $(x^* - \delta, x^* + \delta)$. Rearranging of the terms in (8) and the proof of Theorem 1 part (ii) gives:

$$|x_N^* - x^*| = \frac{|C^N(x_N^*) - C(x_N^*)|}{|f'(\xi_N) - C'(\eta_N)|} \leq \frac{(e^{-\lambda_1 h})^{N-1} \|g + f - V\| \sqrt{p_h(x_N^*, x_N^*)} \sum_{n=1}^{\infty} e^{-\lambda_n h/2}}{|f'(\xi_N) - C'(\eta_N)|}.$$

Finally, if $f'(x^*) \neq C'(x^*)$, $\lim_{N\to\infty} \frac{\sqrt{p_h(x_N^*, x_N^*)}}{|f'(\xi_N) - C'(\eta_N)|} = \frac{\sqrt{p_h(x^*, x^*)}}{|f'(x^*) - C'(x^*)|} < \infty$ implies convergence is R-linear. $\square$

In many applications, the eigenfunctions are differentiable. Under Assumption 1, if for any compact interval $J \subseteq E$, $\sum_{n=1}^{\infty} e^{-\lambda_n h} \|\varphi_n'|_J\|_\infty < \infty$ ($\|\cdot\|_\infty$ is the L^∞ norm), then it can be shown that $\mathcal{P}_h^r f(x)$ is continuously differentiable in E for any $f \in L^2(E, \mathfrak{m})$. This shows the smoothness of $C(x)$ (also $C^N(x)$) since $C(x) = \mathcal{P}_h^r V(x)$ ($C^N(x) = \mathcal{P}_h^r V^{N-1}(x)$). Furthermore $C'(x) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \varphi_n'(x)$ and $(C^N(x))' = \sum_{n=1}^{\infty} c_n^N e^{-\lambda_n t} \varphi_n'(x)$ (we write $(C^N(x))'$ for the derivative of C^N). See Li and Linetsky (2013) Proposition 5.4 for a result in this form. In practice, when N is large enough, the term $f'(\xi_N) - C'(\eta_N)$ in the error bound can be well approximated by $f'(x_N^*) - (C^N(x_N^*))'$. The condition $f'(x^*) \neq C'(x^*)$ is also satisfied in typical financial applications. We remark that the knowledge of $f'(x^*) \neq C'(x^*)$ is not required to apply our result for the error bound. Furthermore, if one observes the error bound decays as N increases, then we must have $f'(x^*) \neq C'(x^*)$, otherwise the right-hand-side of eq.(7) blows up to infinity. Hence in practice, $f'(x^*) \neq C'(x^*)$ can be verified numerically.

4. A Numerical Experiment: Pricing Callable Perpetual Bonds

We assume the bond issuer can make the decision to call the bond at any one time in the set $\{0, h, 2h, \cdots\}$, where $h > 0$ is some time period. If the decision is made to call the bond at time ih, the bond is redeemed for the call price K at time $ih + \delta$ ($\delta \geq 0$). The delay δ is called the *notice period* (the bond issuer makes the decision to call and issues a call notice to the bond holders at time ih, and redeems the bond at time $ih + \delta$ after the notice period). The bond pays periodic coupons of $C > 0$ dollars at coupon times $ih + \delta$, $i = 0, 1, \cdots$, until the bond is called (or in perpetuity, if the bond is never called). If the decision to call is made at time ih, then the last coupon is paid at time $ih + \delta$, along with the call price K.

We assume that under the risk-neutral probability measure the short rate r_t follows a CIR square-root diffusion (Cox et al. (1985)), $dr_t = \kappa(\theta - r_t) + \sigma\sqrt{r_t}dB_t$, $r_0 > 0$, $\kappa, \theta, \sigma > 0$, B is a standard Brownian motion. The speed density of the CIR diffusion is $\mathfrak{m}(x) = (2/\sigma^2)x^{b-1}e^{-2\kappa x/\sigma^2}$, where $b = 2\kappa\theta/\sigma^2$. The CIR SDE has a unique strong solution starting from any positive value. If $b \geq 1$, the solution does not hit zero (zero is an inaccessible boundary point for the diffusion process). If $b < 1$, the solution can hit zero, but is instantaneously reflected from zero (zero is an instantaneously reflecting boundary). In the former case $E = (0, \infty)$. In the latter case $E = [0, \infty)$. Explicit expressions for the eigenvalues and eigenfunctions of the CIR FK semigroup are found by Davydov and Linetsky (2003) (see also Gorovoi and Linetsky (2004), Linetsky (2004) and Linetsky (2008)). In this case Assumptions 1 and 2 can be verified directly, and $\mathcal{P}_t^r f(x)$ is continuously differentiable in x for all $f \in L^2(E, \mathfrak{m})$ and $t > 0$.

For callable bonds the decision maker is the bond issuer who pays the coupons at times $ih + \delta$. Thus, the payments at the call decision times ih have present values $g(x) = -CP(x, \delta)$, where $P(x, \delta)$ is the price of a zero-coupon bond at time 0 with unit face value and maturity δ given $r_0 = x$. The terminal call payment is $f(x) = -KP(x, \delta)$. The negative signs indicate that the decision maker is making the payments. Hence, the value function is also negative. It is straightforward to show that $g, f \in L^2(E, \mathfrak{m})$, hence Assumption 3 holds. The issuer calls the bond back when the short rate declines to a low enough value. Proposition 3.2 in Lim et al. (2012) shows that, for the finite horizon problem (finite-maturity callable bond), the non-linear equation $f(x) = C^N(x)$ has at most one solution in E. If it exists, we denote it by x_N^*. It can be found by a numerical root finding algorithm, such as bisection. In this case $\mathcal{S}^N = (0, x_N^*]$ or $[0, x_N^*]$ and $\mathcal{C}^N = (x_N^*, \infty)$. If $f(x) = C^N(x)$ has no solution, which can be shown to be equivalent to $|f^N(0)| > |C^N(0)|$, then

$\mathcal{S}^{N'} = \varnothing$ for all $N' \geq N$, and hence $\mathcal{S} = \varnothing$. In this case the exercise boundary converges in finite number of steps. Closed-form expressions for g_n, $f_n(\mathcal{S}^N)$ and $\pi_{m,n}(\mathcal{C}^N)$, as well as recursive procedures to calculate them, can be found in Lim et al. (2012), where the algorithm is given to solve the finite-horizon callable bond problem.

To assess the computational performance of our value iteration algorithm, we consider a callable perpetual bond with unit face value, $h = 1$ year and $\delta = 1/6$ year. The CIR process parameters and the coupon C and call price K are randomly sampled from uniform distributions: $\theta \sim \mathcal{U}[0.03, 0.07]$, $\kappa \sim \mathcal{U}[0.1, 0.5]$, $\sigma \sim \mathcal{U}[0.1, 0.4]$, $r_0 = x_0 \sim \mathcal{U}[0.005, 0.05]$, $C \sim \mathcal{U}[\theta - 0.02, \theta + 0.02]$ given θ, $K \sim \mathcal{U}[1.02, 1.07]$. Here $\mathcal{U}[a, b]$ denotes the uniform distribution on $[a, b]$. We generated a sample of 1,000 parameter combinations and valued 1,000 perpetual callable bonds with these parameters. In our valuation algorithm, in each iteration the infinite series were truncated until a given relative error tolerance level was reached. Bisection was used to solve the non-linear equation for the boundary. The error tolerance for the series truncation and bisection was set to 1.0E-10 in our computation. In each iteration we evaluated our error bounds in (4) and (7), with the latter one approximated as discussed above. To calculate $m_h(x)$, the Brent's method for minimization was used (see Brent (1973)). We required accuracy to the fourth decimal place for both the bond value function and the optimal call boundary. (For a unit face value bond, the accuracy to the fourth decimal corresponds to the relative error on the order of 0.01%, or one basis point. This accuracy level is sufficient in applications.) We stopped the value iterations algorithm when the error bounds for both the value function and the optimal call boundary became smaller than the required error tolerance.

The algorithm was coded in C++ and executed on a laptop computer with Intel Core 2 i5-2450M CPU (2.50GHz, 4.00GB RAM) under Linux. For this sample of 1,000 bonds, on average it took 80 iterations to converge to required accuracy with the average computation time of 0.037 seconds. This CPU time includes both the time to perform value iterations, as well as minimization to find $m_h(x)$ to compute our error bound. Our explicit error bounds guarantee that the value iterations algorithm is terminated only after it attains the required level of accuracy. To illustrate the optimal call boundary, Figure 1 shows dependence of the boundary x^* on the call price K for a particular set of parameters.

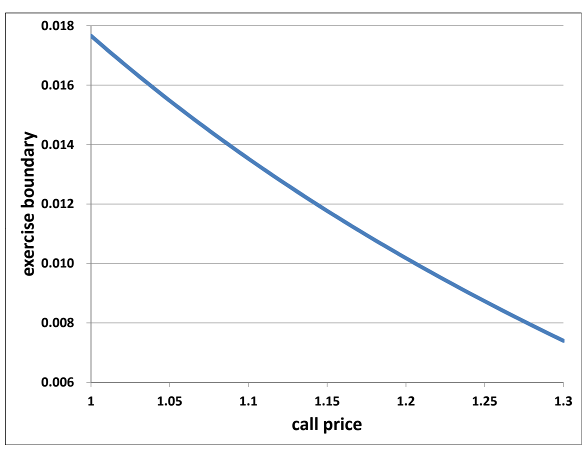

Figure 1: Exercise boundary x^* vs. call price K. $\theta = 0.05$, $\kappa = 0.2$, $\sigma = 0.25$, $C = 0.05$, $h = 1$, $\delta = 1/6$.

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