Optimal Stopping and Early Exercise: An Eigenfunction Expansion Approach*

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Abstract

This paper proposes a new approach to solve finite-horizon optimal stopping problems for a class of Markov processes that includes one-dimensional diffusions, birth-death (BD) processes, and jump-diffusions and continuous-time Markov chains obtained by time changing diffusions and BD processes with Lévy subordinators. When the expectation operator has a purely discrete spectrum in the Hilbert space of square-integrable payoffs, the value function of a discrete optimal stopping problem has an expansion in the eigenfunctions of the expectation operator. The Bellman's dynamic programming for the value function then reduces to an explicit recursion for the expansion coefficients. The value function of the continuous optimal stopping problem is then obtained by extrapolating the value function of the discrete problem to the limit via Richardson extrapolation. To illustrate the method, the paper develops two applications: American-style commodity futures options and Bermudan-style abandonment and capacity expansion options in commodity extraction projects under the subordinate Ornstein-Uhlenbeck model with mean-reverting jumps with the value function given by an expansion in Hermite polynomials.

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1 Introduction

Optimal stopping problems are ubiquitous in financial engineering: exercise of American- and Bermudan-style options on stocks, stock indexes, currencies, commodities, and fixed income instruments, issuer's decision to call a callable bond, bond holder's decision to convert a convertible bond, real options in physical investment projects, such as abandonment, capacity expansion, contraction, extension, and optimal investment and divestment timing in trading strategies are all examples of timing decisions in finance modeled by optimal stopping of a stochastic process. In financial applications, the underlying stochastic process modeling state variables of interest is usually a continuous-time Markov process, while the stopping decision can be made either continuously in time (American-style exercise) or discretely in time (Bermudan-style exercise). The mathematical literature on optimal stopping of continuous-time Markov processes, as well as on financial applications, is very extensive. We do not attempt to survey it here and refer the reader to the monographs Shiryaev (1978) and Peskir and Shiryaev (2006) for mathematical foundations and Karatzas and Shreve (1998), Detemple (2006) and Boyarchenko and Levendorskiĭ (2007b) for financial applications. Extensive bibliographies can be found in these monographs.

In finite horizon, virtually no non-trivial optimal stopping problem is known to admit an analytical solution for the optimal stopping policy and the value function, and numerical methods have to be used. When the stopping decision is made at discrete times, the optimal stopping problem can be solved via Bellman's dynamic programming. At the end of the finite horizon, the value function is equal to the terminal payoff. At each previous decision time, the value function is calculated as the maximum of the payoff or reward and the continuation value, with the latter equal to the conditional expectation of the discounted value function at the previous step, given the value of the state variables at the current step. The conditional expectation involved is computed numerically via a variety of numerical methods, including Markov chain approximations of the underlying Markov process (e.g. Kushner and Dupuis (2001)), binomial and trinomial trees popular in finance (e.g. Nelson and Ramaswamy (1990), Broadie and Detemple (1996)), solving numerically partial differential equation for diffusions or partial integro-differential equations for jump-diffusions (e.g. Achdou and Pironneau (2005) for a survey of numerical PDE methods in finance), fast Fourier and Hilbert transform methods for Lévy processes (Feng and Lin (2009)), fast Gauss transform methods for processes with transition functions given by mixtures of Gaussians (Broadie and Yamamoto (2003)), and Monte Carlo simulation for high-dimensional problems where deterministic numerical schemes suffer from the curse of dimensionality (e.g. Broadie and Glasserman (2004) and Glasserman (2004)).

This paper proposes a new approach to solve finite-horizon optimal stopping problems for an important class of Markov processes, $\mathfrak{m}$*-symmetric Hunt processes taking values in a Borel subset of the real line.* Roughly speaking, Hunt processes are strong Markov processes with sample paths that are right continuous with left limits and quasi-left-continuous. A Markov process is said to be $\mathfrak{m}$-symmetric if there exists a measure $\mathfrak{m}$ on its state space E such that the transition function of the Markov process is symmetric with respect to $\mathfrak{m}$ (see Eq.(2.1)). If a Markov process is a Hunt process and $\mathfrak{m}$-symmetric, then its transition function defines a symmetric semigroup in the Hilbert space $L^2(E, \mathfrak{m})$ of functions square integrable with respect to its symmetry measure $\mathfrak{m}$. Examples of $\mathfrak{m}$-symmetric Hunt processes taking values in a Borel subset of the real line include one-dimensional diffusions, where E is a (finite or infinite) interval, birth-death (BD) processes, where E is a discrete set of points, and jump-diffusions and continuous-time Markov chains (CTMC) obtained by stochastically time changing diffusions and BD processes with Lévy subordinators.

When the transition semigroup operator defined in $L^2(E, \mathfrak{m})$ by the transition function of an $\mathfrak{m}$-symmetric Hunt process has a purely discrete spectrum, the value function of a discrete optimal stopping problem with square-integrable payoffs has the expansion in its eigenfunctions. Under some additional regularity conditions, this eigenfunction expansion converges not only in L^2, but also uniformly on compacts in the state variable. The Bellman's dynamic programming for the value function then reduces to an explicit recursion for the L^2 expansion coefficients of the value function. This result, together with the explicit form of this recursion, constitutes the main theoretical result of this paper (Theorem 3.2). Furthermore, the value function of the continuous optimal stopping problem is then obtained as the limit of the sequence of value functions for discrete optimal stopping problems (Theorem 4.1). Computationally, it can be evaluated by extrapolating the value function of the discrete problem to the limit via Richardson extrapolation (we note that Richardson extrapolation has been first introduced to option pricing by Geske and Johnson (1984) and studied by Broadie and Detemple (1996) in the context of binomial trees). *The present paper can be thought of as the extension of the eigenfunction expansion approach for diffusions and European options (Davydov and Linetsky (2003); see also Linetsky (2004), Linetsky (2008) and the bibliography therein) to optimal stopping and options with early exercise under more general Markov processes.*

Examples of symmetric Hunt processes taking values in a Borel subset of the real line that possess purely discrete spectra and eigenfunctions known in closed form in terms of orthogonal polynomials include Ornstein-Uhlenbeck (OU) processes with eigenfunctions expressed in terms of Hermite polynomials, Cox-Ingersoll-Ross (CIR) and constant elasticity of variance (CEV) diffusions with eigenfunctions expressed in terms of Laguerre polynomials, Jacobi diffusions with eigenfunctions expressed in terms of Jacobi polynomials, and birth-death processes with eigenfunctions expressed in terms of families of discrete polynomials (references can be found in section 2 of this paper). Furthermore, the remarkable fact is that stochastically time changing (*subordinating*) a Markov process possessing an eigenfunction expansion with a Lévy subordinator yields another Markov process with generally very different sample path behavior, but with the *same* eigenfunctions and new eigenvalues $\lambda_n^\phi = \phi(\lambda_n)$, where $\phi(\lambda)$ is the Laplace exponent of the Lévy subordinator appearing in the Lévy-Khintchine theorem, and λ_n are the eigenvalues of the (negative of) the infinitesimal generator of the original Markov process (this important observation goes back to Bochner (1949), who originally introduced the concept of time changes now known as *Bochner's subordination* and observed in Eq.(11) that the subordination preserves the form of the eigenfunction expansion with the old eigenvalues λ_n replaced with the new eigenvalues $\phi(\lambda_n)$). In particular, subordinating diffusions leads to jump-diffusion and pure jump processes with state-dependent jumps (see Albanese and Kuznetsov (2004), Barndorff-Nielsen and Levendorskiǐ (2001), Boyarchenko and Levendorskiǐ (2007a), Mendoza et al. (2010), Li and Linetsky (2012), Lim et al. (2012), Mendoza and Linetsky (2012a), Mendoza and Linetsky (2012b) for applications in finance), while subordinating BD processes leads to continuous-time Markov chains that can generally transition from a given state to any other state. The eigenfunction expansion approach to optimal stopping problems proposed in this paper can be efficiently implemented for all these processes with analytically known eigenfunctions in terms of orthogonal polynomials, by exploiting classical recursions for orthogonal polynomials.

In comparison to purely numerical methods, the eigenfunction expansion approach to optimal stopping proposed in this paper has the following advantages. (1) The method is applicable to a rich class of Markov processes, including one-dimensional diffusions, jump-diffusion and pure jump processes with state-dependent jumps obtained from diffusions by time changing with Lévy subordinators, as well as BD processes and CTMCs obtained from them by subordination. For one-dimensional problems, as long as the spectrum is discrete, and some technical conditions to ensure uniform convergence are satisfied, and the eigenfunctions can be expressed in closed form in terms of special functions for which efficient computational algorithms are available, the method can be implemented to yield an efficient computational algorithm. (2) The value function is given globally on the state space in terms of uniformly convergent eigenfunction expansions, without the need to discretize the state variable. In contrast to numerical methods such as trees, the entire value function is constructed at once, giving, for example, option prices for all values of the underlying asset. (3) Under some mild additional regularity conditions, the derivative of the value function (e.g. the option delta) can be immediately obtained by term-by-term differentiation of the eigenfunction expansion of the value function. (4) The early exercise boundary is determined accurately by finding roots of globally defined non-linear functional equations. This is in contrast to numerical methods that discretize the state space, where accurate determination of the boundary requires considering exceedingly fine step sizes in the discretized state variable.

To illustrate our approach and develop some intuition before presenting the full details, here we informally sketch the pricing of a Bermudan option with two exercise opportunities written on an OU diffusion process X with volatility σ, long-run level θ, and rate of mean reversion κ. The stationary distribution of the OU diffusion is Gaussian $\mathfrak{m}(dx) = \sqrt{\frac{\kappa}{\pi\sigma^2}} e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}} dx$, and the OU diffusion is a symmetric Hunt processes on $\mathbb{R}$ with respect to $\mathfrak{m}$. The Bermudan option can be exercised at its maturity $t > 0$ with the payoff $f(X_t)$, or half way through its life at time $t/2$ with payoff $f(X_{t/2})$. For simplicity we assume that the discount rate is zero. If the payoff is square-integrable with the Gaussian measure, i.e. $f \in L^2(\mathbb{R}, m)$, placing ourselves at time $t/2$, the expected payoff $\mathbb{E}[f(X_t)|X_{t/2}]$ (continuation value at time $t/2$) has the following *eigenfunction expansion* in $L^2(\mathbb{R}, m)$:

$$C^1(x) = \mathbb{E}[f(X_t)|X_{t/2} = x] = \sum_{n=0}^{\infty} e^{-\kappa n t/2} f_n \varphi_n(x),$$

where $\varphi_n(x) = \left(\sqrt{2^n n!}\right)^{-1} H_n(\sqrt{\kappa}(x-\theta)/\sigma)$ are normalized *eigenfunctions* of the conditional expectation operator satisfying:

$$\mathbb{E}[\varphi_n(X_t)|X_s = x] = e^{-\kappa n(t-s)} \varphi_n(x).$$

They form an orthonormal basis in $L^2(\mathbb{R}, m)$. Here H_n are Hermite polynomials. The expansion coefficients are inner products of the payoff and the eigenfunction, $f_n = (f, \varphi_n)$, where $(f, g) = \int_{\mathbb{R}} f(x)g(x)\mathfrak{m}(dx)$ is the inner product. At time $t/2$, the option holder maximizes her payoff. Accordingly, the value function is $V^1(x) = \max(f(x), C^1(x))$. The early exercise (stopping) region at time $t/2$ is $\mathcal{S}^1 = \{x \in \mathbb{R} : f(x) \geq C^1(x)\}$. Assume that the non-linear equation $f(x) - C^1(x) = 0$ has a unique root x^* and, to be specific, the stopping region is $\mathcal{S}^1 = (-\infty, x^*]$. Our method is able to handle more general structure of the stopping region. We make this assumption here to simplify discussion. The root x^* can be found numerically by, say, the bisection algorithm. The value function $V^1(x) = f(x)\mathbf{1}_{(-\infty,x^*]}(x) + C^1(x)\mathbf{1}_{(x^*,\infty)}(x)$ at time $t/2$ is clearly in $L^2(\mathbb{R}, m)$ since both f and C^1 are. Thus, the continuation value at time zero also has an eigenfunction expansion:

$$C^0(x) = \mathbb{E}[V^1(X_{t/2})|X_0 = x] = \sum_{n=0}^{\infty} e^{-\kappa n t/2} c_n^0 \varphi_n(x),$$

where the expansion coefficients satisfy

$$c_n^0 = (f\mathbf{1}_{(-\infty,x^*]}, \varphi_n) + \sum_{m=0}^{\infty} e^{-\kappa n t/2} f_m (\varphi_m \mathbf{1}_{(x^*,\infty)}, \varphi_n).$$

Thus, the algorithm reduces to the following steps: calculate the expansion coefficients of the payoff f_n, determine x^* numerically at time $t/2$, calculate the inner products $(f\mathbf{1}_{(-\infty,x^*]},\varphi_n)$ and $(\varphi_m\mathbf{1}_{(x^*,\infty)},\varphi_n)$ (by using the properties of Hermite polynomials), calculate c_n^0 and, thus, obtain the value function at time zero (in the computation, infinite sums are truncated to a desired tolerance level). This example for an OU process with two exercise dates readily generalizes to N exercise dates and a rich class of Markov processes. Furthermore, the value function converges to the value function of the continuous optimal stopping problem as N tends to infinity, while the Richardson extrapolation is employed to obtain this limit computationally when applicable.

We remark that, while the point of departure of our method is close to Tsitsiklis and Van Roy (1999) and Tsitsiklis and Van Roy (2001), who also consider value functions of discrete optimal stopping problems as elements of a Hilbert space, our method is entirely different. Tsitsiklis and Van Roy (1999) and Tsitsiklis and Van Roy (2001) (also Longstaff and Schwartz (2001)) propose a least-squares Monte Carlo method (generalized by the stochastic mesh method of Broadie and Glasserman (2004)). In this method, the continuation value is approximated by a weighted sum of basis functions, and the weights are determined numerically from a regression. While in the least-squares Monte Carlo the basis functions are generically chosen basis functions in the Hilbert space of square-integrable payoffs, in our eigenfunction expansion method the *basis functions are exact eigenfunctions of the transition semigroup* of the underlying Markov process. Choosing eigenfunctions as basis functions *diagonalizes the expectation operator*, reducing it to an operator of multiplication with an eigenvalue on each of the eigen-subspaces. While the least-squares Monte Carlo is a general-purpose numerical method not requiring any knowledge of eigenfunctions of the expectation operator, when the eigenfunctions *are* known explicitly (such as in examples given in this paper), choosing the eigenfunctions as the basis functions leads to an explicit recursion for the expansion coefficients of the value function in the eigenfunction basis. The advantage of the least-squares Monte Carlo is in treating problems with no explicit knowledge of eigenfunctions of the expectation operator and, in particular, in multi-dimensional problems. In contrast, in problems where the eigenfunctions are explicitly known, our eigenfunction expansion method provides an analytical alternative, where the expansion coefficients satisfy an explicit recursion, and no simulation is required.

The paper is organized as follows. Section 2 defines the class of Markov processes to which our method applies and gives examples important for applications. Section 3 formulates our eigenfunction expansion method for solving discrete optimal stopping problems and proves the key theorem that establishes an eigenfunction expansion of the value function. Section 4 proves that, under some regularity assumptions, the sequence of value functions of discrete optimal stopping problems converges in the limit to the value function of the continuous problem. It also proposes to use Richardson extrapolation to approximate the continuous value function. To illustrate computational performance of our method, Section 5 develops applications to American-style commodity futures options and Bermudan-style abandonment and capacity expansion options in commodity extraction projects under the subordinate Ornstein-Uhlenbeck model with mean-reverting jumps recently proposed by Li and Linetsky (2012) with the value function given by the expansion in Hermite polynomials. Section 6 presents detailed numerical examples. Appendix A contains some definitions from the theory of Markov processes. All proofs are given in Appendix B.

2 The Markovian Set-Up

We first make precise the class of Markov processes we work with, and then present examples important for applications. Let $E \subseteq \mathbb{R}$ be a Borel subset of the real line. To handle killing, we adjoin to E an isolated point ∂ (the *cemetery state*) and and let $E_\partial := E \cup \{\partial\}$. The Borel σ-algebras on E and E_∂ will be denoted by $\mathfrak{B}(E)$ and $\mathfrak{B}(E_\partial)$, respectively. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(X_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F})$ with state space $(E_\partial, \mathfrak{B}(E_\partial))$, and let $(\mathbb{P}_x)_{x\in E_\partial}$ be a family of probability measures on $(\Omega, \mathcal{F})$ parameterized by the initial state x, i.e. such that $\mathbb{P}_x(X_0 = x) = 1$.

Our first assumption is that X is a *Hunt process*. Roughly speaking, Hunt processes are strong Markov processes with sample paths that are right-continuous with left limits and have an additional property of quasi-left-continuity. Hunt processes are a natural class of Markov processes to study optimal stopping in continuous time due to sufficient regularity to work with stopping times (cf. Mordecki and Salminen (2007)). The precise definition of Hunt processes is given in Appendix A.

Our second assumption is that the Hunt process X is $\mathfrak{m}$-*symmetric*. Recall that the *transition function* of a Markov process is defined by $P_t(x, B) := \mathbb{P}_x(X_t \in B)$ for $t \geq 0$, $x \in E$ and $B \in \mathfrak{B}(E)$. It is not conservative in general because $P_t(x, E) = 1 - P_t(x, \{\partial\}) \leq 1$, where $P_t(x, \{\partial\}) = \mathbb{P}_x(X_t = \partial)$ is the probability for the process to end up in the cemetery state by time t (the *killing probability*). For a measurable real-valued function f on E we write $\mathcal{P}_t f(x) := \int_E f(y) P_x(t, dy)$ whenever the integral makes sense. We can also write $\mathcal{P}_t f(x) = \mathbb{E}_x[\mathbf{1}_{\{t<\zeta\}} f(X_t)]$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ and $\mathbf{1}_{\{t<\zeta\}}$ denotes the indicator function equal to one if the process survives to time t and equal to zero if the process is killed prior to or at time t (ζ is the first passage time of X into the cemetery state ∂). If we extend every function on E to E_∂ by setting $f(\partial) := 0$, then $\mathbf{1}_{\{t<\zeta\}}$ inside the expectation can be omitted.

The *transition operators* $(\mathcal{P}_t)_{t\geq 0}$ defined on the Banach space of bounded Borel measurable functions $\mathcal{B}_b(E)$ with the supremum norm form a *strongly continuous contraction semigroup* on $\mathcal{B}_b(E)$. That is, $\lim_{t\to 0} \|\mathcal{P}_t f - f\| = 0$ for all $f \in \mathcal{B}_b(E)$ (strong continuity), $\mathcal{P}_t\mathcal{P}_s = \mathcal{P}_s\mathcal{P}_t = \mathcal{P}_{t+s}$ for all $s, t \geq 0$ (semigroup property), and $\|\mathcal{P}_t f\| \leq \|f\|$ for all $f \in \mathcal{B}_b(E)$ and $t \geq 0$ (contraction property).

Suppose $\mathfrak{m}$ is a positive Radon measure on $(E, \mathfrak{B}(E))$ with full support. The transition function is called $\mathfrak{m}$-*symmetric* if for all non-negative measurable functions f and g and for all $t \geq 0$

$$\int_E \mathcal{P}_t f(x) g(x) \mathfrak{m}(dx) = \int_E f(x) \mathcal{P}_t g(x) \mathfrak{m}(dx). \tag{2.1}$$

A Hunt process with an $\mathfrak{m}$-symmetric transition function is called an $\mathfrak{m}$-*symmetric Hunt process*. The standard references on symmetric Hunt processes are Fukushima et al. (2011) and Chen and Fukushima (2011). Here we follow the exposition in Appendix A.2 and Chapter 12 of Schilling et al. (2010). The transition operators $(\mathcal{P}_t)_{t\geq 0}$ of an $\mathfrak{m}$-symmetric Hunt process restricted to $\mathcal{B}_b(E) \cap L^2(E, \mathfrak{m})$ can be extended uniquely to a *strongly continuous semigroup of symmetric contractions* on the Hilbert space $L^2(E, \mathfrak{m})$ of Borel measurable functions on E square-integrable with $\mathfrak{m}$ and endowed with the inner product $(f, g) = \int_E f(x) g(x) \mathfrak{m}(dx)$ and norm $\|f\|_2 = \sqrt{(f, f)}$. That is, $(\mathcal{P}_t)_{t\geq 0}$ are bounded linear operators on $L^2(E, \mathfrak{m})$ with the following properties: $L^2 - \lim_{t\to 0} \mathcal{P}_t f = f$ (strong continuity), $\mathcal{P}_t\mathcal{P}_s = \mathcal{P}_s\mathcal{P}_t = \mathcal{P}_{t+s}$ for all $s, t \geq 0$ (semigroup property), $\|\mathcal{P}_t f\|_2 \leq \|f\|_2$ for all $f \in L^2(E, \mathfrak{m})$ and $t \geq 0$ (contraction property), and $(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)$ for all $f, g \in L^2(E, \mathfrak{m})$ (symmetry property). We use the same notation $\mathcal{B}_b(E) \cap L^2(E, \mathfrak{m})$ for the semigroup acting on $\mathcal{B}_b(E)$ and on $L^2(E, \mathfrak{m})$.

The *infinitesimal generator* $\mathcal{G}$ of the semigroup $(\mathcal{P}_t)_{t\geq 0}$ on $L^2(E, \mathfrak{m})$ is defined by $\mathcal{G}f := \lim_{t\to 0} \frac{\mathcal{P}_t f - f}{t}$ and is a (generally unbounded) linear operator with the domain $D(\mathcal{G}) = \{f \in L^2(E, \mathfrak{m}) : \mathcal{G}f \text{ exists as a strong limit}\}$. It is a negative semi-definite self-adjoint operator on $L^2(E, \mathfrak{m})$ (i.e., $(f, \mathcal{G}g) = (\mathcal{G}f, g)$ for all $f, g \in D(\mathcal{G})$ (self-adjointness property) and $(f, \mathcal{G}f) \leq 0$ for all $f \in D(\mathcal{G})$ (negative semi-definite property). By the spectral theorem for self-adjoint operators in Hilbert space, the operators $\mathcal{G}$ and $\mathcal{P}_t$ have a spectral decomposition with real spectrum in $(-\infty, 0]$ for $\mathcal{G}$ and in $[0,1]$ for $\mathcal{P}_t$.

Remark 2.1. *Reversible Markov Processes.* For a *conservative* $\mathfrak{m}$-symmetric Markov process X, i.e. $\mathbb{P}_x(\zeta < \infty) = 0$ for all $x \in E$, with the finite symmetry measure $\mathfrak{m}$, i.e. $\mathfrak{m}(E) < \infty$, taking $f(x) = 1_B(x)$ for $B \in \mathfrak{B}(E)$ and $g(x) = 1$ in (2.1), we have $\int_E P_t(x, B)\mathfrak{m}(dx) = \mathfrak{m}(B)$. This implies that X is stationary with stationary distribution $\pi(dx) = \mathfrak{m}(dx)/\mathfrak{m}(E)$. Furthermore, it is shown in Dobrushin et al. (1988) that in this case (2.1) is equivalent to X being reversible. Thus, reversible Markov processes form a subclass of symmetric Markov processes.

In some applications the discount rate is a function of the underlying state variable. To accommodate state-dependent discounting, we consider *Feynman-Kac (FK) operators* (*pricing operators* that include state-dependent discounting of the future payoff, as well as taking the expectation):

$$\mathcal{P}_t^r f(x) := \mathbb{E}_x \left[\exp\left(- \int_0^t r(X_u) du \right) f(X_t) 1_{\{\zeta > t\}} \right], \tag{2.2}$$

where $r(x)$ is a non-negative Borel measurable function on E. These operators define the FK semigroup $(\mathcal{P}_t^r)_{t \geq 0}$ on $L^2(E, \mathfrak{m})$ (*pricing semigroup*). It is also a strongly continuous semigroup of symmetric contractions on $L^2(E, \mathfrak{m})$ (strong continuity follows as a special case of Theorem 1 in Chen (2005), contraction property follows from non-negativity of r). By the spectral theorem, the operators $\mathcal{G}^r$ (the infinitesimal generator of the FK semigroup) and $\mathcal{P}_t^r$ have spectral decompositions with real spectrum (lying in $(-\infty, 0]$ for $\mathcal{G}^r$ and in $[0,1]$ for $\mathcal{P}_t^r$). In general spectral properties of the transition semigroup $\mathcal{P}$ of X and the FK semigroup $\mathcal{P}^r$ with discounting can be quite different and depend on the properties of the discount rate $r(x)$. In the special case when r is constant, we simply have $\mathcal{P}_t^r = e^{-rt}\mathcal{P}_t$.

Remark 2.2. *Discounting as Killing.* The FK semigroup can be turned into the transition semigroup of another process $\hat{X}$ as follows. Let e be a unit-mean exponential random variable independent of X. Define $\hat{\zeta} := \inf\{t \in [0, \zeta] : \int_0^t r(X_u)du \geq e\}$ (by convention, $\hat{\zeta} = \zeta$ if $\int_0^t r(X_u)du < e$ for $t < \xi$). Define a new process $\hat{X}$ by $\hat{X}_t := X_t$ for $t < \hat{\zeta}$ and $\hat{X}_t := \partial$ for $t \geq \hat{\zeta}$. The process $\hat{X}$ is a *subprocess* of X with sample paths of X up to $\hat{\zeta}$ and with the lifetime $\hat{\zeta} \leq \zeta$ when the process $\hat{X}$ is sent to the cemetery state ∂ where it remains for all $t \geq \hat{\zeta}$. It is called the process obtained from X by killing with respect to the positive continuous additive functional (PCAF) $A_t = \int_0^t r(X_u)du$. If X is an $\mathfrak{m}$-symmetric Hunt process, so is $\hat{X}$. Moreover, $(\mathcal{P}_t^r)_{t \geq 0}$ is the transition semigroup of $\hat{X}$. See Chen and Fukushima (2011) pp. 477-8 for the general treatment of killing with respect to a PCAF and Lemma A.3.12 and Theorem A.3.13 for proofs.

Our next assumption is that the Feynman-Kac semigroup is *trace-class*, i.e. for each $t > 0$ the FK operator $\mathcal{P}_t^r$ is trace class. Recall that for a positive semi-definite operator A on a separable Hilbert space $\mathcal{H}$, the trace of A is defined by $\mathrm{tr}A = \sum_{n=1}^\infty (\varphi_n, A\varphi_n) \in [0, \infty]$, where φ_n is an orthonormal basis in $\mathcal{H}$. The trace is independent of the orthonormal basis chosen (e.g. Reed and Simon (1980), p.206). A positive semi-definite operator is called *trace class* if and only if its trace is finite. Our $\mathcal{P}_t^r$ are positive semi-definite. The trace class condition implies that the spectra of $\mathcal{G}^r$ and $\mathcal{P}_t^r$ are purely discrete, and in the eigenfunction basis, the trace class condition reads $\mathrm{tr}\mathcal{P}_t^r = \sum_{n=1}^\infty e^{-\lambda_n t} < \infty$ for all $t > 0$. It implies some properties useful both in the theoretical development and in the computational implementation of the eigenfunction expansion method. It is not strictly necessary for the future development, but will significantly simplify it. It is also satisfied in many applications. Under the trace class assumption, we have the following spectral decomposition of the FK semigroup that follows from Lemma 7.2.1 of Davies (2007).

Proposition 2.1. *Under the trace class assumption, the FK semigroup* $(\mathcal{P}_t^r)_{t\geq 0}$ *on* $L^2(E,\mathfrak{m})$ *and its infinitesimal generator* $\mathcal{G}^r$ *have purely discrete spectra with eigenvalues* $(e^{-\lambda_n t})_{n\in\mathbb{N}_1}$ *(for* $t>0$*) and* $(-\lambda_n)_{n\in\mathbb{N}_1}$*, respectively, and*

$$\mathrm{tr}\mathcal{P}_t^r=\sum_{n=1}^{\infty}e^{-\lambda_n t}<\infty \tag{2.3}$$

for all $t>0$*. Here* $0\leq\lambda_1\leq\lambda_2\leq\ldots$ *are arranged in increasing order and repeated according to multiplicity.* $\mathcal{P}_t^r f(x)$ *has an eigenfunction expansion of the form:*

$$\mathcal{P}_t^r f(x)=\sum_{n=1}^{\infty}f_n e^{-\lambda_n t}\varphi_n(x),\;\; f_n=(f,\varphi_n)\ \textit{for any}\ f\in L^2(E,\mathfrak{m})\ \textit{and all}\ t\geq 0, \tag{2.4}$$

where φ_n *is the eigenfunction corresponding to* λ_n*, i.e.*

$$\mathcal{P}_t^r\varphi_n=e^{-\lambda_n t}\varphi_n \quad \textit{and} \quad \mathcal{G}^r\varphi_n=-\lambda_n\varphi_n, \tag{2.5}$$

the eigenfunctions $(\varphi_n)_{n\in\mathbb{N}_1}$ *form a complete orthonormal basis in* $L^2(E,\mathfrak{m})$*, and* f_n *is the* n*-th expansion coefficient in this basis. Each* $\mathcal{P}_t^r$ *with* $t>0$ *admits a symmetric integral kernel* $p_t(x,y)\in L^2(E\times E,\mathfrak{m}\times\mathfrak{m})$ *with respect to* $\mathfrak{m}$ *(i.e.* $p_t(x,y)=p_t(y,x)$*,* $\mathcal{P}_t^r f(x)=\int_E p_t(x,y)f(y)\mathfrak{m}(dy)$ *for* $f\in L^2(E,\mathfrak{m})$*, and* $\int_{E\times E}p_t^2(x,y)\mathfrak{m}(dx)\mathfrak{m}(dy)<\infty$*), which has the following bilinear expansion:*

$$p_t(x,y)=\sum_{n=1}^{\infty}e^{-\lambda_n t}\varphi_n(x)\varphi_n(y). \tag{2.6}$$

We note that, by Eq.(2.5), the eigenfunctions diagonalize the pricing (FK) semigroup and its generator. The convergence of the eigenfunction expansions in (2.4) and (2.6) in general takes place under the $L^2(E,\mathfrak{m})$ and $L^2(E\times E,\mathfrak{m}\times\mathfrak{m})$ norms, respectively. However, in many cases of interest stronger convergence results are available. In particular, we have the following result useful in applications (which follows from Theorem 7.2.5 of Davies (2007)).

Proposition 2.2. *Suppose the FK semigroup is trace class and, in addition, that for each* $t>0$ *the kernel* $p_t(x,y)$ *is jointly continuous in* x *and* y*. Then*

(1) Each eigenfunction φ_n *is continuous, and satisfies* $|\varphi_n(x)|\leq e^{\lambda_n t/2}\sqrt{p_t(x,x)}$ *for all* n*,* x *and* $t>0$*.*

(2) For any $f\in L^2(E,\mathfrak{m})$*, the expansion* (2.4) *converges uniformly in* x *on compacts, and* $\mathcal{P}_t^r f(x)$ *is continuous in* x*.*

(3) The bilinear expansion (2.6) *converges uniformly on compacts.*

This result gives a sufficient condition for uniform convergence of the eigenfunction expansion that is easy to check in applications. In finance applications we are usually interested in evaluating the value function of a derivative security or of an optimal stopping problem for some range of values of the underlying state variable. The uniform convergence of the eigenfunction expansion allows us to approximate the value function by the eigenfunction expansion truncated to a finite sum with a uniform bound on the truncation error in the domain of interest. Under Proposition 2.2, an estimate of the uniform bound for the truncation error of approximating $\mathcal{P}_t^r f(x)$ by the first N terms in a compact domain D is

$$\max_{x\in D}\sqrt{p_t(x,x)}\|f\|_2\sum_{n=N+1}^{\infty}e^{-\lambda_n t/2},$$

where we used the Cauchy-Schwartz bound $|f_n| \le \|f\|_2$ on the coefficients. Tighter bounds are also often available in specific applications, where sharper bounds on the eigenfunctions are available.

For future convenience, we now summarize our assumptions.

Assumption 1. *In this paper we assume that: (1) X is a Hunt process taking values in $E_\partial = E \cup \{\partial\}$, where $E \subseteq \mathbb{R}$ is a Borel subset of the real line and ∂ is the cemetery state. (2) It is symmetric with respect to a non-negative Radon measure $\mathfrak{m}$ on E with full support. (3) The discount rate $r(x)$ is a non-negative Borel measurable function on E. (4) The FK semigroup is trace class, i.e. for each $t > 0$ the FK operator $\mathcal{P}_t^r$ is trace class (condition (2.3) is satisfied), and, in addition, for each $t > 0$ it possesses a jointly continuous in x and y integral kernel $p_t(x, y)$ with respect to $\mathfrak{m}$.*

Examples of $\mathfrak{m}$-symmetric Hunt processes taking values in a Borel subset of the real line include one-dimensional diffusions (where E is an interval), jump-diffusions and pure jump processes obtained from one-dimensional diffusions by the procedure of Bochner's subordination (time change with a Lévy subordinator), BD processes (where E is a discrete set of points), and CTMCs obtained by subordination of BD processes. We now briefly survey these examples.

Example 2.1. One-Dimensional Diffusions. Consider a time-homogeneous, regular diffusion process X on an interval $E \subseteq \mathbb{R}$ with left and right-end points l and r satisfying $-\infty \le l < r \le \infty$, with volatility $\sigma(x)$, drift $\mu(x)$ and killing rate $k(x)$. For simplicity assume that $\mu(x)$, $\sigma(x)$ and $k(x)$ are continuous and $\sigma(x) > 0$, $k(x) \ge 0$ on (l, r). The infinitesimal generator $\mathcal{G}$ of the transition semigroup $(\mathcal{P}_t)_{t\ge 0}$ acts on functions $f \in C_c^2(E)$ (twice continuously differentiable functions on E with compact support) by the second-order differential operator (Sturm-Liouville operator) in the formally self-adjoint form:

$$\mathcal{G}f(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - k(x)f(x) = \frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{s(x)}\right)' - k(x)f(x),$$

where $s(x) = \exp\left(-\int_{x_0}^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$ (x_0 is an arbitrary point in (l, r)) and $\mathfrak{m}(x) = \frac{2}{\sigma^2(x)s(x)}$ are the *scale* and *speed densities*, respectively. The continuity assumptions on the coefficients are not necessary, and the theory of one-dimensional diffusions can be formulated in much greater generality than presented here (cf. Borodin and Salminen (2002) and Schilling et al. (2010) Chapter 14). We make these assumptions to simplify exposition in view of the fact that they are often satisfied in financial applications.

Feller's classification of boundaries can be formulated in terms of the behavior of μ, σ and k near boundaries l and r (see Borodin and Salminen (2002) for details on 1D diffusions). If any of the boundaries are regular, we specify it either as a killing boundary by sending the process to the cemetery point ∂ or as an instantaneously reflecting boundary. Under these assumptions, X is an $\mathfrak{m}$-symmetric Hunt process with respect to the *speed measure* $\mathfrak{m}(dx) = \mathfrak{m}(x)dx$. For the FK semigroup $(\mathcal{P}_t^r)_{t\ge 0}$ with discounting, the killing rate $k(x)$ in the expression for the infinitesimal generator is replaced with the sum of the killing rate and the discount rate $k(x) + r(x)$.

The general spectral representation for one-dimensional diffusions has been obtained by McKean (1956). He has also proved that the symmetric transition kernel for 1D diffusions with respect to the speed measure is infinitely differentiable and the expansion (2.6) converges uniformly on compacts. Sufficient conditions for the purely discrete spectrum in terms of the behavior of μ, σ and k (or $k + r$ when discounting is considered) near the boundaries can be found in Linetsky (2004), (2007). Many diffusions important in applications satisfy these conditions, including a Brownian motion on a finite interval with killing and/or reflection at the boundaries, OU, CIR, CEV with nonzero drift, and Jacobi diffusions. Linetsky (2004), (2007) give a survey of diffusions with known analytical solutions for their spectral representations, as well as discuss applications in finance. For OU, CIR and CEV, and Jacobi diffusions, the eigenfunctions are expressed in terms of Hermite, generalized Laguerre, and Jacobi polynomials, respectively, and eigenvalues λ_n grow linearly in n in all these cases (and, thus, satisfy the trace class condition (2.4)). In fact, among all the families of orthogonal polynomials, only Hermite, generalized Laguerre, and Jacobi polynomials can serve as eigenfunctions in the expansion in (2.4) for one-dimensional diffusion semigroups (see Mazet (1997) for the proof and Schoutens (2000) for a survey of related topics). Explicit expressions for the eigenvalues and eigenfunctions of the OU, CIR and Jacobi transition semigroups in terms of Hermite, Laguerre and Jacobi polynomials can also be found in these references. In section 5 we present the results for the OU semigroup needed for applications to commodities and real options. Further explicit expressions for the eigenvalues and eigenfunctions of FK semigroups associated with the OU, CIR and CEV processes can be found in Davydov and Linetsky (2003), Gorovoi and Linetsky (2004), Mendoza et al. (2010), Mendoza and Linetsky (2010), Mendoza and Linetsky (2012a), and Mendoza and Linetsky (2012b). Expressions for a FK semigroup associated with the Jacobi diffusion can be found in Delbaen and Shirakawa (2002).

Example 2.2. Birth and Death Processes. For a BD process X, $E = \{0, 1, 2, \cdots\}$. The birth and death rates at state i are denoted by b_i and d_i respectively. If $d_0 > 0$, then the process can be killed and is sent to the cemetery state ∂. If $d_0 = 0$, then 0 is a reflecting state. Define $\pi_0 = 1$, $\pi_i = (b_0 b_1 \cdots b_{i-1})/(d_1 d_2 \cdots d_i)$, $i \geq 1$, and let $P_{ij}(t) = P(X_t = j|X_0 = i)$. Then it can be shown that $\pi_i P_{ij}(t) = \pi_j P_{ji}(t)$ for all $t > 0$. Thus, BD processes are symmetric with respect to the measure π on the discrete set E. BD processes have a wide range of applications, including queueing, biology, demography, etc. For applications in finance see Kou and Kou (2003). Eigenfunction expansion for the transition semigroup of a B-D process was obtained by Karlin and McGregor (1957). Explicit expressions for eigenfunction expansions for the following BD processes are known: $M/M/\infty$ queue in terms of Charlier polynomials, the linear BD process in terms of Meixner polynomials, the quadratic model in terms of Hahn polynomials, and the Ehrenfest model in terms of Krawtchouk polynomials (see Chapter 3 in Schoutens (2000) for explicit expressions).

Example 2.3. Subordinate symmetric Markov processes. Let $(\mathcal{T}_t)_{t\geq 0}$ be a *Lévy subordinator*, i.e. a Lévy process on $[0, \infty)$ starting from the origin and with non-negative drift and positive jumps. The Laplace transform of a subordinator is given by the Lévy-Khintchine formula:

$$\mathbb{E}[e^{-\lambda \mathcal{T}_t}] = e^{-\phi(\lambda)t}, \ \phi(\lambda) = \gamma\lambda + \int_{[0,\infty)} (1 - e^{-\lambda s})\nu(ds), \tag{2.7}$$

where $\phi(\lambda)$ is called the *Laplace exponent* of the subordinator, $\gamma \geq 0$ is the drift and ν is the Lévy measure satisfying the integrability condition $\int_{[0,\infty)} (s \wedge 1)\nu(ds) < \infty$. Standard references on subordinators include Bertoin (1996), Sato (1999) and Schilling et al. (2010).

Subordinators are non-decreasing processes that can be used as random time changes. In particular, let X be a Markov process and define a time changed process $X_t^\phi = X_{\mathcal{T}_t}$ by running X on the new clock $\mathcal{T}_t$, where $\mathcal{T}$ is assumed to be independent of X. This procedure goes back to Bochner (1949) and is called *Bochner's subordination.* The process X^ϕ is a *subordinate process of X with respect to the subordinator $\mathcal{T}$*. The superscript ϕ signifies that the subordination is with respect to the subordinator with Laplace exponent ϕ. An excellent exposition of subordination can be found in Chapter 12 of Schilling et al. (2010). The key result is that if X is a Markov process, X^ϕ is again Markov (cf. Schilling et al. (2010) p.141). The infinitesimal generator $\mathcal{G}^\phi$ of the transition semigroup $(\mathcal{P}_t^\phi)_{t\geq 0}$ of the subordinate process X^ϕ is given explicitly by the Phillips theorem (Phillips (1952), Sato (1999) Theorem 32.1, Schilling et al. (2010) Chapter 12). In the special case when X is a one-dimensional diffusion under assumptions in Example

2.1, the subordinate diffusion X^ϕ has the infinitesimal generator explicitly given as an integro-differential operator when acting on $C_c^2(E)$ functions:

$$\mathcal{G}^\phi f(x) = \frac{\gamma}{2}\sigma^2(x)f''(x) + \mu^\phi(x)f'(x) - k^\phi(x)f(x)$$
$$+ \int_E \left(f(x+y) - f(x) - \mathbf{1}_{\{|y|\le 1\}} y f'(x)\right)\pi(x,y)\mathfrak{m}(dy)$$

with

$$\mu^\phi(x) = \gamma\mu(x) + \int_{(0,\infty)} \left(\int_{\{|y|\le 1\}} y p_s(x, x+y)\mathfrak{m}(dy)\right)\nu(ds),$$

the killing rate given by

$$k^\phi(x) = \gamma k(x) + \int_{(0,\infty)} P_s(x, \{\partial\})\nu(ds),$$

where $P_s(x,\{\partial\}) = 1 - \int_E p_s(x,y)\mathfrak{m}(dy)$ is the killing probability (the probability of X ending up in the cemetery state by time t if started at x at time zero), and the state-dependent symmetric Lévy density

$$\pi(x,y) = \pi(y,x) = \int_{(0,\infty)} p_s(x, x+y)\nu(ds)$$

with respect to the speed measure $\mathfrak{m}(dy)$ of the diffusion X, where $p_s(x,y) = p_s(y,x)$ is the continuous symmetric transition density of X with respect to the speed measure $\mathfrak{m}$ (such a density always exists for one-dimensional diffusions due to the result proved by McKean (1956)). By examining the infinitesimal generator $\mathcal{G}^\phi$ we see that the subordinate process X^ϕ is either a jump-diffusion ($\gamma > 0$) or a pure jump process ($\gamma = 0$). When the diffusion X is not a Brownian motion with drift on $\mathbb{R}$, the Lévy density $\pi(x,y)$ is state dependent in the sense that it depends *both* on the jump size y, as well as on the pre-jump state x. Subordinate diffusions are thus better candidates than Lévy processes which have state-homogeneous jumps for modeling phenomena where jumps depend on the state, such as mean-reverting jumps (see Li and Linetsky (2012) and Mendoza and Linetsky (2012b) for details).

If X is an $\mathfrak{m}$-symmetric Hunt process with the transition semigroup $(\mathcal{P}_t)_{t\ge 0}$ on $L^2(E,\mathfrak{m})$, then X^ϕ is an $\mathfrak{m}$-symmetric Hunt process with the transition semigroup $(\mathcal{P}_t^\phi)_{t\ge 0}$ on $L^2(E,\mathfrak{m})$ with the spectral representation of $\mathcal{P}_t^\phi$ determined in terms of the spectral representation of $\mathcal{P}_t$ and the Laplace exponent ϕ of the subordinator (subordination of symmetric Markov processes is studied in Albeverio and Rüdiger (2003), Albeverio and Rüdiger (2005), Okura (2002), Chen and Song (2005) and Chen and Song (2006) (see also Chapters 12 and 13 in Schilling et al. (2010)). In particular, when the spectrum of $\mathcal{P}_t$ is purely discrete with the eigenfunction expansion of the form (2.4), $\mathcal{P}_t^\phi$ also has the eigenfunction expansion of the form (2.4) with the same eigenfunctions φ_n of $\mathcal{P}_t$ and with the eigenvalues $e^{-\lambda_n^\phi t}$ with $\lambda_n^\phi = \phi(\lambda_n)$, where $\phi(\lambda)$ is the Laplace exponent. This remarkable fact makes the subordinate process X^ϕ as analytically tractable as the original process X. The only modification is the replacement of the eigenvalues λ_n of the negative of the infinitesimal generator $\mathcal{G}$ of the original process X with the the eigenvalues $\lambda_n^\phi = \phi(\lambda_n)$ of the negative of the infinitesimal generator of the subordinate process X^ϕ. It is remarkable in view of the fact that sample path behavior of the subordinate process X^ϕ can drastically differ from the behavior of the original process X. In particular, subordinate diffusions are pure jump or jump-diffusion processes, while subordinate BD processes are CTMCs that can transit into any state, rather than just the nearest neighbors.

From the applied point of view, starting from a diffusion (or a BD process) with the known eigenfunction expansion (2.4), one can construct rich families of processes with state-dependent jumps (or CTCM) by time changing with a subordinator with the known Laplace exponent and enjoy immediate analytical tractability with the eigenfunction expansion in the same eigenfunctions by simply replacing λ_n with $\phi(\lambda_n)$. This idea has been applied in Albanese and Kuznetsov (2004) for equity modeling, in Mendoza and Linetsky (2012b) for credit modeling, in Mendoza et al. (2010) and Mendoza and Linetsky (2012a) for unified credit-equity modeling, in Boyarchenko and Levendorskiĭ (2007a) and Lim et al. (2012) for interest rate modeling, and in Li and Linetsky (2012) for commodity modeling.

3 Discrete Time Optimal Stopping Problems

Consider a discrete optimal stopping problem for a Hunt process X with a finite horizon $T > 0$ where the stopping is allowed at discrete times $0 = t_0 < t_1 < \cdots < t_N = T$. Without loss of generality and to simplify notation we assume that the time points are equally spaced with the interval h, i.e. $t_i = ih$. If the stopping occurs at time t_i, the payoff $f(X_{t_i}, t_i)$ is received. We assume that $f^i(x) \equiv f(x, t_i)$ are real-valued Borel measurable functions on E_∂. If the process is killed prior to or at T, i.e. $\zeta \le T$, then the game is automatically terminated with zero payoff at the next time t_i such that $t_{i-1} < \zeta \le t_i$. That is, we assume that $f^i(\partial) = 0$ for all $i = 0, 1, ..., N$. To simplify exposition and notation, in this paper we restrict our attention to zero payoff in the cemetery state. It is possible to extend the formulation to include a non-zero payoff in the cemetery state (such as a rebate payment in the context of barrier options or a recovery payment in the event of default in the credit risk context).

Let $\mathcal{T}_h$ be the collection of all stopping times with respect to the minimal completed admissible filtration $(\mathcal{F}_t)_{t\ge 0}$ that take values in $\{t_0, t_1, \cdots, t_N\}$. Let ζ' be the discretely observed lifetime over $[0, T]$, i.e. $\zeta' = t_{i+1}$ if $t_i < \zeta \le t_{i+1}$ for $i = 0, 1, \cdots, N-1$, and $\zeta' = \zeta$ if $\zeta = 0$ or $\zeta > T$. It is easy to see that ζ' is a stopping time. Define $\mathcal{T}'_h := \{\tau \wedge \zeta' : \tau \in \mathcal{T}_h\}$. We wish to find the value function

$$V(x) = \sup_{\tau \in \mathcal{T}'_h} \mathbb{E}_x \left[\exp\left(-\int_0^\tau r(X_u)du \right) f(X_\tau, \tau) \right], \quad x \in E. \tag{3.1}$$

We have the following dynamic programming formulation (cf. Shiryaev (1978), Section 2.2 or Peskir and Shiryaev (2006), Theorem 1.7).

Theorem 3.1. (Backward Induction) *Suppose for each $k = 0, 1, \cdots, N-1$,*

$$\mathbb{E}_x \left[\sup_{0 \le i \le N-k} \left| \exp\left(-\int_0^{ih} r(X_u)du \right) f^{k+i}(X_{ih}) \right| \right] < \infty \tag{3.2}$$

for all $x \in E$. Let $\mathcal{P}^r_h$ be the FK operator in (2.2) with $t = h$. Define two sequences of functions $(V^i(x))_{0\le i\le N}$ and $(C^i(x))_{0\le i\le N-1}$ on E_∂ recursively:

(1) $V^N(x) := f^N(x)$, $x \in E_\partial$.

(2) For $i = N-1, \cdots, 0$, $C^i(x) := \mathcal{P}^r_h V^{i+1}(x)$ for $x \in E$ and $C^i(\partial) := 0$ and $V^i(x) := \max\left(f^i(x), C^i(x)\right)$ for $x \in E_\partial$.

Then $V(x) = V^0(x)$ and $\tau^* = \inf\{0 \le t_i \le T : V^i(X_{t_i}) = f^i(X_{t_i})\}$ is an optimal stopping time in $\mathcal{T}'_h$.

The functions $C^i(x)$ and $V^i(x)$ are interpreted as the continuation value function and the value function of the optimal stopping problem at time t_i (maximum of the payoff and the continuation value), respectively. We extend the definition of C^i to the cemetery state by setting $C^i(\partial) := 0$ by convention. Then $V^i(\partial) = 0$ due to the assumed zero payoff in the cemetery state. We also define for all $i = 0, 1, \cdots, N-1$ the following Borel subsets of E:

$$\mathcal{S}^i := \{x \in E : f^i(x) \geq C^i(x)\} = \{x \in E : f^i(x) = V^i(x)\}, \tag{3.3}$$

$$\mathcal{C}^i := \{x \in E : f^i(x) < C^i(x)\} = \{x \in E : f^i(x) < V^i(x)\}, \tag{3.4}$$

so that $\mathcal{C}^i \cup \mathcal{S}^i = E$ for all $i = 0, 1, ..., N-1$. Then the $\mathcal{C}^i$ and $\mathcal{S}^i$ are the continuation and stopping regions in E at time t_i, respectively. It is possible that $\mathcal{C}^i$ or $\mathcal{S}^i$ is an empty set for some i.

We are now ready to formulate our main result for discrete time optimal stopping problems.

Theorem 3.2. (Backward Induction in $L^2(E, \mathfrak{m})$) *Suppose Assumption 1 is in force, so that Propositions 2.1 and 2.2 hold. Suppose further that $f^i \in L^2(E, \mathfrak{m})$ for every $i = 0, 1, \cdots, N$. For every Borel subset $A \subseteq E$ define*

$$\pi_{m,n}(A) := (1_A \varphi_m, \varphi_n), \;\; m, n = 1, 2, \cdots,$$

where φ_n are the eigenfunctions of the FK semigroup $(\mathcal{P}^r_t)_{t \geq 0}$ in $L^2(E, \mathfrak{m})$ and $1_A(x)$ is the indicator function of the set A. For every $f \in L^2(E, \mathfrak{m})$ and every Borel set $A \subseteq E$ define

$$f_n(A) := (1_A f, \varphi_n) = \sum_{m=1}^{\infty} f_m \pi_{m,n}(A) = f_n - \sum_{m=1}^{\infty} f_m \pi_{m,n}(E \backslash A), \;\; n = 1, 2, \cdots. \tag{3.5}$$

Then the following results hold:
(i) $C^i \in L^2(E, \mathfrak{m})$ for all $i = 0, 1, \cdots, N-1$ and $V^i \in L^2(E, \mathfrak{m})$ for all $i = 0, 1, \cdots, N$.
(ii) C^i have the $L^2(E, \mathfrak{m})$ eigenfunction expansions for all $i = 0, 1, \cdots, N-1$:

$$C^i(x) = \sum_{n=1}^{\infty} c^i_n e^{-\lambda_n h} \varphi_n(x) \tag{3.6}$$

with the expansion coefficients satisfying the following recursion:

$$c^{N-1}_n = f^N_n, \quad c^i_n = f^{i+1}_n\left(\mathcal{S}^{i+1}\right) + \sum_{m=1}^{\infty} c^{i+1}_m e^{-\lambda_m h} \pi_{m,n}\left(\mathcal{C}^{i+1}\right) \text{ for } i = N-2, \cdots, 0. \tag{3.7}$$

(iii) For $i = 0, 1, \cdots, N-1$, the expansion in (3.6) *converges in x uniformly on compacts, and C^i is a continuous function.*

Theorem 3.2 reduces the backward induction for the value function to the backward induction for its $L^2(E, \mathfrak{m})$ coefficients in the complete orthonormal basis of eigenfunctions of the FK semigroup $(\mathcal{P}^r)_{t \geq 0}$. It starts with the coefficients $c^{N-1}_n = f^N_n$ of the continuation value function at time t_{N-1} equal to the coefficients of the payoff at time t_N. The next step is to determine the stopping region (3.3) at time t_{N-1} by comparing the payoff $f^{N-1}(x)$ and the continuation value function $C^{N-1}(x)$ given by the expansion (3.6) that converges uniformly on compacts. Given $\mathcal{C}^{N-1}$, the coefficients c^{N-2}_n are then determined by (3.7), and the recursion is continued until time zero is reached. The value function $V(x)$ is then computed via $V(x) = V^0(x) = \max(f^0(x), C^0(x))$. Under the assumption in Proposition 2.2, the infinite eigenfunction expansion can be truncated and the truncation error can be estimated uniformly on compacts.

We now discuss computational implementation of the recursion in Theorem 3.2 in the case when $E \subseteq \mathbb{R}$ is an interval on the real line with the end-points $-\infty \le l$ and $r \le \infty$ with the symmetry measure m absolutely continuous with respect to the Lebesgue measure, as in the case of diffusions in Example 2.1 or subordinate diffusions in Example 2.3. Computing the (truncated) sums in the recursion given by Eq.(3.7) requires computing the quantities $\pi_{m,n}(A)$ and $f_n^i(A)$. In many financial applications the stopping region (and hence the continuation region) is one-sided, i.e. one continues when the process is above or below some threshold level and stops otherwise. In this case, A is an interval. However, problems where A is a union of disjoint intervals also appear in applications (cf. Guo and Shepp (2001) and Dayanik and Karatzas (2003)). It is one of the strengths of our approach that we can handle the multiple-sided case as easily as the one-sided case by the linearity of integrals. To illustrate, suppose $A = \bigcup_{j=1}^J B_j$ where B_j are disjoint intervals. Then we have

$$\pi_{m,n}(A) = \sum_{j=1}^J \pi_{m,n}(B_j), \quad f_n^i(A) = \sum_{j=1}^J f_n^i(B_j).$$

Therefore it is sufficient to do the calculation for the case where $A = (a, b)$. We simply write $f_n^i(a, b)$ for $f_n^i\left((a, b)\right)$ and $\pi_{m,n}(a, b)$ for $\pi_{m,n}((a, b))$ due to the absolute continuity of $\mathfrak{m}$. By the linearity of integrals, $\pi_{m,n}(a, b) = \pi_{m,n}(l, b) - \pi_{m,n}(l, a)$, thus we only need to consider $\pi_{m,n}(l, x)$ for $x \in (l, r)$. Alternatively we can calculate $\pi_{m,n}(x, r)$, since $\pi_{m,n}(l, x) = \delta_{m,n} - \pi_{m,n}(x, r)$ ($\delta_{m,n}$ is the Kronecker delta) due to the orthonormality of eigenfunctions. When the eigenfunctions are known in closed form, the integral $\int_l^x \varphi_m(y)\varphi_n(y)m(y)dy$ can often be calculated in closed form as well. Furthermore, for eigenfunctions expressed in terms of orthogonal polynomials (as is the case for OU, CIR, CEV, and Jacobi diffusions and the corresponding processes with jumps obtained by subordination), using the backward shift property of orthogonal polynomials, integration by parts, and the forward shift property, one can obtain computationally efficient recursive algorithms for evaluating $\pi_{m,n}(l, x)$. For a detailed treatment of orthogonal polynomials see Koekoek et al. (2010). In section 5 we consider applications to commodity options and real options that involve Hermite polynomials.

The coefficients $f_n^i(a, b)$ can also often be explicitly computed in applications either by first evaluating the expansion coefficients f_n^i of the payoffs $f^i(x)$ in the eigenfunction basis $\varphi_n(x)$ and then computing $f_n^i(a, b)$ via the expansion as in (3.5), or by directly calculating the integral $\int_a^b f^i(x)\varphi_n(x)\mathfrak{m}(x)dx$ in closed form. When no closed form solutions are available for the integrals in $\pi_{m,n}(l, x)$ and $f_n^i(a, b)$, they can be computed via numerical integration.

The previous discussion focuses on computing the quantities $\pi_{m,n}(A)$ and $f_n^i(A)$ given A. At the step i of the recursion in (3.7) the knowledge of the continuation region $\mathcal{C}^{i+1}$ is required. It can be computed by solving the equation

$$f^{i+1}(x) - C^{i+1}(x) = 0 \tag{3.8}$$

for the boundary between the continuation and the stopping regions. By Proposition 2.1, the continuous function $C^{i+1}(x)$ is approximated by the truncated eigenfunction expansion with the coefficients computed at the previous step, with the truncation error uniformly controlled on compacts. In applications, the structure of the continuation and stopping regions can often be determined from the structure of the payoff function, with the precise location of the roots of (3.8) determined by a numerical root finding algorithm, such as the bisection method. In particular the root is unique for problems with one-sided continuation regions.

The discussion above is specific to the implementation when E is an interval on the real line. When E is a discrete set, such as for BD processes and CTMC obtained by their subordination, the implementation is generally similar, with the sums replacing the integrals in expressions for $\pi_{m,n}(A)$ and $f_n^i(A)$. The decomposition of E into the continuation and stopping regions is accomplished by considering the function $f^{i+1}(x) - C^{i+1}(x)$ for x taking values in the discrete set E.

4 Continuous Time Optimal Stopping Problems

We now allow the decision maker to stop at any time in the interval $[0, T]$. The game is automatically terminated when the process is killed. We again assume that the payoff in the cemetery state is zero, $f(\partial, t) = 0$ for all $t \in [0, T]$. Let $\mathcal{T}$ be the collection of all $(\mathcal{F}_t)_{t\geq 0}$-stopping times taking values in $[0, T]$, and define $\mathcal{T}' := \{\tau \wedge \zeta : \tau \in \mathcal{T}\}$. We are interested in determining the value function of the continuous time optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}'} \mathbb{E}_x \left[\exp\left(-\int_0^\tau r(X_u)du \right) f(X_\tau, \tau) \right]. \tag{4.1}$$

Here we consider convergence of the sequence of value functions of the discrete time optimal stopping problems to the continuous time optimal stopping value function as the number of exercise opportunities goes to infinity. Consider a sequence of sets $\{\mathcal{D}_N\}_{N\geq 1}$, where $\mathcal{D}_N = \{t_0^N, t_1^N, \cdots, t_N^N\} \subset [0, T]$, $t_N^N = T$, $t_{n_1}^N < t_{n_2}^N$ for $n_1 < n_2$. Define $h_N := \max_{0\leq n\leq N-1}(t_{n+1}^N - t_n^N)$, and suppose $\lim_{N\to\infty} h_N = 0$. Let V_N be the value function of the discrete time optimal stopping problem with stopping allowed at times in the set $\mathcal{D}_N$. We have the following results.

Theorem 4.1. *Suppose X is a Hunt process, $r(x)$ is a non-negative Borel-measurable function on E, the payoff $f(x, t)$ is a continuous function on $E \times [0, T]$, and*

$$\mathbb{E}_x \left[\sup_{t\in[0,T]} \left| \exp\left(-\int_0^t r(X_u)du \right) f(X_t, t) \right| \right] < \infty \tag{4.2}$$

for all $x \in E$. Assume there exists an optimal stopping time τ^* in $\mathcal{T}'$ for the continuous optimal stopping problem, so that

$$V(x) = \mathbb{E}_x \left[\exp\left(-\int_0^{\tau^*} r(X_u)du \right) f(X_{\tau^*}, \tau^*) \right] \text{ for all } x \in E.$$

Then

$$\lim_{N\to\infty} V_N(x) = V(x) \text{ for all } x \in E_\partial.$$

Theorem 4.1 suggests an approximation for the value function $V(x)$ of the continuous time optimal stopping problem with the value function $V_N(x)$ of the corresponding discrete time optimal stopping problem for sufficiently large N. For payoffs $f(\cdot, t) \in L^2(E, \mathfrak{m})$ for every $t \in [0, T]$, we can then compute $V_N(x)$ by Theorem 3.2. To *a priori* understand how many discrete stopping points N suffice to produce an acceptable approximation, some information on the rate of convergence of $V_N(x)$ to $V(x)$ as $N \to \infty$ is helpful. Dupuis and Wang (2005) proved the linear convergence rate for infinite horizon optimal stopping problems for one-dimensional diffusions under a restrictive set of sufficient conditions both on the payoff and on the drift and volatility functions of the diffusion:

$$V(x) = V_N(x) + \frac{c(x)}{N} + o\left(\frac{1}{N}\right). \tag{4.3}$$

To the best of our knowledge, Dupuis and Wang (2005) is the best result so far in the literature, with their proof already highly non-trivial even under their set of restrictive sufficient conditions.

The method of the present paper is for finite, rather than infinite, horizon problems and for a much wider class of Markov processes, as well as a wider class of payoff functions. Nevertheless, we have conducted extensive numerical experiments and verified computationally that linear convergence holds in many of the examples we have considered. In particular, it holds for finite horizon problems for OU and SubOU processes in section 5. When linear convergence is verified, Richardson extrapolation (RE) can be applied to accelerate convergence as follows (see Sidi (2003) for a detailed account of RE and the general procedure when $1/N$ is replaced by $(1/N)^k$ in (4.5) with some $k > 0$). To simplify presentation, set $t_i^N = ih$, where $h := T/N$, so the discrete stopping dates in $\mathcal{D}^N$ are equally spaced. Let $\{V_N\}_{N\geq 1}$ be the sequence of the value functions of the corresponding discrete time optimal stopping problems. Here we consider the simplest version of RE. Consider a sequence $\{V_N^{RE}\}_{N\geq 1}$ constructed according to:

$$V_N^{RE}(x) = (N+1)V_{N+1}(x) - NV_N(x). \tag{4.4}$$

It is easy to see from (4.3) that

$$V(x) = V_N^{RE}(x) + o\left(\frac{1}{N}\right), \tag{4.5}$$

where the linear term with $1/N$ is cancelled out. Thus, the series $V_N^{RE}(x)$ constructed by the RE procedure (4.4) converges to $V(x)$ faster than the original series $V_N(x)$. In practice, convergence of the extrapolated series is often orders of magnitude faster. It may thus be sufficient to compute $V_n^{RE}(x)$ with $n \ll N$ to approximate $V(x)$ by $V_n^{RE}(x)$ to the same accuracy as the approximation of $V(x)$ by $V_N(x)$.

Remark 4.1. *Reverse Extrapolation.* Suppose we are interested in computing the value function of the discrete optimal stopping problem with large but finite N. We can also approximate the value function $V_N(x)$ with large N as follows. Suppose (4.5) holds. Take some $n \ll N$ and compute $[(N-n)(n+1)V_{n+1}(x) - n(N-n-1)V_n(x)]/N$. The error of the approximation of $V_N(x)$ with this linear combination of $V_n(x)$ and $V_{n+1}(x)$ is of the order $o(1/n)$.

5 Applications to Commodity Options and Real Options

5.1 The Commodity Model

We now give an application of our method to American-style commodity futures options and commodity extraction projects with abandonment or expansion options under the commodity model with mean-reverting jumps introduced in Li and Linetsky (2012) based on the subordinate OU process (SubOU). Under the risk-neutral measure $\mathbb{Q}$ chosen by the market, Li and Linetsky (2012) model the spot commodity price by:

$$S_t = F(0,t)e^{X_t^\phi - G(t)}, \tag{5.1}$$

where $\{F(0,t) : t \geq 0\}$ is the initial futures curve at time zero, $X_t^\phi = X_{\mathcal{T}_t}$ is an OU diffusion X_t with constant volatility $\sigma > 0$, rate of mean reversion $\kappa > 0$ and the long-run level $\theta \in \mathbb{R}$ (so that the OU drift is $\mu(x) = \kappa(\theta - x)$) time changed with an independent Lévy subordinator $\mathcal{T}_t$ with drift $\gamma \geq 0$, Lévy measure ν and the Laplace exponent $\phi(\lambda)$ given by (2.7). $G(t)$ is a deterministic function of time which satisfies $G(t) = \ln \mathbb{E}[e^{X_t^\phi}]$ in order to ensure that the model is consistent with the initial futures curve, i.e. $\mathbb{E}[S_t] = F(0,t)$ under $\mathbb{Q}$.

Li and Linetsky (2012) give a detailed treatment of SubOU processes, including their semimartingale sample path decomposition and equivalent measure transformations. SubOU processes are ergodic processes on $E = \mathbb{R}$ with the stationary Gaussian density $\mathfrak{m}(x) = \sqrt{\frac{\kappa}{\pi\sigma^2}}e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}$ and are $\mathfrak{m}$-symmetric Hunt processes with respect to this Gaussian measure.

The OU transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ on $L^2(\mathbb{R},\mathfrak{m})$ with the inner product defined by the Gaussian measure m given above has the eigenfunction expansion (2.4) with $\lambda_n = \kappa n$ and eigenfunctions expressed in terms of Hermite polynomials (c.f. Lebedev (1972)):

$$\varphi_n(x) = \left(\sqrt{2^n n!}\right)^{-1} H_n(\sqrt{\kappa}(x-\theta)/\sigma), \; n = 0, 1, \cdots . \tag{5.2}$$

The OU semigroup eigenvalues and eigenfunctions are usually indexed starting from $n = 0$ in the literature. We follow this convention in section 5 and 6. This is different from the indexing we did in the theoretical part, which starts from 1, also the convention in the literature for theoretical derivations. Since $\sum_{n=0}^{\infty} e^{-\kappa nt} < \infty$ for all $t > 0$, the OU semigroup is trace class. Its symmetric transition kernel $p_t(x, y)$ with respect to the Gaussian stationary measure $\mathfrak{m}$ is jointly continuous in x and y ($p(t, x, y) = p_t(x, y)\mathfrak{m}(y)$ is the well-known transition density of the OU diffusion). Hence, Proposition 2.2 applies and the eigenfunction expansion (2.4) converges uniformly in x on compacts for all $t > 0$ and all $f \in L^2(\mathbb{R},\mathfrak{m})$.

The SubOU semigroup $(\mathcal{P}^\phi_t)_{t\geq 0}$ on $L^2(\mathbb{R},\mathfrak{m})$ has the same eigenfunctions φ_n and $\lambda^\phi_n = \phi(\kappa n)$. When the Laplace exponent of the subordinator satisfies

$$\sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} < \infty, \tag{5.3}$$

the SubOU semigroup is trace class, its symmetric transition kernel $p^\phi_t(x, y)$ is jointly continuous in x and y, and Proposition 2.2 applies and the eigenfunction expansion converges uniformly in x for all $t > 0$ and all $f \in L^2(\mathbb{R},\mathfrak{m})$. The continuity of the SubOU kernel is verified as follows. On any compact interval I, there exists a constant C_I independent of n such that

$$|\varphi_n(x)| \leq C_I n^{-\frac{1}{4}}$$

for all $n \geq 1$ (Nikiforov and Uvarov (1988) p.54). (5.3) also implies $\sum_{n=1}^{\infty} e^{-\phi(\kappa n)t} n^{-1/2} < \infty$. Since $p^\phi_t(x, y) = \int_{(0,\infty)} p_s(x, y)\pi_t(ds)$, with π_t the transition function of the subordinator, this condition allows us to calculate the symmetric kernel as $p^\phi_t(x, y) = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)t}\varphi_n(x)\varphi_n(y)$ with the bilinear expansion convergent uniformly on compacts. The kernel $p^\phi_t(x, y)$ is then continuous due to the continuity of the eigenfunctions.

From now on we will assume that the Laplace exponent of the subordinator satisfies the trace class condition (5.3). It is clear that (5.3) is satisfied in the jump-diffusion case with non-zero subordinator drift $\gamma > 0$. If $\gamma = 0$, so that X^ϕ is a pure jump process, this condition may or may not hold, depending on the behavior of the Lévy measure. In particular, it holds for all tempered stable subordinators with Lévy measures of the form $\nu(ds) = Cs^{-1-p}e^{-\eta s}ds$, where $C > 0$, $0 < p < 1$, $\eta \geq 0$, and $\phi(\lambda) = -C\Gamma(-p)[(\lambda + \eta)^p - \eta^p]$ ($\Gamma(\cdot)$ is the Gamma function). $p = \frac{1}{2}$ corresponds to the Inverse Gaussian subordinator popular in the finance literature (cf. Barndorff-Nielsen (1998)). Eigenfunction expansion truncation error estimates for the OU and SubOU semigroups are given in Li and Linetsky (2012) Remark 2.9.

The futures price process in the spot price model (5.1) is obtained in Li and Linetsky (2012). Let $F(x, s, t)$ denote the t-maturity futures price as seen at time $s \in [0, t]$ if $X^\phi_s = x$. Then

$$F(x, s, t) = F(0, t)e^{-G(t)} \sum_{n=0}^{\infty} e^{-\phi(\kappa n)(t-s)} F_n \varphi_n(x), \quad s \in [0, t], \tag{5.4}$$

$$F_n = e^{\theta + \frac{\sigma^2}{4\kappa}} \frac{1}{\sqrt{n!}} \left(\frac{\sigma}{\sqrt{2\kappa}}\right)^n, \quad e^{G(t)} = \mathbb{E}[e^{X^\phi_t}] = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} F_n \varphi_n(x_0). \tag{5.5}$$

Then the futures price process $\{F(s,t) = F(X_s^\phi, s, t), s \in [0,t]\}$ is a martingale under $\mathbb{Q}$ starting with $F(X_0^\phi, 0, t) = F(0,t)$ (the initial futures price at time zero).

When $\mathcal{T}_t = t$ (i.e. no time change), so that $\phi(\lambda) = \lambda$ and $X_t^\phi = X_t$ is an OU diffusion, the model (5.1) reduces to the standard exponential OU model:

$$S_t = F(0,t) \exp\left\{ X_t - x_0 e^{-\kappa t} - \theta(1 - e^{-\kappa t}) - \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t}) \right\}. \tag{5.6}$$

For an OU diffusion X_t solving the SDE $dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$, applying Itô's formula we obtain the spot price SDE:

$$dS_t = \kappa(\Theta(t) - \ln S_t)S_t dt + \sigma S_t dB_t, \tag{5.7}$$

where $\Theta(t) = \frac{1}{\kappa}\left(\frac{d}{dt}\ln F(0,t) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa t})\right) + \ln F(0,t)$. This is essentially the same SDE as the Model 1 in Schwartz (1997) with the long run level $\Theta(t)$ taken to be a deterministic function of time completely determined by the initial futures curve. This is a popular model in commodity markets (e.g. Clelow and Strickland (1999) and Hull (2011)). In this case, applying the well known formula for the generating function of Hermite polynomials (e.g. Lebedev (1972) p.60), the eigenfunction expansion for the futures price (5.4) collapses to

$$F(s,t) = F(0,t) \exp\left\{ X_s e^{-\kappa(t-s)} - x_0 e^{-\kappa t} - \theta(e^{-\kappa(t-s)} - e^{-\kappa t}) - \frac{\sigma^2}{4\kappa}(e^{-2\kappa(t-s)} - e^{-2\kappa t}) \right\}. \tag{5.8}$$

Applying Itô's formula, we obtain that $dF(s,t) = \sigma e^{-\kappa(t-s)} F(s,t) dB_s$, $s \in [0,t]$.

5.2 Commodity Futures Options

While Li and Linetsky (2012) give closed-form solutions for European-style futures options, commodity futures options listed on futures exchanges are American-style and allow early exercise at any time prior to option expiration. Following our approach in sections 3 and 4, we first consider Bermudan-style options with expiration t written on t^*-maturity futures with $t^* > t$. The American-style option can then be approximated by Richardson extrapolation.

Suppose the option holder can exercise at times $0 = t_0 < t_1 < \cdots < t_N = t$. Then the call (put) option payoff is $f^i(x) = (F(x, t_i, t^*) - K)^+$ $(f^i(x) = (K - F(x, t_i, t^*))^+)$. We have the following results.

Proposition 5.1. *(i) The put and the call payoffs are in $L^2(\mathbb{R}, \mathfrak{m})$. (ii) If $r > 0$, the early exercise region is one-sided for both futures calls and puts. If $r = 0$, the early exercise region is empty, i.e. early exercise is never optimal.*

When $r = 0$, it is also easy to prove by arbitrage arguments that it is never optimal to exercise American futures options early (see also Kim (1994)). Our result provides an alternative verification. In order to implement the recursion in Theorem 3.2 in the (Sub)OU model, we need to efficiently compute $\pi_{m,n}$, f_n^i and eigenfunctions φ_n. We start by recalling that Hermite polynomials can be efficiently computed using the following classical recursion (cf. Lebedev (1972) p.61):

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x),\ n \geq 2.$$

Based on this recursion, we can drive the recursion for $\varphi_n(x)$:

$$\varphi_0(x) = 1, \varphi_1(x) = \frac{\sqrt{2\kappa}}{\sigma}(x - \theta), \varphi_n(x) = \sqrt{\frac{2}{n}}\frac{\sqrt{\kappa}}{\sigma}(x - \theta)\varphi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\varphi_{n-2}(x), \quad n \geqslant 2.$$

$\pi_{m,n}(x,\infty)$ can be calculated as in the following proposition. For $\pi_{m,n}(-\infty,x)$, we note that $\pi_{m,n}(-\infty,x) = 1-\pi_{m,n}(x,\infty)$ for $m=n$ and $\pi_{m,n}(-\infty,x) = -\pi_{m,n}(x,\infty)$ for $m \neq n$.

Proposition 5.2.

$$\pi_{0,0}(x,\infty) = \Phi\left(-\frac{\sqrt{2\kappa}(x-\theta)}{\sigma}\right), \pi_{n,n}(x,\infty) = \pi_{n-1,n-1}(x,\infty) + \frac{1}{\sqrt{2\pi n}}\varphi_{n-1}(x)\varphi_n(x)e^{-\frac{\kappa}{\sigma^2}(x-\theta)^2}, n \geq 1,$$

$$\pi_{m,n}(x,\infty) = \frac{\sqrt{n+1}\varphi_m(x)\varphi_{n+1}(x) - \sqrt{m+1}\varphi_n(x)\varphi_{m+1}(x)}{\sqrt{2\pi}(m-n)}e^{-\frac{\kappa}{\sigma^2}(x-\theta)^2}, \quad m \neq n, m \geq 0, n \geq 0.$$

where $\Phi(x)$ is the standard normal CDF.

Next we show how to compute f_n^i for the call and put payoffs. Assume $r > 0$. Define $F := F(0,t^*)$ and $\alpha := \frac{\sigma}{\sqrt{2\kappa}}$. For calls, when $r > 0$ it is optimal to exercise at time t_i if $X_{t_i}^{\phi} \geq x_i^*$, where x_i^* is the unique root of the equation $f^i(x) - C^i(x) = 0$. Hence to value calls we need to compute $f_n^i(x_i^*,\infty)$. From the results for European futures options in Li and Linetsky (2012), for the call payoff we have:

$$f_n^i(x,\infty) = Fe^{\theta+\frac{\sigma^2}{4\kappa}-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)(t^*-t_i)}\frac{\alpha^m}{\sqrt{m!}}\pi_{m,n}(y,\infty) - K\pi_{0,n}(y,\infty),$$

for $n = 0,1,\cdots$ and generic x ($y := \sqrt{\kappa}(x-\theta)/\sigma$). For puts, it is optimal to exercise at time t_i if $X_{t_i}^{\phi} \leq x_i^*$. Hence, to value the puts we need to compute $f_n^i(-\infty,x_i^*)$. The following gives the expression for generic x ($\delta_{m,n}$ is the Kronecker delta).

$$f_n^i(-\infty,x) = K(\delta_{0,n} - \pi_{0,n}(y,\infty)) - Fe^{\theta+\frac{\sigma^2}{4\kappa}-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)(t^*-t_i)}\frac{\alpha^m}{\sqrt{m!}}(\delta_{m,n} - \pi_{m,n}(y,\infty)).$$

American-style futures options permit exercise at any time during the option's life. The call and put payoffs are continuous in the futures price, which is, in turn, continuous in x. We further have the following result.

Proposition 5.3. *Let $V(x,t)$ be the American-style call or put price at time $t \leq T$ given $X_t^{\phi} = x$ (T is the expiration date). Then, under the SubOU model, $\tau^* := \inf\{t \geq 0 : V(X_t^{\phi},t) = f(X_t^{\phi},t)\}$ is an optimal stopping time.*

Thus, by Theorem 4.1, the Bermudan option value converges to the American option value.

We also note that under an additional technical condition on the Laplace exponent ϕ of the subordinator, the Bermudan option delta can be computed analytically by term-by-term differentiation of the eigenfunction expansion.

Proposition 5.4. *Suppose the Laplace exponent of the subordinator satisfies*

$$\sum_{n=1}^{\infty} e^{-\phi(\kappa(n+1))t} n^{\frac{1}{4}} < \infty \tag{5.9}$$

for all $t > 0$. Then

$$\frac{d}{dx}\mathcal{P}_t^{\phi} f(x) = \frac{\sqrt{2\kappa}}{\sigma}\sum_{n=0}^{\infty}\sqrt{n+1}f_{n+1}e^{-\phi(\kappa(n+1))t}\varphi_n(x) \tag{5.10}$$

converges in x uniformly on compacts for all $t > 0$ and $f \in L^2(\mathbb{R},\mathfrak{m})$.

The condition (5.9) on the Laplace exponent is automatically satisfied if $\gamma > 0$. If $\gamma = 0$, it is satisfied for all tempered stable subordinators with $p > 0$.

Now consider the calculation of the delta at time $s \geq 0$ for a Bermudan option that expires at time $t > s$ with the underlying futures contract maturing at time $t^* > t$. Suppose this option has N remaining exercise dates after time s (and, without loss of generality, s is not an exercise date). Let $V_N(x, s, t)$ be the value function as a function of the state $X_s^\phi = x$ of the SubOU process driving the model. Then we have $V_N(x, s, t) = \sum_{n=0}^{\infty} c_n^0 e^{-\phi(\kappa n)(t-s)/N} \varphi_n(x)$. Denote its delta *with respect to the futures price* $F(s, t)$ at time s by $\Delta_N(x, s, t)$. Assuming the Laplace exponent of the subordinator satisfies the assumption in Proposition 5.4, we can calculate the Bermudan option delta by the chain rule and the term-by-term differentiation of eigenfunction expansions:

$$\Delta_N(x, s, t) = \frac{\sqrt{2\kappa}}{\sigma} \frac{\sum_{n=0}^{\infty} \sqrt{n+1} c_{n+1}^0 e^{-\phi(\kappa(n+1))(t-s)/N} \varphi_n(x)}{F(0, t^*) e^{-G(t^*)} \sum_{n=0}^{\infty} F_n e^{-\phi(\kappa(n+1))(t^*-s)} \varphi_n(x)}, \tag{5.11}$$

where F_n is given in (5.5). The expression in the denominator is obtained by term-by-term differentiation of the eigenfunction expansion (5.4) for the futures price. It is a significant advantage of the eigenfunction expansion approach over purely numerical approaches, such as Monte Carlo, that the value function is given analytically, allowing to obtain hedges by analytical differentiation.

5.3 Real Options

Many physical investment projects involve options that can add substantial value to the project, such as options to abandon, to expand, to contract, to defer, to extend, etc. (see Dixit and Pindyck (1994), Trigeorgis (1996), Boyarchenko and Levendorskiĭ (2007b) and Hull (2011) Chapter 34 for surveys of real options).

In this section we follow the setting in Hull (2011) Section 34.5 and consider a firm that has to decide whether to invest an amount I (in millions of dollars) to extract QN million units of a commodity from a certain source at the rate of Q million units per accounting period (such as a year) for the next N periods. Assume all cash flows except the initial investment I occur at the end of each period. Let h denote the length of the time period, $T = Nh$ the project horizon, and $t_i = ih$, $i = 0, 1, ..., N$, accounting dates when the cash flows are recorded. Since the commodity source has limited reserves, a finite time horizon of T periods is considered. Let c_v and c_f denote variable costs (in dollars) per unit of the commodity extracted and fixed costs per period (in millions of dollars), respectively. Our analysis will focus on the valuation of projects with abandonment or expansion. Other types of real options can be treated in a similarly way.

The abandonment option allows the firm to sell or close down a project and mitigates the impact of very poor investment outcomes and increases the initial valuation of a project. Let K_a (in millions of dollars) denote the liquidation value of the project net of costs. The abandonment option can be valued as a Bermudan put on the residual value of the project with strike price K_a and exercise dates t_i, $i = 1, 2, ..., N-1$. The residual value of the project at time t is the present value of all cash flows generated by the project after t.

The expansion option allows the manager to make further investments in the project to increase the project's output, if market conditions are favorable. Let K_e (in millions of dollars) denote the costs incurred to create additional capacity at the time the decision to expand is made. Then the expansion option can be valued as a Bermudan call on the value of the additional capacity with strike price K_e and exercise dates t_i with $i = 0, 1, ..., N-1$. We assume the additional capacity is a fraction s of the base project. Then the value of addition capacity at time t is equal to s times the residual value of the project at time t.

We shall assume the commodity spot price S_t follows the model (5.1). The analysis in Hull (2011) Section 34.5 uses the pure diffusion exponential OU model, that is a special case of our flexible exponential SubOU jump-diffusion model. Let P_{t_i} be the residual value at time t_i, $1 \le i \le N-1$, of the project. Then

$$\begin{aligned} P_{t_i} &= \sum_{k=1}^{N-i} e^{-rkh} \left[Q(\mathbb{E}[S_{t_{i+k}} | \mathcal{F}_{t_i}] - c_v) - c_f \right] \\ &= Q \sum_{k=1}^{N-i} e^{-rkh} F(X^{\phi}_{t_i}, t_i, t_{i+k}) - (Qc_v + c_f) \sum_{k=1}^{N-i} e^{-rkh}, \end{aligned} \tag{5.12}$$

where $F(X^{\phi}_{t_i}, t_i, t_{i+k}) = \mathbb{E}[S_{t_{i+k}} | \mathcal{F}_{t_i}]$ is the t_{i+k}-maturity futures price of the commodity as seen at time t_i. Using similar arguments to section 5.2, we can show the following result.

Proposition 5.5. *(i) Both the abandonment payoff $(K_a - P_{t_i})^+$ $(i = 1, 2, \cdots, N-1)$ and the expansion payoff $(sP_{t_i} - K_e)^+$ $(i = 0, 1, \cdots, N-1)$ are in $L^2(\mathbb{R}, \mathfrak{m})$.*
(ii) For both options, the early exercise region is one-sided.

In contrast to futures options in section 5.2, early exercise of abandonment or expansion options may be optimal even when $r = 0$. Define $\alpha := \frac{\sigma}{\sqrt{2\kappa}}$, $y := \frac{\sqrt{\kappa}(x-\theta)}{\sigma}$, $K^i_a := K_a + (Qc_v + c_f) \sum_{k=1}^{N-i} e^{-rkh}$, $K^i_e := K_e + s(Qc_v + c_f) \sum_{k=1}^{N-i} e^{-rkh}$ and $b^i_m := \sum_{k=1}^{N-i} F(0, (i+k)h) e^{\theta + \frac{\sigma^2}{4\kappa} - G((i+k)h)} e^{-(r+\phi(\kappa m))kh}$, where $F(0, t)$ is the initial futures curve. For the abandonment option, the exercise region at time $t_i = ih$ is $(-\infty, x^*_i]$, and we need to compute $f^i_n(-\infty, x^*_i)$. Similar to the futures put in section 5.2, for generic x and $n = 0, 1, \cdots$ we have

$$f^i_n(-\infty, x) = K^i_a(\delta_{0,n} - \pi_{0,n}(y, \infty)) - Q \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} b^i_m (\delta_{m,n} - \pi_{m,n}(y, \infty)).$$

For the expansion option, the exercise region at time $t_i = ih$ is $[x^*_i, \infty)$, and we need to compute $f^i_n(x^*_i, \infty)$. Similar to the futures call, for generic x and $n = 0, 1, \cdots$,

$$f^i_n(x, \infty) = sQ \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} b^i_m \pi_{m,n}(y, \infty) - K^i_e \pi_{0,n}(y, \infty).$$

Now we can calculate the option value through the backward induction of Theorem 3.2. Then the initial project valuation with the option taken into account is equal to the the base project value $P_0 = \sum_{k=1}^{N} e^{-rkh} [Q(F(0, kh) - c_v) - c_f] - I$ plus the option value determined by the backward induction.

6 Numerical Illustrations

All computations in this section were performed on a laptop computer with Intel Core 2 i5-2450M CPU at 2.50GHz with 4.00GB RAM under Linux. All codes were written in C++ and compiled with g++ 4.3.6. The infinite sums in eigenfunction expansions were truncated when the given relative error tolerance e_1 was reached. The bisection algorithm was used to find the root of (3.8) with the given absolute error tolerance e_2.

Our first example considers the pure diffusion OU commodity model (5.1) with no time change and evaluates abandonment and capacity expansion options of section 5.3. We take the same model parameters as in Hull (2011) section 34.5. The duration of the project is three years and the decisions to exercise real options are made once per year at the end of each year, i.e. $N = 3$ years and $h = 1$ year in this example of Bermudan-style real options. The initial commodity price is $S_0 = 20$, the initial future curve is $F(0, 1) = 22$, $F(0, 2) = 23$, and $F(0, 3) = 24$, and the OU process parameters are $\kappa = 0.1$, $\sigma = 0.2$, $\theta = 0$, and $x_0 = 0$. When the initial futures curve is specified, θ and x_0 do not affect the dynamics of S_t in the pure diffusion case by (5.7) and are, thus, set to zero. The parameters of the real options in section 5.3 are taken as in Hull: $I = 15$, $Q = 2$, $c_v = 17$, $c_f = 6$ and $r = 10\%$. For the abandonment option, $K_a = 0$. For the expansion option, $s = 0.2$ and $K_e = 2$.

We first compute accurate benchmarks for the values of abandonment and expansion options using our eigenfunction expansion algorithm by setting $e_1 = 10^{-15}$ and $e_2 = 10^{-15}$. The CPU times for these exceedingly tight error tolerances are around 0.02 seconds. The results are displayed in Table 1.

Type	Value (in millions)	EEB at year 1	EEB at year 2
Abandonment	1.385613	17.537	18.744
Expansion	1.189545	24.544	21.847

Table 1: Value and Early Exercise Boundary (EEB) for the Abandonment and Expansion Option.

To show convergence of the algorithm as the error tolerance in truncating the infinite sums is decreased, we fix $e_2 = 10^{-6}$ and run the algorithm with $e_1 = 10^{-5}, 10^{-6}, \cdots, 10^{-12}$. Figure 1 presents the computational performance of the eigenfunction expansion algorithm for the abandonment option. Similar results are also observed for the expansion option. The figure plots the absolute pricing error (in millions of dollars) in evaluating the abandonment option value vs. the CPU time, as the error tolerance parameter e_1 is decreased from 10^{-5} to 10^{-12}. The CPU time ranges from 0.001 seconds for $e_1 = 10^{-5}$ to 0.007 seconds for $e_1 = 10^{-12}$. The absolute pricing error is around 10^{-4} at $e_1 = 10^{-5}$ and rapidly decreases to 10^{-11} at $e_1 = 10^{-12}$. Figure 1 also presents the computational performance of the trinomial tree algorithm for this problem. Hull (2011) uses an alternative approach to evaluate these real options by the trinomial tree algorithm. We implemented the trinomial tree algorithm for the OU process X with M time steps in each year (see Hull and White (1993) and Hull (2011)). The options are then evaluated by going backwards through the tree. At the exercise times, we evaluate the project's residual value for each node using (5.8) and (5.12). We also find the node where it first becomes optimal to exercise. The spot price for this node is calculated by (5.6), and this value is used as an approximation for the early exercise boundary. We ran the trinomial tree algorithm for $M = 250, 500, 1000$ time steps and then increased the number of time steps by 1000 each time until $10,000$. The computation time for the trinomial tree algorithm ranges from 0.04 seconds to 42 seconds. We observe from the plot that the eigenfunction expansion algorithm converges orders of magnitude faster due to the fact that no discretizations of the state variable and time (inside the time step h between the Bermudan-style option exercise dates) are required.

For the early exercise boundary, when the error tolerance in determining the root of the equation (3.8) is $e_2 = 10^{-6}$ in the OU state variable and the error tolerance in evaluating the expansions is $e_1 = 10^{-7}$, the eigenfunction expansion algorithm computes the values of the boundary (the critical commodity spot price) at the end of year one and at the end of year two with the accuracy of five decimal places. For $e_1 = 10^{-5}$, the result is accurate to the 3rd and 4th decimal place for the boundary at year one and year two, respectively. In contrast, for the trinomial tree, no time step M considered computes the boundaries at both year one and year two accurate to even the 2nd decimal place, as the spacing between two adjacent nodes in the tree for the OU state variable is only 1.1×10^{-3} in the value of the OU state variable even when the number of time steps is $M = 10,000$. When it is transformed into the spot commodity price, the spacing becomes even coarser. We also note that while the trinomial tree algorithm is limited to pure diffusions, the eigenfunction expansion algorithm is equally applicable to processes with state-dependent jumps obtained by subordination.

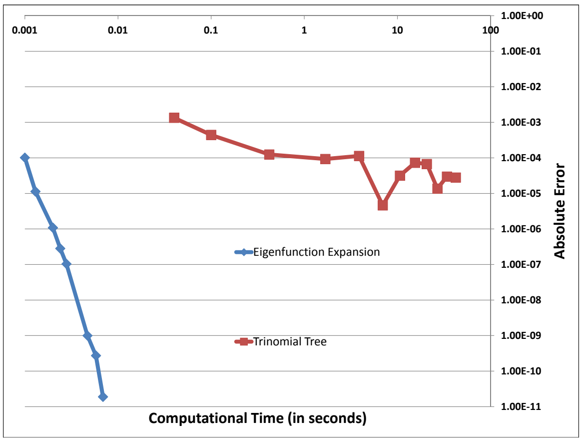

Figure 1: Eigenfunction Expansion vs. the Trinomial Tree for the Abandonment Option (on log-log scale)

We next value American-style commodity futures options by applying Richardson extrapolation to Bermudan-style futures options. We present convergence results for an in-the-money futures put option under pure diffusion, jump-diffusion and pure jump model specifications for the SubOU commodity model. Similar results hold for puts with different parameters, as well as for calls. The time change used in this example is the Inverse Gaussian (IG) subordinator $\mathcal{T}_t$ with the added drift γ and with the Lévy measure $\nu(ds) = \mu\sqrt{\frac{\mu}{2\pi v}} s^{-\frac{3}{2}} \exp\left(-\frac{\mu}{2v}s\right) ds$, where $\mu = \mathbb{E}[\mathcal{T}_1]$ and $v = \mathrm{Var}[\mathcal{T}_1]$. In the example, we price at time zero a put option with expiration $t = 1$ year and the underlying futures contract maturity $t^* = 1.04$ (which corresponds to futures maturing approximately two weeks after the option expires, a typical example for commodity futures). The initial futures price is $F(0, t^*) = 100$, the strike is $K = 120$ (the option is in-the-money), and $r = 0.05$. The OU process parameters are $\kappa = 0.2$, $\sigma = 0.4$, $\theta = -0.3$, and $x_0 = 0$. For the IG subordinator, the parameters of the Lévy measure are chosen $\mu = 1$ and $\nu = 2$. We consider two cases of the drift: $\gamma = 0.2$ for the jump-diffusion case and $\gamma = 0$ for the pure jump case. This is a representative set of parameters from our calibration experience to different commodities in Li and Linetsky (2012).

We compute the Bermudan put price for N from 2 to 10 with the increment of 1 and from 10 to 50 with the increment of 10. Figure 2 plots the put option value vs. the step size parameter $1/N$. It is clear that the convergence rate is linear in all cases. The linear regression fits have the R-square $\geq 99.9\%$ in all three cases.

To demonstrate the effectiveness of the Richardson extrapolation, we consider the pure diffusion case, where an independent benchmark can be obtained by an alternative method. We price the American-style futures options in the pure diffusion OU model (5.8) with a 15,000-step trinomial tree to provide the benchmark. In our numerical experiments we follow the approach of Broadie and Detemple (1996) and consider a sample of options with randomly sampled parameters. Our sample size is 500 options as in Broadie et al. (1999). The parameters are sampled from the following distributions: $K \sim \mathcal{U}[70, 130]$, $r \sim \mathcal{U}[0.01, 0.1]$, $\kappa \sim \mathcal{U}[0.1, 1]$, $\theta \sim$

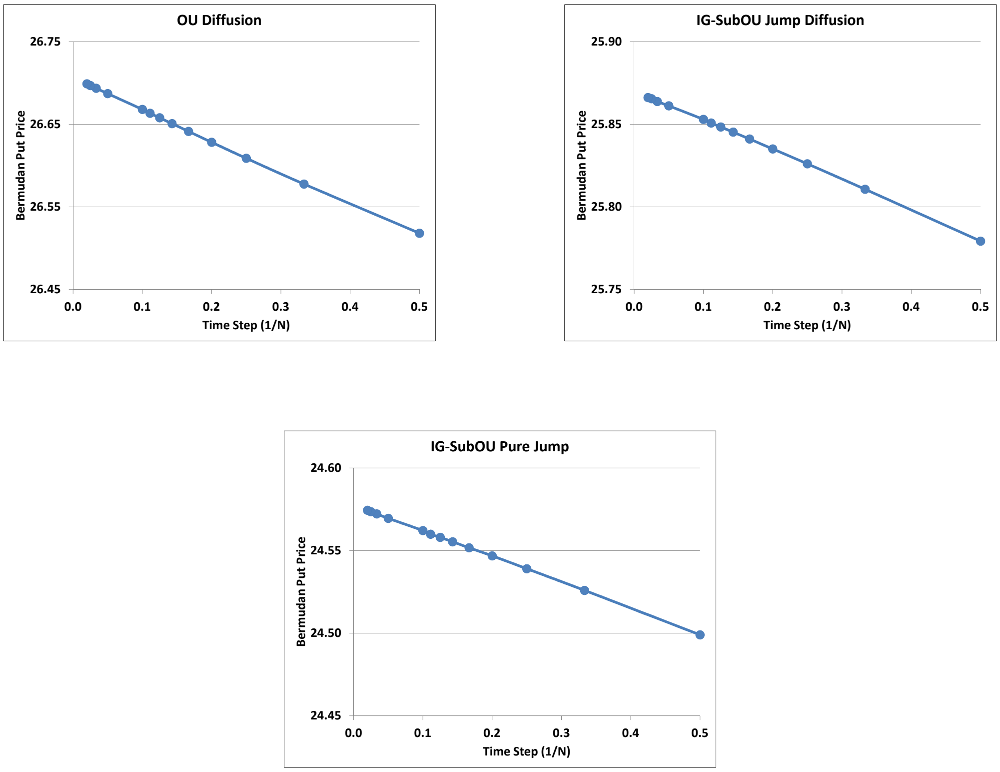

Figure 2: Bermudan put price vs. step size

$\mathcal{U}[-0.5, 0.5]$, $\sigma \sim \mathcal{U}[0.1, 0.6]$, where $\mathcal{U}[a, b]$ refers to uniform distribution on $[a, b]$. These intervals cover wide parameter ranges of practical interest. We set $t = 1$, $t^* = 1.04$, $F(0, t^*) = 100$, $x_0 = 0$ and set error tolerances to $e_1 = e_2 = 10^{-6}$. Under each set of parameters, we compute V_N^{RE} with $N = 2, 3$, which correspond to 2,3-point RE and 3,4-point RE, and then calculate their relative pricing error. We measure the overall performance by RMS (root mean squared) relative error. Following Broadie and Detemple (1996) we exclude lower priced options with prices less than 0.5 from our calculation of the relative pricing error. After applying this filter, there are 472 options in our sample. Table 2 presents the results for the option price and the average CPU time.

	2,3-RE	3,4-RE
RMS relative error	0.174%	0.097%
Time (in seconds)	0.0017	0.0039

Table 2: RMS and computation time for 2,3-point and 3,4-point Richardson extrapolation

The data suggest that the value V_3^{RE} computed from the prices of two Bermudan options with 3 and 4 early exercise opportunities already approximates American option prices with relative pricing error of less than 0.1%, which is sufficient for practical applications (see e.g. Broadie and Detemple (1996) for a discussion of desirable accuracy levels in option pricing). The CPU time is about 4 milliseconds per option at this level of accuracy. This computational speed would allow practitioners essentially real time pricing, as around 250 options can be priced in one second using (3,4)-extrapolation. Using (2,3)-extrapolation increases the pricing error to 0.174%, while cutting CPU time down to 1.7 milliseconds per option, or allowing one to price about 600 options per one second. In practice CPU time can be further reduced by using faster computers (all the experiments in this paper have been conducted on a personal laptop computer). The RE procedure can also be easily implemented via parallel computing. For (3,4)-extrapolation, one can calculate Bermudan prices with 3 and 4 exercise opportunities in parallel. The CPU time then decreases to the time needed to price one Bermudan option with 4 exercise opportunities, rather than two options — one with 4 and one with 3 opportunities.

Remark 6.1. When programming the method, care must be taken to deal with possible overflow and underflow issues. In our simulation experiment, which covers many cases of practical interests, all quantities in our computation stay in the range of double precision. However, in parameter scenarios when $\frac{\kappa}{\sigma^2}$ is extremely large (due to κ being sufficiently large and/or σ sufficiently small) and the option is very deep out of money, when computing $\pi_{m,n}(x,\infty)$, the products $\varphi_m(x)\varphi_{n+1}(x)$ and $\varphi_n(x)\varphi_{m+1}(x)$ may overflow, while the exponential $\exp\left(-\frac{\kappa}{\sigma^2}(x-\theta)^2\right)$ may underflow. A direct implementation of the formulas in Proposition 5.2 produces nan (not a number) in C++. This can be avoided by taking the logarithms as follows. To be specific, consider the expression $\varphi_m(x)\varphi_{n+1}(x)\exp\left(-\frac{\kappa}{\sigma^2}(x-\theta)^2\right)$. It can be written as

$$sgn(\varphi_m(x))sgn(\varphi_{n+1}(x))\exp\left(\ln\left(|\varphi_m(x)|\right)\ln\left(|\varphi_{n+1}(x)|\right)-\frac{\kappa}{\sigma^2}(x-\theta)^2\right),$$

where $sgn(\cdot)$ is the sign function. This avoids the overflow/underflow issues in many cases. If the computations of eigenfunctions $\varphi_n(x)$ also overflow, we use the GNU Multiple Precision Arithmetic Library (GMP). In our implementation, we start with the direct computation with double precision. If overflow or underflow occurs, we switch to the implementation with the logarithm. If the overflow or underflow still occurs, we switch to the GMP library. One can also simplify the implementation and always use the GMP library, rather than start with double precision. However this is less computationally efficient as the GMP library requires additional CPU time than double precision. For commodities, these extreme scenarios are not interesting and barely encountered in practice, so double precision seems to be sufficient for practical needs.

7 Conclusions

This paper proposes a new approach to solve finite-horizon optimal stopping problems for a rich class of Markov processes that includes one-dimensional diffusions, birth-death processes, and jump-diffusions and continuous-time Markov chains obtained by time changing diffusions and BD processes with Lévy subordinators. The method is directly applicable when the expectation operator (or the pricing operator in case of state-dependent discounting) is a trace class operator on the Hilbert space of square-integrable payoffs, ensuring purely discrete spectrum and the finiteness of the sum of all eigenvalues. This leads to the eigenfunction expansion of the value function for the optimal stopping problem with the discrete set of decision dates, with the expansion coefficients satisfying a backward recursion. The value function of the continuous optimal stopping problem is then obtained in the limit of the time step between decision points shrinking to zero, and can be efficiently computed via Richardson extrapolation. This paper illustrates the method with two applications to commodity futures options and real options in commodity extraction in the (subordinate) Ornstein-Uhlenbeck model. The method proves to be fast and accurate in these applications. In future research, we plan to apply the method to a wide range of further applications in financial engineering with optimal stopping and early exercise, as well as extend the computational implementation of the method to multi-dimensional processes and processes with continuous spectra.

A Hunt Processes

While in the main body of this paper the state space E is a Borel subset of the real line, here E is a locally compact separable metric space and $E_\partial := E \cup \{\partial\}$ its one-point compactification. If E is already compact, then ∂ is added as an isolated point. The family $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E_\partial})$ is called a *(time homogeneous) Markov process* if: (1) $x \to \mathbb{P}_x(X_t \in B)$ is Borel measurable for every $B \in \mathfrak{B}(E)$. (2) There exists a filtration $(\mathcal{G}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$ such that X is adapted to it and $\mathbb{P}_x(X_{t+s} \in B|\mathcal{G}_t) = \mathbb{P}_{X_t}(X_s \in B)$, $\mathbb{P}_x$ almost surely for every $x \in E$, $s \geq 0$ and $B \in \mathfrak{B}(E)$. (3) $\mathbb{P}_\partial(X_t = \partial) = 1$ for all $t \geq 0$. (4) $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E$. Condition (4) says that X starts at x under $\mathbb{P}_x$. Condition (3) says that the *cemetery state* ∂ is an absorbing state for X. Condition (2) is the Markov property of X with respect to $(\mathcal{G}_t)_{t\geq 0}$.

A Markov process with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ is called a *strong Markov process* if the filtration is right-continuous and $\mathbb{P}_\mu(X_{T+s} \in B|\mathcal{G}_T) = \mathbb{P}_{X_T}(X_s \in B)$, $\mathbb{P}_\mu$-almost surely for every $(\mathcal{G}_t)_{t\geq 0}$-stopping time T such that $\mathbb{P}_\mu(T < \infty) = 1$, and for all initial distributions μ on E_∂, $B \in \mathfrak{B}(E_\partial)$ and $s \geq 0$.

A Markov process X with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ is said to be *quasi left-continuous* if for any sequence $(T_n)_{n\geq 1}$ of $(\mathcal{G}_t)_{t\geq 0}$-stopping times increasing to a stopping time T it holds that $\mathbb{P}_\mu(\lim_{n\to\infty} X_{T_n} = X_T, T < \infty) = \mathbb{P}_\mu(T < \infty)$ for all initial distributions μ. Here $\mathbb{P}_\mu(\Lambda) := \int_{E_\partial} \mathbb{P}_x(\Lambda)\mu(dx)$ is the probability measure on $(\Omega, \mathcal{F})$ corresponding to starting the process with the initial distribution μ on E_∂.

A strong Markov process on $(E, \mathfrak{B}(E))$ with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ is called a *Hunt process* if it is quasi left-continuous and satisfies the following conditions: (1) $X_t(\omega) = \partial$ for every $t \geq \zeta(\omega)$, where $\zeta(\omega) = \inf\{t \geq 0 : X_t = \partial\}$ is the *lifetime* of X (the first time the process reaches the cemetery state). (2) For each $t \geq 0$, there exists a map (called the *shift operator*) $\theta_t : \Omega \to \Omega$ such that $X_t \circ \theta_s = X_{t+s}$, $s \geq 0$. (3) For each $\omega \in \Omega$, the sample path $t \to X_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$.

Let $(\mathcal{F}_t^0 := \sigma(X_s : 0 \leq s \leq t))_{t\geq 0}$ denote the filtration generated by X and let $(\mathcal{F}_t)_{t\geq 0}$ denote its completion with respect to $\mathbb{P}_\mu$ for all initial distributions μ (here $\mathbb{P}_\mu(\Lambda) := \int_{E_\partial} \mathbb{P}_x(\Lambda)\mu(dx)$ is the probability measure on $(\Omega, \mathcal{F})$ corresponding to starting the process with the initial distribution μ on E_∂). $(\mathcal{F}_t)_{t\geq 0}$ is called the *minimal completed admissible filtration.*

The process X is a Hunt process if, and only if, it is a strong Markov process and *quasi left-continuous* with respect to the minimal completed admissible filtration (cf. Theorem A.2.1 in Fukushima et al. (2011) or Theorem A.1.24 in Chen and Fukushima (2011) or Schilling et al. (2010) p.285).

B Proofs

Proposition 2.2. Part (1) and (3) come from Davies (2007) Theorem 7.2.5. Part (2) is shown by the following calculation using (2.6):

$$\begin{aligned}\mathcal{P}_t^r f(x) &= \int_E f(y)p_t(x,y)\mathfrak{m}(dy) = \int_E f(y)\mathfrak{m}(dy)\sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y) \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\int_E f(y)\varphi_n(y)\mathfrak{m}(dy) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\varphi_n(x).\end{aligned}$$

The interchange of integration and summation is justified by: that

$$\left|\int_E f(y)\mathfrak{m}(dy)\sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)\right| \leq \sum_{n=1}^{\infty} e^{-\lambda_n t}\left|\varphi_n(x)\right|\int_E \left|f(y)\varphi_n(y)\right|\mathfrak{m}(dy)$$

$$\le \sum_{n=1}^{\infty} e^{-\lambda_n t} |\varphi_n(x)| \, ||f||_2 \cdot ||\varphi_n||_2 = ||f||_2 \sum_{n=1}^{\infty} e^{-\lambda_n t} |\varphi_n(x)| \le ||f||_2 \sqrt{p_t(x,x)} \sum_{n=1}^{\infty} e^{-\lambda_n t/2} < \infty.$$

In the second step, we used the Cauchy-Schwartz inequality. In the third step, we used the bound on eigenfunctions in part (1). Finally the trace class assumption implies the bound is finite. This implies that $|\mathcal{P}_t f(x)| < \infty$ for each x and $t > 0$. Second, we can apply the Dominated Convergence Theorem with the dominating function $\sum_{n=1}^{\infty} e^{-\lambda_n t} |\varphi_n(x)| \, |f(y)\varphi_n(y)| \, \mathfrak{m}(dy)$ to justify the interchange, so pointwise convergence holds. The uniform convergence on compacts easily follows from part (1) and the trace class assumption. The continuity of $\mathcal{P}_t^r f(x)$ follows from the continuity of the eigenfunctions and the uniform convergence on compacts. □

Theorem 3.1. The assumption implies that for all $\tau \in \mathcal{T}_h'$, $\mathbb{E}_x[f(X_\tau, \tau)]$ as well as all other expectations used in the proof are well defined. Since for the time-dependent payoff the time variable can be added to the state space as an extra state variable, it is sufficient to consider time-independent payoffs in the proof. Define the operator Q by $Qg(x) = \max\{g(x), \mathcal{P}_h^r g(x)\}$ for $x \in E_\partial$. Then Q has all properties of the operator Q defined in Shiryaev (1978) Section 2.2. The rest of the proof can be carried out using Q in the same manner as Shiryaev (1978) Section 2.2 Theorem 1. □

Theorem 3.2. We first show that condition (3.2) in Theorem 3.1 is satisfied. It suffices to show that for any Borel measurable functions f_1 and f_2 such that $f^1, f^2 \in L^2(E, \mathfrak{m})$, and each $t_1, t_2 > 0$ and $x \in E$,

$$\mathbb{E}_x \left[\max\left\{ \exp\left(-\int_0^{t_1} r(X_u)du \right) |f^1(X_{t_1})|, \exp\left(-\int_0^{t_2} r(X_u)du \right) |f^2(X_{t_2})| \right\} \right] < \infty.$$

This quantity is bounded by $\mathcal{P}_{t_1}^r |f^1|(x) + \mathcal{P}_{t_2}^r |f^2|(x)$. This is a continuous function of x by Proposition 2.2 and, hence, finite for each $x \in E$.

Part (i) follows from the definition of C^i and V^i in Theorem 3.1 and the facts that $\mathcal{P}_h^r : L^2(E, \mathfrak{m}) \mapsto L^2(E, \mathfrak{m})$ and $\max(f, g) \in L^2(E, \mathfrak{m})$ for two Borel measurable functions f and g on E that satisfy $f, g \in L^2(E, \mathfrak{m})$.

To prove part (ii), from the definitions in Theorem 3.1, (3.3) and (3.4), we have for $i = 0, 1, ..., N-2$, $C^i(x) = \mathcal{P}_h^r \max(f^{i+1}, C^{i+1})(x) = \mathcal{P}_h^r(\mathbf{1}_{\mathcal{S}^{i+1}} f^{i+1} + \mathbf{1}_{\mathcal{C}^{i+1}} C^{i+1})(x)$. From (2.4), C^i has the eigenfunction expansion (3.6) with

$$\begin{aligned} c_n^i &= (\varphi_n, \mathbf{1}_{\mathcal{S}^{i+1}} f^{i+1} + \mathbf{1}_{\mathcal{C}^{i+1}} C^{i+1}) = f_n^{i+1}(\mathcal{S}^{i+1}) + (\varphi_n, \mathbf{1}_{\mathcal{C}^{i+1}} C^{i+1}) \\ &= f_n^{i+1}(\mathcal{S}^{i+1}) + (\varphi_n, \mathbf{1}_{\mathcal{C}^{i+1}} \sum_{m=1}^{\infty} c_m^{i+1} e^{-\lambda_m h} \varphi_m) = f_n^{i+1}(\mathcal{S}^{i+1}) + \sum_{m=1}^{\infty} c_m^{i+1} e^{-\lambda_m h} \pi_{m,n}\left(\mathcal{C}^{i+1}\right), \end{aligned}$$

where in the last equality we used the continuity of the inner product, i.e. if $g_n \to g$ in $L^2(E, \mathfrak{m})$, then $\lim_{n\to\infty}(g_n, h) = (g, h)$ for any $h \in L^2(E, \mathfrak{m})$. The second and third equalities in (3.5) for $f_n(A)$ can be shown similarly using the continuity of the inner product and the orthonormality of the eigenfunctions. Part (iii) follows from Proposition 2.2. □

Theorem 4.1. Consider the sequence of random times $\{\tau_N\}$ defined by $\tau_N := t_{n+1}^N$ if $t_n^N \le \tau^* < t_{n+1}^N$ for some $n \le N-1$, otherwise $\tau_N := T$. Then $\tau_N \ge \tau^*$, and because $h_N \to 0$, $\tau_N \to \tau^*$. Let us verify that τ_N is a stopping time. Define $\gamma_N(t) := \max\{t_n^N : t_n^N \le t, n = 0, 1, \cdots, N\}$, then for $t < T$, $\{\tau_N \le t\} = \{\tau_N \le \gamma_N(t)\} = \{\tau^* < \gamma_N(t)\} \in \mathcal{F}_{\gamma_N(t)} \subseteq \mathcal{F}_t$. Obviously, $\{\tau_N \le T\} = \{\tau^* \le T\} \subseteq \mathcal{F}_T$. Hence τ_N is a stopping time. Since X is a Hunt process, we have

$\lim_{N\to\infty} X_{\tau_N} = X_{\tau^*}$ due to the right continuity of sample paths. On one hand,

$$\begin{aligned}\lim_{N\to\infty} V_N(x) &\geq \lim_{N\to\infty} \mathbb{E}_x \left[\exp\left(-\int_0^{\tau_N} r(X_u)du\right) f(X_{\tau_N}, \tau_N) 1_{\{\tau_N < \zeta'\}}\right] \\ &= \mathbb{E}_x \left[\lim_{N\to\infty} \exp\left(-\int_0^{\tau_N} r(X_u)du\right) f(X_{\tau_N}, \tau_N) 1_{\{\tau_N < \zeta'\}}\right] \\ &= \mathbb{E}_x \left[\exp\left(-\int_0^{\tau^*} r(X_u)du\right) f(X_{\tau^*}, \tau^*) 1_{\{\tau^* < \zeta\}}\right] = V(x).\end{aligned}$$

In the first step, we used the definition of $V_N(x)$. In the second step, we apply the dominated convergence theorem according to condition (4.2). In the third step, we used the continuity of the payoff on $E \times [0,T]$, and $\lim_{N\to\infty} 1_{\{\tau^N < \zeta'\}} = 1_{\{\tau^* < \zeta\}}$. In the final step, we used $f(X_\zeta, \zeta) = 0$. On the other hand, it is obvious that $\lim_{N\to\infty} V_N(x) \leq V(x)$. This shows $\lim_{N\to\infty} V_N(x) = V(x)$. □

Proposition 5.1. The put payoff is bounded by K, and m is Gaussian, so it is in $L^2(E, \mathfrak{m})$. We now consider the call payoff. For any $s \in [0,t]$, note that $(F(x,s,t^*) - K)^+ \leq F(x,s,t^*)$, and $F(x,s,t^*) = F(0,t^*)e^{-G(t^*)}\mathbb{E}_x[e^{X^\phi_{t^*-s}}]$. Since $e^x \in L^2(E,\mathfrak{m})$, $\mathbb{E}_x[e^{X^\phi_{t^*-s}}] \in L^2(E,\mathfrak{m})$, and the assertion for the call payoff follows.

Now we consider the shape of the stopping region. For notational simplicity, we assume the exercise dates are equally spaced with increment h. Our proof consists of two steps.
Step 1: We first show that the SubOU model satisfies the following property:

(i) The futures price is an increasing function of the SubOU state variable.

(ii) For a Bermudan put (call), the continuation value at t_i is a decreasing (increasing) function of the futures price at t_i $(i = 0, 1, \cdots, N-1)$.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypothesis. B_t is a standard Brownian motion and T_t is a Lévy subordinator defined on this space. Let X_t^1 and X_t^2 be two OU diffusions driven by B_t with the same κ, θ and σ, $X_0^1 = x_1$, $X_0^2 = x_2$. If $x_1 \leq x_2$, then by Theorem 1.1 of Ikeda and Watanabe (1977), $\mathbb{P}[X_t^1(\omega) \leq X_t^2(\omega) \text{ for all } t \geq 0] = 1$. Define $X_t^{\phi,1} = X^1_{T_t}$, $X_t^{\phi,2} = X^2_{T_t}$. Then it is easy to see that we have the following comparison result

$$\mathbb{P}[X_t^{\phi,1}(\omega) \leq X_t^{\phi,2}(\omega) \text{ for all } t \geq 0] = 1. \tag{B.1}$$

This shows that for a SubOU process X^ϕ, the function $g(x) := E_x[f(X_t^\phi)]$ $(t > 0)$ is increasing (decreasing), if f is increasing (decreasing). Property (i) then follows from $F(x,s,t^*) = F(0,t^*)e^{-G(t^*)}\mathbb{E}_x[e^{X^\phi_{t^*-s}}]$ and that e^x is increasing. (i) implies that the futures process satisfies similar comparison result as in (B.1):

$$\mathbb{P}[F^1(\omega,s,t^*) \leq F^2(\omega,s,t^*) \text{ for all } u \leq s \leq t^*] = 1 \ \text{ if } F^1(u,t^*) \leq F^2(u,t^*), u \geq 0, \tag{B.2}$$

where $F^i(\omega,s,t^*) = F^i(X_s^{\phi,i}(\omega), s, t^*)$ with the RHS given by (5.4). From now on to simplify notation we shall write F_s for $F(s,t^*)$ and bear in mind that the maturity for the futures contract is t^*. (B.2) implies that $g(y) := \mathbb{E}[f(F_s)|F_u = y]$ $(0 \leq u < s \leq t^*)$ is increasing (decreasing) if f is increasing (decreasing).

Next we prove property (ii) for the put option. The proof for the call option is similar. Since $C^{N-1}(y) = \mathbb{E}\left[(K - F_t)^+ | F_{t_{N-1}} = y\right]$ and $(K-y)^+$ is decreasing, $C^{N-1}(y)$ is decreasing by the results above. The rest of the proof is by induction. Suppose (ii) holds for $C^{i+1}(y)$. We have $C^i(y) = \mathbb{E}\left[\max\{(K-F_{t_{i+1}})^+, C^{i+1}(F_{t_{i+1}})\}|F_{t_i}=y\right]$, and $\max\{(K-y)^+, C^{i+1}(y)\}$ is decreasing, hence $C^i(y)$ is decreasing.

Step 2: By (i) we can consider the futures price as the underlying state variable instead of the SubOU variable. The option matures at t $(t<t^*)$. The put payoff $f(y)=(K-y)^+$ and the call payoff $f(y)=(y-K)^+$. In step 2 we show the solution to $C^i(y)=f(y)$ exists and is unique for $i=N-1,\cdots,1,0$ when $r>0$. The one-sided structure of the exercise region then follows because both C^i and f are continuous functions. We only prove for the put. The proof for the call can be carried out in a similar way.

(1) At time t_{N-1}, the equation is $e^{-rh}\mathbb{E}\left[(K-F_t)^+ \mid F_{t_{N-1}}=y\right] = (K-y)^+$. It is easy to see that equivalently we can solve $e^{-rh}\mathbb{E}\left[(K-F_t)^+ \mid F_{t_{N-1}}=y\right] = K-y$. Define $A^{N-1}(y) = \mathbb{E}\big[\big(K-F_t\big)^+\big|F_{t_{N-1}}=y\big]$, and $g(y)=K-C^{N-1}(y)=K-e^{-rh}A^{N-1}(y)$. The function $A^{N-1}(\cdot)$ is the continuation value at time t_{N-1} without discounting. Then the equation becomes $g(y)=y$. Now we wish to show that $g(\cdot)$ is a contraction. Let $y_1\le y_2$ then $A^{N-1}(y_1)\ge A^{N-1}(y_2)$. Hence

$$\big|g(y_2)-g(y_1)\big| = e^{-rh}\big|A^{N-1}(y_1)-A^{N-1}(y_2)\big| = e^{-rh}\big(A^{N-1}(y_1)-A^{N-1}(y_2)\big)$$

We consider the probability setting in step 1 again. Since $y_1\le y_2$, we have $F_t^1(\omega)\le F_t^2(\omega)$. Hence

$$\begin{aligned}\big(K-F_t^1(\omega)\big)^+ - \big(K-F_t^2(\omega)\big)^+ &= \begin{cases} F_t^2(\omega)-F_t^1(\omega), & F_t^1(\omega)\le F_t^2(\omega)\le K \\ K-F_t^1(\omega), & F_t^1(\omega)\le K < F_t^2(\omega)\end{cases} \\ &\le F_t^2(\omega)-F_t^1(\omega).\end{aligned}$$

This implies that

$$\begin{aligned}A^{N-1}(y_1)-A^{N-1}(y_2) &= \mathbb{E}\big[\big(K-F_t^2(\omega)\big)^+ - \big(K-F_t^1(\omega)\big)^+|\mathcal{F}_t\big] \\ &\leqslant \mathbb{E}\big[F_t^2(\omega)-F_t^1(\omega)|\mathcal{F}_t\big] = y_2-y_1,\end{aligned}$$

where in the last equality we have used the fact that futures process is a martingale.

Therefore for $g(\cdot)$ we have

$$\big|g(y_2)-g(y_1)\big| = \big|C^{N-1}(y_1)-C^{N-1}(y_2)\big| = e^{-rh}\big(A^{N-1}(y_1)-A^{N-1}(y_2)\big) \le e^{-rh}(y_2-y_1).$$

Since $r>0$, $0<e^{-rh}<1$, therefore $g(\cdot)$ is a contraction. By the Banach Fixed Point Theorem, the solution to $g(y)=y$ exists and is unique.

(2) The rest of the proof is by induction. Note that at time t_{N-1}, we have proved that the continuation value satisfies $\big|C^{N-1}(y_2)-C^{N-1}(y_1)\big| \leqslant e^{-rh}\big|y_2-y_1\big|$. At time t_{i+1} $(i <= N-2)$, assume the equation $C^{i+1}(y)=f(y)$ has a unique solution, and $|C^{i+1}(y_2)-C^{i+1}(y_1)| \leqslant e^{-rh}\big|y_2-y_1\big|$. We wish to show that $C^i(y)=f(y)$ has a unique solution and $|C^i(y_2)-C^i(y_1)| \leqslant e^{-rh}\big|y_2-y_1\big|$. Similar to the arguments at t_{N-1}, equivalently we can solve $C^i(y)=K-y$. Define $g(y)=K-C^i(y)=K-e^{-rh}A^i(y)$, where $A^i(y)$ is the continuation value at time t_i without discounting. Suppose $y_1\le y_2$, then $A^i(y_1)\ge A^i(y_2)$ and

$$\big|g(y_2)-g(y_1)\big| = e^{-rh}\big|A^i(y_1)-A^i(y_2)\big| = e^{-rh}\big(A^i(y_1)-A^i(y_2)\big).$$

We have $F_{t_{i+1}}^1(\omega)\le F_{t_{i+1}}^2(\omega)$, and $C^{i+1}(F_{t_{i+1}}^1(\omega))\le C^{i+1}(F_{t_{i+1}}^2(\omega))$.

$$\begin{aligned}&\max\big\{K-F_{t_{i+1}}^1(\omega), C^{i+1}\big(F_{t_{i+1}}^1(\omega)\big)\big\} - \max\big\{K-F_{t_{i+1}}^2(\omega), C^{i+1}\big(F_{t_{i+1}}^2(\omega)\big)\big\} \\ &\le \max\Big\{F_{t_{i+1}}^2(\omega)-F_{t_{i+1}}^1(\omega), K-F_{t_{i+1}}^1(\omega)-C^{i+1}\big(F_{t_{i+1}}^2(\omega)\big),\end{aligned}$$

$$
\begin{aligned}
&C^{i+1}\big(F^1_{t_{i+1}}(\omega)\big) - C^{i+1}\big(F^2_{t_{i+1}}(\omega)\big), C^{i+1}\big(F^1_{t_{i+1}}(\omega)\big) - \big(K - F^2_{t_{i+1}}(\omega)\big)\Big\} \\
&\le \max\Big\{F^2_{t_{i+1}}(\omega) - F^1_{t_{i+1}}(\omega), C^{i+1}\big(F^1_{t_{i+1}}(\omega)\big) - C^{i+1}\big(F^2_{t_{i+1}}(\omega)\big)\Big\} \\
&\le \max\{F^2_{t_{i+1}}(\omega) - F^1_{t_{i+1}}(\omega), e^{-rh}(F^2_{t_{i+1}}(\omega) - F^1_{t_{i+1}}(\omega))\} \\
&\le F^2_{t_{i+1}}(\omega) - F^1_{t_{i+1}}(\omega),
\end{aligned}
$$

where the second to the last inequality is from the induction assumption. Hence

$$
\begin{aligned}
A^i(y_1) - A^i(y_2) &= \mathbb{E}\big[\max\{K - F^1_{t_{i+1}}(\omega), C^{i+1}\big(F^1_{t_{i+1}}(\omega)\big)\}|\mathcal{F}_{t_i}\big] \\
&\quad - \mathbb{E}\big[\max\{K - F^2_{t_{i+1}}(\omega), C^{i+1}\big(F^2_{t_{i+1}}(\omega)\big)\}|\mathcal{F}_{t_i}\big] \\
&\leqslant \mathbb{E}\big[F^2_{t_{i+1}}(\omega) - F^1_{t_{i+1}}(\omega)|\mathcal{F}_{t_i}\big] = y_2 - y_1.
\end{aligned}
$$

Again we have used the martingale property of the futures price process. Finally

$$
\big|g(y_2) - g(y_1)\big| = \big|C^i(y_1) - C^i(y_2)\big| = e^{-rh}\big(A^i(y_1) - A^i(y_2)\big) \le e^{-rh}\big(y_2 - y_1\big).
$$

So $g(\cdot)$ is a contraction, and the existence and uniqueness follows from the Banach Fixed Point Theorem.
Step 3: Finally we consider the case $r = 0$. From Jensen's inequality,

$$
C^i(y) = \mathbb{E}\left[\big(K - F_{t_{i+1}}\big)^+ |F_{t_i} = y\right] > \big(K - \mathbb{E}[F_{t_{i+1}}|F_{t_i} = y]\big)^+ = (K - y)^+ = f(y).
$$

The inequality is strict because $K - F_{t_{i+1}}$ is not almost surely equal to a constant. Therefore the early exercise region is empty. □

Proposition 5.2. We define $a_{m,n}(x) = \int_x^\infty H_m(u)H_n(u)e^{-u^2}du$. Since $H_0(x) = 1$, it is easy to see that $a_{0,0}(x) = \sqrt{\pi}\Phi(-\sqrt{2}x)$. For Hermite polynomials, the forward shift property reads $\frac{d}{dx}H_n(x) = 2nH_{n-1}(x) \quad (n \ge 1)$, and the backward shift property reads $\frac{d}{dx}\left[e^{-x^2}H_n(x)\right] = -e^{-x^2}H_{n+1}(x)$. We first apply the backward shift, integrate by parts and then apply the forward shift.

$$
\begin{aligned}
a_{m+1,n+1}(x) &= \int_x^\infty H_{m+1}(u)H_{n+1}(u)e^{-u^2}du = -\int_x^\infty H_{m+1}(u)d(e^{-u^2}H_n(u)) \\
&= -H_{m+1}(u)H_n(u)e^{-u^2}\,|_x^\infty + \int_x^\infty e^{-u^2}H_n(u)d(H_{m+1}(u)) \\
&= H_{m+1}(x)H_n(x)e^{-x^2} + \int_x^\infty e^{-u^2}H_n(u)2(m+1)H_m(u)du \\
&= H_{m+1}(x)H_n(x)e^{-x^2} + 2(m+1)a_{m,n}(x) \qquad \text{(B.3)}
\end{aligned}
$$

Exchanging the role of m and n, we have

$$
a_{m+1,n+1}(x) = H_{n+1}(x)H_m(x)e^{-x^2} + 2(n+1)a_{m,n}(x). \qquad \text{(B.4)}
$$

If $m \ne n$, then subtracting (B.4) from (B.3) and rearranging leads to

$$
a_{m,n}(x) = \frac{H_m(x)H_{n+1}(x) - H_n(x)H_{m+1}(x)}{2(m-n)}e^{-x^2}, \quad m \ne n, m \ge 0, n \ge 0. \qquad \text{(B.5)}
$$

Setting $m = n$ in (B.4) we obtain the recursion for $a_{n,n}(y)$, which is

$$
a_{n,n}(x) = 2na_{n-1,n-1}(x) + H_{n-1}(x)H_n(x)e^{-x^2}, n \ge 1. \qquad \text{(B.6)}
$$

By change of variable it is easy to see that

$$\pi_{m,n}(x,\infty) = \frac{1}{\sqrt{\pi 2^m m! 2^n n!}} a_{m,n}\left(\sqrt{\kappa}(x-\theta)/\sigma\right). \tag{B.7}$$

Eqs.(B.5), (B.6), (B.7) and (5.2) together give the results in Proposition 5.2. □

Proposition 5.3. We first verify the following two conditions.

(1) $\mathbb{E}_x\left[\sup_{0\le s\le t}(K-F(X_s,s,t^*))^+\right]<\infty$, and $\mathbb{E}_x\left[\sup_{0\le s\le t}(F(X_s,s,t^*)-K)^+\right]<\infty$.

(2) The continuous time value function V is lower semi-continuous.

To verify (1), observe that $\mathbb{E}_x\left[\sup_{0\le s\le t}(K-F(X_s,s,t^*))^+\right]\le K$ for the put. To verify for the call, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\sup_{0\le s\le t}(F(X_s,s,t^*)-K)^+\right] \le \mathbb{E}_x\left[\sup_{0\le s\le t}F(X_s,s,t^*)\right]\\
&\le \frac{e}{e-1}\left(1+\mathbb{E}_x\left[F(X_t,t,t^*)\left(\ln F(X_t,t,t^*)\right)^+\right]\right) \le \frac{e}{e-1}\left(1+\mathbb{E}_x\left[F^2(X_t,t,t^*)\right]\right)\\
&\le \frac{e}{e-1}\left(1+F^2(0,t)e^{-2G(t)}\mathbb{E}_x\left[\left(\mathbb{E}_x[e^{X^\phi_{t^*}}|X^\phi_t]\right)^2\right]\right) \le \frac{e}{e-1}\left(1+F^2(0,t)e^{-2G(t)}\mathbb{E}_x\left[e^{2X^\phi_{t^*}}\right]\right),
\end{aligned}$$

which is finite since $\mathbb{E}_x\left[e^{2X^\phi_{t^*}}\right]<\infty$. We used Doob's inequality for nonnegative submartingale with $p=1$ in the second step, and Jensen's inequality in the fifth step.

To verify (2), note that $C_b(E)\subset L^2(E,\mathfrak{m})$, so from Proposition 2.2, $\mathcal{P}^\phi_t$ maps $C_b(E)$ into $C_b(E)$. Hence both put and call payoff are continuous, so from Peskir and Shiryaev (2006) (2.2.80) V is lower-semi-continuous.

Conditions (1) and (2) together allow us to apply Peskir and Shiryaev (2006) Chapter I Corollary 2.9, which proves the claim in the proposition. □

Proposition 5.4. By Cauchy-Schwartz inequality, we have $|f_n|\le ||f||_{L^2}$ for all n. Since $H_n'(x)=2nH_{n-1}(x)$ $(n\ge1)$, we have $\varphi_n'(x)=\varphi_{n-1}(x)\frac{\sqrt{2\kappa n}}{\sigma}$ $(n\ge1)$. On any compact interval I, there exists a constant C such that $|\varphi_n(x)|\le C/n^{\frac14}$ for $n\ge1$. From these results we have for any $x\in I$,

$$\begin{aligned}
\left|\sum_{n=1}^\infty f_n e^{-\phi(\kappa n)t}\varphi_n'(x)\right| &= \frac{\sqrt{2\kappa}}{\sigma}\left|\sum_{n=0}^\infty \sqrt{n+1}f_{n+1}e^{-\phi(\kappa(n+1))t}\varphi_n(x)\right|\\
&\le C_1 + C_2\sum_{n=1}^\infty e^{-\phi(\kappa(n+1))t}\frac{\sqrt{n+1}}{n^{\frac14}},
\end{aligned}$$

for some constants C_1 and C_2. Hence the assumption implies $\sum_{n=1}^\infty f_n e^{-\phi(\kappa n)t}\varphi_n'(x)$ converges uniformly on any compacts. This allows us to interchange summation and differentiation, which results in (5.10). □

Proposition 5.5. The abandonment payoff is bounded by K, so it is in $L^2(\mathbb{R},\mathfrak{m})$. For the expansion payoff, $(sP_{t_i}-K_e)^+\le sP_{t_i}$. P_{t_i} is in $L^2(\mathbb{R},\mathfrak{m})$ by (5.12) and the fact that the futures price is in $L^2(\mathbb{R},\mathfrak{m})$, as shown in Proposition 5.1.

Next we consider the shape of the stopping region. For notational simplicity, we assume the exercise dates are equally spaced with increment h. We use the same probabilistic setting as in

Proposition 5.1. It is easy to see from (5.12) and property (i) in the proof of Proposition 5.1 that P_{t_i} is an increasing function of the SubOU variable at t_i. The discrete time residual value process $\{P_{t_i} : i = 0, 1, \cdots, N-1\}$ satisfies the following comparison result

$$\mathbb{P}[P^1_{t_j}(\omega) \le P^2_{t_j}(\omega) \text{ for all } i \le j \le N-1] = 1 \ \text{ if } P^1_{t_i} \le P^2_{t_i}, i \ge 0. \tag{B.8}$$

(B.8) implies that $g(z) := \mathbb{E}[f(P_s)|P_u = z]$ $(0 \le u < s \le t_{N-1})$ is increasing (decreasing) if f is increasing (decreasing). Similar to Proposition 5.1, this allows us to conclude that each C^i is a decreasing (increasing) function of the residual value at t_i for the abandonment (expansion) option.

To show the one-sided structure of the stopping region, we consider the residual value as the underlying state variable instead of the SubOU variable. The abandonment payoff $f(z) = (K_a - z)^+$ and the expansion payoff $f(z) = (sz - K_e)^+$. We show the solution to $C^i(z) = f(z)$ exists and is unique for $i = N-2, \cdots, 1, 0$. The one-side structure of the exercise region then follows because both C^i and f are continuous functions. We only prove for the abandonment option. The proof for the expansion option can be carried out in a similar way.

(1) At time t_{N-2}, the equation is

$$e^{-rh}\mathbb{E}\left[\left(K_a - P_{t_{N-1}}\right)^+ \mid P_{t_{N-2}} = z\right] = (K_a - z)^+ .$$

To show the existence of the solution, we first note that $C^{N-2}(z) > 0$ while $(K_a - z)^+ = 0$ for $z > K_a$. As $z \to 0$,

$$\begin{aligned} C^{N-2}(z) &\sim e^{-rh}\mathbb{E}\left[K_a - P_{t_{N-1}}|P_{t_{N-2}} = z\right] \\ &= e^{-rh}K_a - \left\{z - e^{-rh}(QF(t_{N-2}, t_{N-1}) - (c_f + Qc_v))\right\} \end{aligned}$$

Hence $\lim_{z\to 0} C^{N-2}(z) = e^{-rh}(K_a - c_f - Qc_v) < K_a$ since $c_f, c_v > 0$. Therefore by the intermediate value theorem for continuous functions, a solution to the equation exists.

To show the uniqueness, let $z_1 < z_2$. Then similar to the derivation in Proposition 5.1, we can show that

$$\begin{aligned} \left|C^{N-2}(z_2) - C^{N-2}(z_1)\right| &\le e^{-rh}\mathbb{E}\left[P^2_{t_{N-1}}(\omega) - P^1_{t_{N-1}}(\omega)|\mathcal{F}_{t_{N-2}}\right] \\ &= z_2 - z_1 - e^{-rh}Q(F^2(t_{N-2}, t_{N-1}) - F^1(t_{N-2}, t_{N-1})) \\ &< |z_2 - z_1|. \end{aligned}$$

For the last step, we used $F^2(t_{N-2}, t_{N-1}) > F^1(t_{N-2}, t_{N-1})$ since $z_2 > z_1$. Now suppose there are two different solutions z_1^* and z_2^* to the equation. Then they are both less than K_a. From the above

$$|(K_a - z_2^*) - (K_a - z_1^*)| = |z_2^* - z_1^*| < |z_2^* - z_1^*|,$$

which is a contradiction. This implies the solution must be unique.

(2) The rest of the proof is by induction. At time t_{i+1} $(i <= N-3)$, assume the stopping region is one-sided and $|C^{i+1}(z_2) - C^{i+1}(z_1)| < |z_2 - z_1|$ $(z_2 \ne z_1)$. We wish to show that $C^i(z) = f(z)$ has a unique solution and $|C^i(z_2) - C^i(z_1)| < |z_2 - z_1|$.

$C^i(z) = \mathbb{E}\left[\max\{(K_a - P_{t_{i+1}})^+, C^{i+1}(P_{t_{i+1}})\}|P_{t_i} = z\right]$. First, we have $C^i(z) > 0$ while $(K_a - z)^+ = 0$ for $z > K_a$. By the induction hypothesis, the stopping region at t_{i+1} is one-sided, and exercise occurs for z sufficiently small. Hence for small z

$$\begin{aligned} C^i(z) &\sim e^{-rh}\mathbb{E}\left[K_a - P_{t_{N-1}}|P_{t_{N-2}} = z\right] \\ &= e^{-rh}K_a - \left\{z - e^{-rh}(QF(t_{N-2}, t_{N-1}) - (c_f + Qc_v))\right\} \end{aligned}$$

Thus $\lim_{z\to 0} C^{N-2}(z) = e^{-rh}(K_a - c_f - Qc_v) < K_a$. The solution exists due to the intermediate value theorem for continuous functions.

Suppose $z_1 < z_2$. Similar to the proof in Proposition 5.1, we can show

$$\max\left\{K_a - P^1_{t_{i+1}}(\omega), C^{i+1}\left(P^1_{t_{i+1}}(\omega)\right)\right\} - \max\left\{K - P^2_{t_{i+1}}(\omega), C^{i+1}\left(P^2_{t_{i+1}}(\omega)\right)\right\} \leq P^2_{t_{i+1}}(\omega) - P^1_{t_{i+1}}(\omega).$$

Hence following the calculation in (1) we must have $\left|C^i(z_1) - C^i(z_2)\right| < |z_2 - z_1|$. This shows the uniqueness of the solution. □

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