OPTION PRICING IN SOME NON-LÉVY JUMP MODELS

LINGFEI LI* AND GONGQIU ZHANG†

Abstract. This paper considers pricing European options in a large class of one-dimensional Markovian jump processes known as subordinate diffusions, which are obtained by time changing a diffusion process with an independent Lévy or additive random clock. These jump processes are non-Lévy in general, and they can be viewed as natural generalization of many popular Lévy processes used in finance. Subordinate diffusions offer richer jump behavior than Lévy processes and they have found a variety of applications in financial modelling. The pricing problem for these processes presents unique challenges as existing numerical PIDE schemes fail to be efficient and the applicability of transform methods to many subordinate diffusions is unclear. We develop a novel method based on finite difference approximation of spatial derivatives and matrix eigendecomposition, and it can deal with diffusions that exhibit various types of boundary behavior. Since financial payoffs are typically not smooth, we apply a smoothing technique and use extrapolation to speed up convergence. We provide convergence and error analysis and perform various numerical experiments to show the proposed method is fast and accurate. Extension to pricing path-dependent options will be investigated in a follow-up paper.

Key words. jump processes, option pricing, time change, finite difference, matrix eigendecomposition.

AMS subject classifications. 60J60, 60J75, 91G20, 91G60.

1. Introduction.

Jump processes are an essential modelling tool in finance and popular financial models with jumps are often based on Lévy processes. There is an extensive literature on the study of Lévy processes and their applications in finance (see the monograph [6] and [13]). To price options in Lévy-driven models, numerical methods based on transforms (e.g., [11], [43], [20, 21], [22, 24], [7]) and numerical schemes for partial integro-differential equations (PIDEs) (e.g., [1], [16], [14], [23]) are developed. However, jumps of Lévy processes are independent of the state, which can be quite unrealistic in some applications.

This paper develops a fast and accurate numerical method for pricing options in models based on one-dimensional (1D) subordinate diffusions. The problem presents unique challenges as existing numerical PIDE schemes fail to be efficient and the applicability of transform methods to many subordinate diffusions is unclear. A 1D subordinate diffusion is obtained by time changing a 1D diffusion process (it will be called background diffusion hereafter) with an independent Lévy or additive random clock (i.e., a nonnegative Lévy/additive process; also called Lévy/additive subordinator in the literature). We shall add "Lévy" or "additive" before "subordinate diffusion" to indicate which type of subordinator is used if needed. Subordinate diffusions form a large class of Markovian jump processes whose jumps are generally *state-dependent*, hence they offer richer jump behavior than Lévy processes. Their jumps can exhibit a variety of interesting behavior. For example, jumps can have finite or infinite activity, and have finite or infinite variation (recent high-frequency statistical analysis favors infinite activity pure jump processes with infinite variation in some applications; see [56]). Jumps of subordinate diffusion are mean-reverting if the background diffusion is so, and this fact has been exploited by [36] for commodity modelling. Applying

*Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: lfli@se.cuhk.edu.hk.

†Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: gqzhang@se.cuhk.edu.hk.

subordination to diffusions on a bounded interval is also a natural approach to construct jump processes with bounds, which are useful for modelling financial variables that move in bounded zones (e.g., asset prices in a highly competitive market and exchange rates in a target zone). A series of papers have already established the usefulness of subordinate diffusions in financial applications, and in our opinion they can be viewed as nice additions to Lévy processes for modelling jumps. See [3], [46], [36], [5], [41] and [48] for applications of Lévy subordinate diffusions in a variety of markets and [38], [32], [33] for applications of additive subordinate diffusions. Many popular Lévy processes in finance can be represented as a Brownian motion time changed by an independent Lévy subordinator, including VG, NIG, CGMY, hyperbolic and Meixner processes ([44]). Therefore subordinate diffusions can be viewed as a natural generalization of many Lévy processes by time changing more general diffusions.

Existing applications of subordinate diffusions in finance focus on analytically tractable specifications whose transition operator admits an eigenfunction expansion with known eigenvalues and eigenfunctions. European options can be priced analytically using eigenfunction expansions in these models. This approach has been further extended in several works ([34, 35, 37, 40]) to price Bermudan, American, barrier, swing and real options in these models.

Restricting to analytically tractable specifications of subordinate diffusions for modelling limits the choice of the background diffusion, and such models may fail to capture important features. The analytical approach using eigenfunction expansions is not applicable to general subordinate diffusions, which motivates us to develop a fast and accurate numerical method that is generally applicable.

The rest of the paper is organized as follows. In Section 2, we introduce subordinate diffusions and explain why existing numerical methods are not applicable or do not solve our problem efficiently. Section 3 presents our numerical method for pricing European options and provide convergence and error analysis. Section 4 presents various numerical examples which confirm the computational efficiency and accuracy of our method, and comparison to a popular existing PIDE scheme is given. Section 5 concludes the paper and the appendix contains the proof for lemmas.

2. Subordinate Diffusions. Let X be a one-dimensional time-homogeneous diffusion living in an interval I with end-points l and r ($-\infty \leq l < r \leq \infty$), and for $f \in C_c^2((l,r))$ (twice-continuously differentiable functions on (l,r) with compact support), its infinitesimal generator $\mathcal{G}$ takes the form

$$\mathcal{G}f(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - k(x)f(x) \text{ for } f \in C_c^2((l,r)),$$

where $\mu(x), \sigma^2(x), k(x)$ are known as the drift, diffusion coefficient and killing rate, respectively. We assume that for $x \in (l,r)$, $\mu(x)$, $\sigma(x)$ and $k(x)$ are continuous, and $\sigma(x) > 0$, $k(x) \geq 0$. We also assume that X is regular, i.e., for any $x, y \in (l,r)$, X can reach y starting from x in finite time with positive probability.

Whether an end-point is included in I depends on the boundary behavior. If it is infinite, we assume it is inaccessible, i.e., starting from any point in (l,r), X cannot reach the boundary in finite time with positive probability. If it is finite, the boundary behavior can be either natural, exit, entrance, regular specified as killing or reflecting. We refer readers to e.g., [4, Chapter II.6] for Feller's classification of boundaries and conditions to determine the boundary behavior. Upon hitting the exit and killing boundary, X is sent immediately to the cemetery state Δ. Alternatively X can be killed by the additive functional $\int_0^t k(X_u)du$, where k is the killing rate, i.e., X is sent

to Δ at time $\tau_k := \inf\left\{t \geq 0 : \int_0^t k(X_u)du \geq e\right\}$, where e is an exponential random variable with unit mean and independent of X. The lifetime of X, denoted by ζ, is equal to the first time X is killed at the boundary or τ_k, whichever is smaller. Note that killing is a natural way to model bankruptcy risk (see e.g., [10]). This framework encompasses many diffusions used in finance.

A Lévy subordinator is a stochastically continuous process with independent and stationary increments that starts at zero and is non-negative (see [13, Definition 3.1]). Non-negativity implies that it is non-decreasing ([13, Proposition 3.10]). Consider a Lévy subordinator T and let $q_t(\cdot)$ be the probability distribution of T_t, which is unknown in general. However the Laplace transform of T is known and given by the Lévy-Khintchine formula ([13, Proposition 3.10])

$$E[e^{-\lambda T_t}] = \int_{[0,\infty)} e^{-\lambda s} q_t(ds) = e^{-\phi(\lambda)t},\ \phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\nu(ds),$$

Here $\gamma \geq 0$ is the drift of the subordinator and ν is the Lévy measure satisfying $\int_{(0,\infty)}(s\wedge 1)\nu(ds) < \infty$. For typical Lévy subordinators used in finance, the integral in $\phi(\lambda)$ can be further reduced to a closed-form expression. Popular choices are tempered stable Lévy subordinators, whose Lévy measures are given by $\nu(ds) = Cs^{-p-1}e^{-\eta s}$ with $C > 0$, $0 < p < 1$, $\eta > 0$. For them, $\phi(\lambda) = \gamma\lambda - C\Gamma(-p)[(\lambda+\eta)^p - \lambda^p]$ where $\Gamma(\cdot)$ is the gamma function.

We assume X and T are independent and define $X_t^\phi := X_{T_t}$. This time change technique is called Bochner's subordination and has been extensively studied in the mathematics literature (see [54]). X^ϕ is called Lévy subordinate diffusion, and the superscript ϕ indicates the Laplace exponent of T. It is a time-homogenous Markov process with lifetime given by $\zeta^\phi = \inf\{t \geq 0 : T_t \geq \zeta\}$. The infinitesimal generator of X^ϕ is given in [34, p.631], which is an integro-differential operator. In particular its jump density is state-dependent in general and is given by

$$\pi^\phi(x, y) = \int_{(0,\infty)} p(s, x, y)\nu(ds), \tag{2.1}$$

where $p(s, x, y)$ is the transition probability density of X.

[32] proposes to construct time dependent jump processes by time changing a diffusion X with an independent additive subordinator T (an additive subordinator is basically a Lévy subordinator without stationary increments). The resulting processes, called additive subordinate diffusions, generalize Lévy subordinate diffusions by having time dependence and they provide good fit to the implied volatility surface while Lévy subordinate diffusions typically only fit volatility smiles/skews of a single maturity well. We refer readers to [32] for detailed discussion. Like Lévy subordinators, the Laplace transform of additive subordinators is known analytically. In the rest of the paper, we focus on Lévy subordinate diffusions. Our method extends directly to additive subordinate diffusions and the only change is to replace the Laplace transform of the Lévy subordinator by that of the additive subordinator.

To obtain the price of an European option with payoff function f, we need to compute $e^{-rt}E_x[f(X_t^\phi)]$ under the risk-neutral measure (the risk-free interest rate r is assumed to be constant). This expectation can be decomposed into two parts:

$$\begin{aligned} E_x[f(X_t^\phi)] &= E_x[f(X_t^\phi)1_{\{\zeta^\phi > t\}}] + f(\Delta)E_x[1_{\{\zeta^\phi \leq t\}}] \\ &= E_x[f(X_t^\phi)1_{\{\zeta^\phi > t\}}] + f(\Delta) - f(\Delta)E_x[1_{\{\zeta^\phi > t\}}]. \end{aligned}$$

Therefore to price European options, we only need to compute

$$u^{\phi}(t,x) := E_x[f(X_t^{\phi})1_{\{\zeta^{\phi}>t\}}]. \tag{2.2}$$

To illustrate the inefficiency of existing numerical PIDE schemes in obtaining the European option price $u^{\phi}(t,x)$, let's for the moment, assume that $u^{\phi}(t,x)$ is a classical solution to a PIDE (in general the European option price is not a classical solution to a PIDE; sufficient conditions for it to hold can be found in e.g., [15, Proposition 2] for Lévy processes and [52, Theorem 7] for general SDEs with jumps). There already exist many numerical schemes for solving PIDEs. See, for example, [1], [14], [16] and [23]. To apply these schemes requires the value of the jump density $\pi^{\phi}(x,y)$ for different x and y on a grid. In our case, $\pi^{\phi}(x,y) = \int_{(0,\infty)} p(\tau,x,y)\nu(d\tau)$ (see (2.1)), which cannot be obtained in closed form in general. One can certainly compute $\pi^{\phi}(x,y)$ by numerical integration, which requires the value of the diffusion transition density $p(\tau,x,y)$ for different τ in a large region to obtain high accuracy (here $\nu(\cdot)$ is known). But $p(\tau,x,y)$ is in general unknown, and has to be computed by numerically solving the PDE it satisfies. If the forward PDE is used, for fixed x, we need to solve the PDE numerically over a long time horizon, and this has to be repeated for different x on the grid. Apparently, this procedure is inefficient as it requires very intensive computations. The above discussion was based on the assumption that $u^{\phi}(t,x)$ is a classical solution to a PIDE. More generally, $u^{\phi}(t,x)$ may not even be differentiable in x for typical payoffs in finance, let alone being a classical solution. An example is given by the pure jump VG process, which is obtained by time changing a BM with a driftless Gamma subordinator ([15, p.307]). In general, $u^{\phi}(t,x)$ can only be interpreted as a viscosity solution to the PIDE. See [15, Proposition 8] for results in exponential Lévy models and a convergent scheme is developed in [14] to find the viscosity solution in these models. For some existing PIDE schemes, since it is not clear whether they are convergent for finding the viscosity solution, their applicability to pricing European options for general subordinate diffusions is even called into question. As for transform methods, they can be applied to subordinate diffusions with an explicit expression for the characteristic function. This is true in e.g., many subordinate Brownian motion models. However, for many other subordinate diffusions, it is not clear how to compute the characteristic function, so the applicability of transform methods remains to be a question (see Remark 1). In this paper, we will develop an efficient method which does not require the smoothness of $u^{\phi}(t,x)$ (in particular our method applies to the VG model) and the value of the jump density $\pi^{\phi}(x,y)$.

Remark 1 (characteristic function for subordinate diffusions). *When X is a Brownian motion with drift μ and volatility σ, the characteristic function $E[e^{iuX_t^{\phi}}] = E[E[e^{iuX_{T_t}}|T_t]] = E[e^{iu\mu T_t-(1/2)u^2\sigma^2 T_t}] = e^{-\phi(1/2u^2\sigma^2-iu\mu)}$, which has a closed-form expression if ϕ does. When X is an affine diffusion, its characteristic function can be computed by solving the associated Riccati equations and if q_t, the distribution of T_t is known, one can use numerical quadrature to compute $E[e^{iuX_t^{\phi}}] = \int_{[0,\infty)} E[e^{iuX_s}]q_t(ds)$ (this idea has been used in [39] where the Gamma subordinator is used; in this case q_t is a Gamma distribution and Gauss-Laguerre quadrature can be applied to efficiently compute the integral). For these cases, one can apply various transform methods to compute option prices efficiently and the method developed in this paper can be viewed as an alternative. However, there are also many financial applications where the diffusion is non-affine (see e.g., [12], [46]) and the distribution of the subordinator is unknown (this is the case for tempered stable subordinators with $p \neq 1/2$). So for*

these cases, it is unclear how the characteristic function of X_t^ϕ can be computed and hence the applicability of transform methods is unclear.

3. European Option Pricing in Subordinate Diffusion Models. We want to compute $u^\phi(t,x)$ defined in (2.2) for a general Lévy subordinate diffusion X^ϕ. Define $u(s,x) = E_x[f(X_s)1_{\{\zeta>s\}}]$, where ζ is the lifetime of the background diffusion X. Since $X^\phi = X_{T_t}$ with X, T independent, we have

$$u^\phi(t,x) = \int_{[0,\infty)} u(s,x) q_t(ds).$$

For the class of diffusions we consider (see Section 2), $u(s,x)$ is the solution to the following PDE for a large class of payoff functions (for instance when f is continuous and bounded on (l,r); see [45]):

$$\begin{aligned} \partial_t u(t,x) &= \mu(x)\partial_x u(t,x) + \frac{1}{2}\sigma^2(x)\partial_{xx}u(t,x) - k(x)u(t,x),\ t>0, x\in(l,r), \quad (3.1)\\ u(0,x) &= f(x),\ x\in(l,r). \end{aligned}$$

We will specify boundary conditions for the PDE to correctly capture the boundary behavior in Section 3.2. The key idea of our numerical method is to develop an approximation for the value function of the diffusion PDE that is easy to generalize after subordination. To do that, we first localize the problem when infinite boundary points are present. We then consider the diffusion PDE problem on a finite interval. We approximate the spatial derivatives by finite difference which leads us to an ODE system in time, for which we find the solution using matrix eigendecomposition. For an $n\times n$ matrix, eigendecomposition typically requires $O(n^3)$ works. We show how to do it in $O(n^2)$ in our case. Since time enters the approximate solution for the diffusion PDE in an exponential form, using the analytical knowledge of the Laplace transform of the subordinator, we immediately obtain an approximation for the value function of the subordinate diffusion. The proposed method will be termed as the finite difference-eigendecomposition (FDEIG) algorithm.

3.1. Localization of infinite boundaries. We start by considering the case $l=-\infty$ and $r=\infty$ (the case where only one of them is infinite can be dealt with similarly). By our assumption, they are inaccessible. We consider a new diffusion X^A with lower boundary $-A$ and upper boundary A, where A is sufficiently large. X^A has the same drift, diffusion coefficient and killing rate as X in $(-A,A)$, and $-A$ and A are regular boundaries which can be either specified as killing or reflecting. Let $\zeta^A, \zeta^{\phi,A}$ be the lifetime of process X_t^A and $X_t^{\phi,A} := X_{T_t}^A$, respectively. Define

$$u_A(t,x) := E_x[f(X_t^A)1_{\{\zeta^A>t\}}],\ u_A^\phi(t,x) := E_x[f(X_t^{\phi,A})1_{\{\zeta^{\phi,A}>t\}}].$$

We introduce the pre-default process of X, denoted by $\hat{X}$, which has the same drift and diffusion coefficient as X but zero killing rate, hence its lifetime is infinite. We assume that $-\infty,\infty$ are inaccessible boundaries of $\hat{X}$. Similarly, $\hat{X}^A$ is the pre-default process of X^A. However, unlike $\hat{X}$, $\hat{X}^A$ can have finite lifetime because it is killed on the boundary if A and $-A$ are specified as killing. Let $M_{s,x} = \sup_{0\le u\le s}|\hat{X}_u|$. We first provide an estimate for $|u_A(s,x)-u(s,x)|$ in Lemma 3.1, based on which we show the convergence of $u_A^\phi(t,x)$.

LEMMA 3.1. *Assume that f is nonnegative and monotone on $\mathbb{R}$. Then for any $(s,x)\in(0,\infty)\times(-A,A)$,*

$$|u^A(s,x)-u(s,x)| \le E_x[(f(\hat{X}_s)+f(M_{s,x})+f(-M_{s,x}))1_{\{M_{s,x}\ge A\}}].$$

When A and $-A$ are killing boundaries, $f(M_{s,x})+f(-M_{s,x})$ on the RHS is not needed. Alternatively, assume that f is bounded on $\mathbb{R}$ and we denote its L^∞-norm by $\|f\|_\infty$. Then for any $(s,x) \in (0,\infty)\times(-A,A)$,

$$|u^A(s,x) - u(s,x)| \le 2\|f\|_\infty P_x\left(M_{s,x} \ge A\right). \tag{3.2}$$

PROPOSITION 3.2. *Assume either f is nonnegative and monotone on $\mathbb{R}$ and for any $(s,x) \in [0,\infty)\times\mathbb{R}$, $E_x[f(\hat{X}_s)]$, $E_x[f(-M_{s,x})]$, $E_x[f(M_{s,x})]$ are finite or assume that f is bounded on $\mathbb{R}$. Then for any $(t,x) \in [0,\infty)\times\mathbb{R}$, $u_A^\phi(t,x) \to u^\phi(t,x)$ when $A \to \infty$.*

Proof. We first prove the claim under the second assumption. Since $-\infty, \infty$ are inaccessible for $\hat{X}$, $P_x(M_{s,x} \ge A) \to 0$ as $A \to \infty$. From (3.2), this implies $u_A(s,x) \to u(s,x)$. From the time-change construction, we have

$$|u_A^\phi(t,x) - u^\phi(t,x)| \le \int_{[0,\infty)} |u_A(s,x) - u(s,x)| q_t(ds). \tag{3.3}$$

Since f is bounded, u and u_A are bounded. Hence by dominated convergence theorem, $u_A^\phi(t,x) \to u^\phi(t,x)$. Under the first assumption, by dominated convergence theorem,

$$E_x[f(\hat{X}_s)1_{\{M_{s,x}\ge A\}}] \to 0,\ E_x[f(M_{s,x})1_{\{M_{s,x}\ge A\}}] \to 0,\ E_x[f(-M_{s,x})1_{\{M_{s,x}\ge A\}}] \to 0$$

when $A \to \infty$. Then from Lemma 3.1 and monotone convergence theorem,

$$\begin{aligned}|u_A^\phi(t,x) - u^\phi(t,x)| &\le \int_{[0,\infty)} E_x[f(\hat{X}_s)1_{\{M_{s,x}\ge A\}}] q_t(ds) \\ &+ \int_{[0,\infty)} E_x[f(M_{s,x})1_{\{M_{s,x}\ge A\}}] q_t(ds) + \int_{[0,\infty)} E_x[f(-M_{s,x})1_{\{M_{s,x}\ge A\}}] q_t(ds)\end{aligned}$$

which goes to 0 when $A \to \infty$. □

We call $|u_A^\phi(t,x) - u^\phi(t,x)|$ as the localization error. Under further conditions on μ and σ, we show that the localization error decays exponentially based on an estimate of $P_x(M_{s,x} \ge A)$.

PROPOSITION 3.3. *Suppose $\hat{X}$ is the unique weak solution in law to the SDE $d\hat{X}_t = \mu(\hat{X}_t)dt + \sigma(\hat{X}_t)dW_t$ with $\hat{X}_0 = x$. Assume that f is bounded on $\mathbb{R}$ and that there exists $\beta > 0$ such that $\int_{(1,\infty)} e^{\beta s}\nu(ds) < \infty$. Further assume that $\mu(x)$ and $\sigma(x)$ are bounded on $\mathbb{R}$. Then for any $x \in (-A,A)$, there exist $\alpha, C > 0$ independent of x, A such that*

$$|u_A^\phi(t,x) - u^\phi(t,x)| \le Ce^{-\alpha(A-|x|)}. \tag{3.4}$$

Proof. Let $\bar{\mu} := \|\mu\|_\infty$, $\bar{\sigma} := \|\sigma\|_\infty$. From (3.2) and (3.3), we have

$$\begin{aligned}|u_A^\phi(t,x) - u^\phi(t,x)| &\le 2\|f\|_\infty \int_{[0,\infty)} P_x\left[M_{s,x} \ge A\right] q_t(ds) \\ &\le 2\|f\|_\infty e^{-\alpha A} \int_{[0,\infty)} E_x\left[\exp(\alpha M_{s,x})\right] q_t(ds).\end{aligned} \tag{3.5}$$

for $\alpha > 0$. In the second step, we used the Chebyshev inequality. The choice of α will be specified later. Below we estimate $E_x\left[\exp(\alpha M_{s,x})\right]$. Notice that for any $s \geq 0$,

$$\begin{aligned}\sup_{0\leq u\leq s}|\hat{X}_u| &= \sup_{0\leq s\leq u}\left|x+\int_0^s \mu(\hat{X}_u)du+\int_0^s \sigma(\hat{X}_u)dW_u\right| \\ &\leq |x|+\sup_{0\leq u\leq s}\left|\int_0^s \sigma(\hat{X}_u)dW_u\right|+\bar{\mu}s,\end{aligned}$$

which implies $P_x(M_{s,x}\geq z)\leq P_x\left(\sup_{0\leq u\leq s}\left|\int_0^s \sigma(\hat{X}_u)dW_u\right|\geq z-\bar{\mu}s-|x|\right)$. Hence,

$$\begin{aligned}&E_x[\exp(\alpha M_{s,x})] = \int_0^\infty P_x(\exp(\alpha M_{s,x})\geq z)\,dz \\ &\leq \int_0^\infty P_x\left(\sup_{0\leq u\leq s}\left|\int_0^s \sigma(\hat{X}_u)dW_u\right|\geq \ln z/\alpha-\bar{\mu}s-|x|\right)dz \\ &\leq \int_0^\infty \min\left(1, 2\exp\left(-\frac{(\ln z/\alpha-\bar{\mu}s-|x|)^2}{2\bar{\sigma}^2 s}\right)\right)dz.\end{aligned}$$

In the last step, we used Corollary 9.29 in [50]. Now let $z^* = \exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))$.

$$E_x[\exp(\alpha M_{s,x})]\leq z^*+\int_{z^*}^\infty 2\exp\left(-\frac{(\ln z/\alpha-\bar{\mu}s-|x|)^2}{2\bar{\sigma}^2 s}\right)dz.$$

Let $y = \ln z/\alpha-\bar{\mu}s-|x|$ and $y^* = 2\bar{\sigma}\sqrt{s\ln 2}$. Then,

$$\begin{aligned}&E_x[\exp(\alpha M_{s,x})]\leq \exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))+\int_{y^*}^\infty 2\alpha\exp\left(-\frac{y^2}{2\bar{\sigma}^2 s}\right)\exp\left(\alpha(\bar{\mu}s+|x|+y)\right)dy \\ &\leq \exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))+2\alpha\exp\left(\alpha(\bar{\mu}s+|x|)\right)\int_{-\infty}^\infty \exp\left(-\frac{y^2}{2\bar{\sigma}^2 s}+\alpha y\right)dy \\ &= \exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))+2\alpha\exp\left(\alpha(\bar{\mu}s+|x|)+\bar{\sigma}^2\alpha^2 s/2\right)\int_{-\infty}^\infty \exp\left(-\frac{(y-\alpha\sigma^2 s)^2}{2\bar{\sigma}^2 s}\right)dy \\ &= \exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))+2\alpha\exp\left(\alpha(\bar{\mu}s+|x|)+\bar{\sigma}^2\alpha^2 s/2\right)\sqrt{2\pi\bar{\sigma}^2 s}.\end{aligned}$$

Now we choose an $\alpha > 0$ such that $\bar{\mu}\alpha+\bar{\sigma}^2\alpha^2/2 < \beta$. It is easy to see that there exists a constant $C_0 > 0$ such that

$$\exp(\alpha(|x|+\bar{\mu}s+\sqrt{2\bar{\sigma}^2 s\ln 2}))+2\alpha\exp\left(\alpha(\bar{\mu}s+|x|)+\bar{\sigma}^2\alpha^2 s/2\right)\sqrt{2\pi\bar{\sigma}^2 s}\leq C_0 e^{\alpha|x|+\beta s}.$$

By [53, Theorem 25.3], the condition $\int_{(1,\infty)} e^{\beta s}\nu(ds) < \infty$ is equivalent to $E[e^{\beta T_t}] = \int_{[0,\infty)} e^{\beta s}q_t(ds) < \infty$ for all $t \geq 0$. Hence $\int_{[0,\infty)} q_t(ds)E_x[\exp(\alpha M_{s,x})]\leq C_1 e^{\alpha|x|}$ for some $C_1 > 0$. Substituting this into (3.5), we arrive at (3.4) for some $C > 0$. □

REMARK 2. *The existence of $\int_{(1,\infty)} e^{\beta s}\nu(ds) < \infty$ for some $\beta > 0$ is satisfied if $\nu(ds)$ has an exponentially decaying tail, which is true in e.g., tempered stable subordinators.*

REMARK 3. *Let $\hat{X}$ be a diffusion with linear mean-reverting drift with bounded $\sigma(x)$, i.e., $d\hat{X}_t = \kappa(\theta-\hat{X}_t)dt+\sigma(\hat{X}_t)dW_t$ ($\kappa > 0$). An important example in finance is given by the Ornstein-Uhlenbeck process for which $\sigma(x)$ is a constant. For these*

processes, $\mu(x)$ is not bounded. Hence Proposition 3.3 cannot be directly applied. However, in this case

$$\hat{X}_t = x_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \int_0^t \sigma(\hat{X}_s) e^{-\kappa(t-s)} dW_s.$$

Therefore we can proceed in a way similar to the proof of Proposition 3.3 and show that for such process, the localization error still decays exponentially as in (3.4).

In Section 4, we will see that even if the diffusion under consideration does not satisfy the bounded coefficient condition in Proposition 3.3, the localization error can still converge exponentially.

3.2. Boundary conditions for the diffusion PDE at finite boundaries. From now on, we consider a diffusion X living on a finite interval with end-point l and r (if infinite boundaries are present in the problem, we localize them and consider the localized diffusion X^A). In general, the boundary behavior can be either natural, entrance, exit, killing or reflecting. We impose the following assumption from Section 3.2 to 3.5. In Section 3.6, we will discuss how to extend our method when Assumption 1 is violated.

ASSUMPTION 1. *$\sigma(x) \in C^4([l,r])$, $\mu(x) \in C^3([l,r])$, $k(x) \in C^2([l,r])$, $\sigma(x) > 0$ for all $x \in [l,r]$.*

Here $C^n([l,r])$ is the space of functions that are n times continuously differentiable on (l,r) and the function and its derivatives up to the n-th order have continuous extensions to l and r. The smoothness assumptions on the diffusion characteristics are naturally satisfied in many financial models. From the conditions that determine the boundary behavior in [4, Chapter II.6], it is easy to see that l and r can only be killing or reflecting boundaries under Assumption 1.

We discuss what boundary conditions should be imposed for the diffusion PDE (3.1) so that the boundary behavior is correctly captured. In fact under Assumption 1, the PDE (3.1) holds on (l,r) for any $f \in L^2(I,m) := \{f \text{ is measurable} : \int_I f^2(x)m(dx) < \infty\}$, where m is the speed measure of X ([42, Eq.(3.3)]) and I is the state-space (we prove it in Proposition 3.5). Below we only specify results for the left boundary l. Results for r are entirely similar.

When l is an killing boundary, it is easy to see that

$$u(t,l) = 0 \text{ for all } t \geq 0,$$

because if the process is already at l, it is sent to the cemetery state and $1_{\{\zeta > t\}} = 0$. We next derive the boundary condition for the reflecting case which is less obvious.

For the class of diffusion process we introduce in Section 2 (not necessarily satisfying Assumption 1), its infinitesimal generator $\mathcal{G}$ can be extended uniquely to $L^2(I,m)$. If there are no natural boundaries, the spectrum of $\mathcal{G}$ is purely discrete and simple ([42, Theorem 3.2]), and for payoff function $f \in L^2(I,m)$,

$$u(t,x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n t} \varphi_n(x), \;\; f_n = \int_I f(x)\varphi_n(x)m(dx). \tag{3.6}$$

Here λ_n and φ_n are the n-th eigenvalue and eigenfunction for $\mathcal{G}$, respectively, with $\lambda_n \leq 0$ and $\int_I \varphi_n^2(x)m(dx) = 1$ (i.e., φ_n is normalized). They are solutions to the Sturm-Liouville (S-L) eigenvalue problem

$$\frac{1}{2}\sigma^2(x)\varphi''(x) + \mu(x)\varphi'(x) - k(x)\varphi(x) = \lambda\varphi(x)$$

with appropriate boundary conditions at l and r (see e.g., [42] Section 3.3):

- l is killing: $\varphi(l) = 0$.
- l is reflecting: $\lim_{x\to l} \frac{\varphi'(x)}{s(x)} = 0$, where $s(x)$ is the scale density of X.

In general, the eigenvalues and eigenfunctions cannot be found out explicitly, but one can obtain estimates for them. To derive the boundary condition for $u(t,x)$ at reflecting boundaries, we make use of (3.6) and these estimates. Under Assumption 1, the S-L problem becomes a regular one that can be converted into Liouville normal form ([25, Eq. (2.1i) to (2.9)]), and the condition $\lim_{x\to l} \varphi'(x)/s(x) = 0$ is equivalent to $\varphi'(l) = 0$ (as $\lim_{x\to l} s(x) \neq 0$). The next lemma obtains estimates for λ_n, φ_n and its derivatives based on results from regular S-L theory, which are crucial for deriving the PDE boundary condition. We will also need them later when analyzing the convergence rate of the discretization scheme.

LEMMA 3.4. *Suppose Assumption 1 hold. Consider the regular S-L problem*

$$\begin{cases} \frac{1}{2}\sigma^2(x)\varphi''(x) + \mu(x)\varphi'(x) - k(x)\varphi(x) = -\lambda\varphi(x), \ x \in [l,r], \\ \varphi(x_0) = 0, \ \textit{if } x_0 \textit{ is killing}, \ x_0 \in \{l,r\}, \\ \varphi'(x_0) = 0, \ \textit{if } x_0 \textit{ is reflecting}, \ x_0 \in \{l,r\}. \end{cases}$$

Its eigenvalues satisfy $0 \geq \lambda_1 > \lambda_2 > \cdots$, *and* $\lambda_n = O(-n^2)$. *For the corresponding normalized eigenfunctions* $(\varphi_n)_{n\geq 1}$, *there exists a constant* $C > 0$ *such that for all* $n \geq 1$, $\|\varphi_n^{(k)}\|_\infty \leq Cn^k$ *(*$k = 0,1,2,3,4$*). Here* $\varphi^{(k)}$ *is the* k*-th order derivative of* φ *for* $k \geq 1$ *and* $\varphi^{(0)}$ *is* φ *itself.*

Using Lemma 3.4, we now prove the following.

PROPOSITION 3.5. *Under Assumption 1, the PDE* (3.1) *is valid on* $[l,r]$ *for any* $f \in L^2(I,m)$. *If* l *is reflecting, for any* $t > 0$, $\partial_x u(t,l) = 0$ *and the PDE at* l *reduces to* $\partial_t u(t,l) = \frac{1}{2}\sigma^2(l)\partial_{xx}u(t,l) - k(l)u(t,l)$.

Proof. For $f \in L^2(I,m)$, $u(t,x)$ is represented by (3.6) for all $x \in [l,r]$. By Cauchy-Schwartz inequality, $|f_n| \leq \|f\|_2$ (the L^2-norm of f). From the estimates of λ_n and φ_n in Lemma 3.4, the series

$$\sum_{n=1}^{\infty} f_n e^{\lambda_n t}\varphi_n'(x), \ \sum_{n=1}^{\infty} f_n e^{\lambda_n t}\varphi_n''(x), \ \sum_{n=1}^{n} f_n\lambda_n e^{\lambda_n t}\varphi_n(x)$$

converge absolutely and uniformly in x. Hence we can interchange summation and differentiation and for $x \in [l,r]$,

$$\partial_x u(t,x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n t}\varphi_n'(x), \partial_{xx}u(t,x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n t}\varphi_n''(x), \partial_t u(t,x) = \sum_{n=1}^{n} f_n\lambda_n e^{\lambda_n t}\varphi_n(x).$$

Using the S-L equation for φ_n shows that the PDE holds on $[l,r]$. If l is reflecting, since $\varphi_n'(l) = 0$, we have $\partial_x u(t,l) = 0$ and the PDE at l becomes $\partial_t u(t,l) = \frac{1}{2}\sigma^2(l)\partial_{xx}u(t,l) - k(l)u(t,l)$. □

REMARK 4. *When* l *is a reflecting boundary, the condition* $\partial_x u(t,l) = 0$ *alone suffices to characterize the boundary behavior. This condition can also be found in [28, p.330], but no proof is given there. The fact that the PDE holds at* l *is not given in [28] and we do not see any proof for this fact in the literature. From the numerical perspective, it is important to know the PDE is valid at* l, *which will be used in our discretization scheme.*

To summarize, the following boundary conditions are imposed for the diffusion PDE:

- l is killing: $u(t,l)=0$ for $t\geq 0$.
- l is reflecting: $\partial_t u(t,l)=\frac{1}{2}\sigma^2(l)\partial_{xx}u(t,l)-k(l)u(t,l)$ for $t>0$, $u(0,l)=f(l)$.

Note that when l is killing, in general $u(0,l)\neq f(l)$ as $u(0,l)=0$ while $f(l)$ is not necessarily zero.

3.3. Discretization by finite difference. We continue our discussions under Assumption 1. To obtain a numerical approximation for $u(t,x)$, we discretize the space variable x in the diffusion PDE. To simplify the exposition, we consider a uniform grid on $[l,r]$, with $h=(r-l)/N$ and $x_i=l+ih$ for $i=0,1,\cdots,N$. However, non-uniform grids can also be used in our method and the extension is straightforward. We approximate the partial derivatives w.r.t. x by central differences, i.e. for $1\leq i\leq N-1,\ t>0$,

$$\begin{aligned}&\mu(x_i)\partial_x u(t,x_i)+\frac{1}{2}\sigma^2(x_i)\partial_{xx}u(t,x_i)-k(x_i)u(x_i)\\&\approx\mu(x_i)\frac{u(t,x_{i+1})-u(t,x_{i-1})}{2h}+\frac{1}{2}\sigma^2(x_i)\frac{u(t,x_{i+1})-2u(t,x_i)+u(t,x_{i-1})}{h^2}-k(x_i)u(t,x_i).\end{aligned}$$

To complete the discretization, we need to deal with boundary conditions. When l is killing boundary, we simply set $u(t,x_0)=0$. When l is reflecting, we approximate $\frac{1}{2}\sigma^2(l)\partial_{xx}u(t,l)-k(l)u(t,l)$ as follows:

$$\frac{1}{2}\sigma^2(x_0)\partial_{xx}u(t,x_0)-k(x_0)u(t,x_0)\approx\sigma^2(x_0)\frac{u(t,x_1)-u(t,x_0)}{h^2}-k(x_0)u(t,x_0),$$

where to approximate $\partial_{xx}u(t,x_0)$, we expand $u(t,x_1)$ at $x=x_0$ and use $\partial_x u(t,x_0)=0$. The discretization scheme yields a semi-discrete system in $\mathbb{R}^{N+1}$:

$$\frac{d}{dt}u_h(t)=Gu_h(t),\ u_h(0)=f_h. \tag{3.7}$$

Here G is an $(N+1)\times(N+1)$ tridiagonal matrix and it has the following entries: for $1\leq i\leq N-1$, $G_{i,i\pm1}=[\pm\mu(x_i)/h+\sigma^2(x_i)/h^2]/2$, $G_{i,i}=-\sigma^2(x_i)/h^2-k(x_i)$. If x_0 is killing, then $G_{0,i}=0$ for $0\leq i\leq N$. If x_0 is reflecting, $G_{0,0}=-\sigma^2(x_0)/h^2-k(x_0)$, $G_{0,1}=\sigma^2(x_0)/h^2$. In both cases, $G_{0,i}=0$ for $2\leq i\leq N$. The formulae for entries at x_N are entirely similar and hence they are omitted here. f_h is a $(N+1)\times 1$ vector, with $f_{h,i}=f(x_i)$ for $1\leq i\leq N-1$. When l,r are killing, $f_{h,0}=f_{h,N}=0$, and when l,r are reflecting, $f_{h,0}=f(l)$ and $f_{h,N}=f(r)$.

3.4. Eigendecomposition. The solution to the semi-discrete system (3.7) is given by $u_h(t)=e^{Gt}f_h$, where e^{Gt} is the matrix exponential defined by

$$e^{Gt}=\sum_{n=0}^{\infty}\frac{t^n}{n!}G^n. \tag{3.8}$$

In the following, we derive another representation for $u_h(t)$, using which we can compute $\int_{[0,\infty)}u_h(s)q_t(ds)$ analytically using the Laplace transform of the subordinator.

Before proceeding, we observe that when x_0 is a killing boundary, the value for the entries of e^{Gt}_{ij} with $1\leq i,j\leq N$ (the (i,j)-th entry of e^{Gt}) does not depend on the 0-th column and the 0-th row of G. Furthermore, $f_h^0=0$ (the 0-th entry of f_h). Therefore, to compute the i-th entry of $u_h(t)$ for $1\leq i\leq N$, the row and the column in G that correspond to x_0 are not needed. The same reasoning applies to

x_N when it is killing. Hence in the following, we can eliminate killing boundaries from consideration. We let H be the matrix constructed from G by eliminating rows and columns which correspond to killing boundaries, i.e., H is an $n \times n$ matrix with $n = N + 1 - \#$ of killing boundaries. In all cases, we start the index of H from 1 (whereas in G indexing starts from 0), and we have

$$\text{for } 1 \le i, j \le n,\ H_{i,j} = \begin{cases} G_{i,j}, & l, r \text{ are killing,} \\ G_{i-1,j-1}, & l, r \text{ are reflecting.} \end{cases}$$

Note that H is also tridiagonal. For simplicity, we abuse the notation and still use $u_h(t)$ and f_h to denote the solution and payoff vector without entries for the killing boundaries. Then $u_h(t) = e^{Ht} f_h$. Our next result characterizes H.

PROPOSITION 3.6. *Suppose the off-diagonal entries of H are all positive (Under Assumption 1, this is true when h is small enough). Then H has n simple and real eigenvalues with $0 \ge \Lambda_1 > \Lambda_2 > \cdots > \Lambda_n$, where Λ_i is the i-th eigenvalue.*

Proof. H is a tridiagonal matrix with $H_{i,i+1} H_{i+1,i} > 0$ for all $i = 1, 2, \cdots, n-1$, so it has n simple real eigenvalues by [57, Theorem 1]. In addition, H is diagonally dominant because its off-diagonal entries are positive, its diagonal entries are negative and the sum of each row is nonpositive. Thus by Gershgorin Circle Theorem, all its eigenvalues lies at the left half of the complex plane. Since they are real, they are nonpositive. □

Let $D = \text{diag}(\Lambda_1, \cdots, \Lambda_n)$ and $V = (\Phi_1, \cdots, \Phi_n)$, where Φ_i is the eigenvector corresponding to Λ_i. Then $H = VDV^{-1}$. Using (3.8), it is easy to see that $u_h(t) = Ve^{Dt}V^{-1} f_h$. Since D is a diagonal matrix, we have

$$\int_{[0,\infty)} e^{Ds} q_t(ds) = \int_{[0,\infty)} \text{diag}(e^{\Lambda_1 s}, \cdots, e^{\Lambda_n s}) q_t(ds) = e^{-\phi(-D)t},$$

where the matrix function $-\phi(-D) = \text{diag}(-\phi(-\Lambda_1), \cdots, -\phi(-\Lambda_n))$. Hence

$$u_h^\phi(t) := \int_{[0,\infty)} u_h(s) q_t(ds) = Ve^{-\phi(-D)t} V^{-1} f_h,$$

approximates $u^\phi(t, x)$ on the grid.

Now we need to find out all the eigenvalues and eigenvectors of H. For a general $n \times n$ matrix, this requires $O(n^3)$ operations which is expensive when n is large. Below we show that in our problem, this can actually be done in $O(n^2)$ operations!

Define recursively

$$j_1 = 1,\ j_k = j_{k-1} H_{k-1,k} / H_{k,k-1},\ 2 \le k \le n,$$

and put $J = \text{diag}(j_1, \cdots, j_n)$. Direct calculation shows that JH is symmetric and tridiagonal and H is similar to $\overline{H} := J^{-\frac{1}{2}} JH J^{-\frac{1}{2}}$, which is again symmetric and tridiagonal (note that these hold also for non-uniform grids). $\overline{H}$ has the same eigenvalues as H and its k-th eigenvector, denoted by Ψ_k is related to Φ_k by $\Phi_k = J^{-\frac{1}{2}} \Psi_k$. For an $n \times n$ symmetric and tridiagonal matrix, its eigenvalues and eigenvectors can be found out efficiently by the MR3 algorithm which is stable and has worst case complexity $O(n^2)$ ([17]). We can now apply this algorithm to find out the eigendecomposition of $\overline{H}$. Let $W = (\Psi_1, \cdots, \Psi_n)$. Now $u_h^\phi(t)$ becomes

$$u_h^\phi(t) = J^{-1/2} W e^{-\phi(-D)t} W^T J^{1/2} f_h, \tag{3.9}$$

where the superscript T denotes transpose. We derive an alternative expression for (3.9). We consider the normalized Ψ_k, i.e., $\sum_{j=1}^n \Psi_{k,j}^2 = 1$, and since $\overline{H}$ is symmetric, Ψ_k and Ψ_l are orthogonal for $k \neq l$. As $\Phi_k = J^{-\frac{1}{2}}\Psi_k$, it is easy to see that $\Phi_k^T J \Phi_l = \delta_{k,l}$ (the Kronecker delta), i.e., $\{\Phi_1, \cdots, \Phi_n\}$ are orthonormal w.r.t. J. Some algebra shows that

$$u_h^\phi(t) = \sum_{k=1}^n e^{-\phi(-\Lambda_k)t} f_{h,k} \Phi_k, \quad f_{h,k} = \Phi_k^T J f_h. \tag{3.10}$$

This expression can be thought as the discrete analogue of the exact eigenfunction expansion

$$u^\phi(t,x) = \sum_{k=1}^\infty e^{-\phi(-\lambda_k)t} f_k \varphi_k(x), \quad f_k = \int_I f(x)\varphi_k(x) m(dx).$$

Recall that λ_k is the k-th eigenvalue of the diffusion generator and φ_k is the corresponding normalized eigenfunction ($\int_I \varphi_k^2(x) m(dx) = 1$). In the discrete case, the diagonal of J plays the role of the speed measure m in the continuous case.

To evaluate (3.9), we go from right to left. Note that $J^{1/2}$, $e^{-\phi(-D)t}$ and $J^{-1/2}$ are diagonal, so multiplying them by a vector only requires $O(n)$ operations. Since W is dense, its multiplication with a vector requires $O(n^2)$ operations. So the overall time complexity to calculate (3.9) is $O(n^2)$. Further adding up the complexity of the MR3 algorithm for eigendecomposition, the total time complexity of the proposed scheme is $O(n^2)$, where n is the number of grid points excluding killing boundaries. Since both the MR3 algorithm and matrix-vector multiplication can be parallelized (see [31] for how to parallelize the MR3 algorithm), further reduction in computation time can be expected.

Remark 5 (Subordinate Brownian Motion Models). *Consider the subordinate Brownian motion models discussed in Section 3, where the background diffusion X is a BM with drift μ and volatility σ. Suppose we specify $-A$ and A to be killing boundaries. In this case, we can obtain analytical formulas for the eigenvalues and eigenvectors of $\overline{H}$. Let $a = \sqrt{\frac{\sigma^4}{4h^4} - \frac{\mu^2}{4h^2}}$, $b = -\frac{\sigma^2}{h^2}$. Then $\overline{H}_{i,i} = b$, $\overline{H}_{i,i-1} = a$ and $\overline{H}_{i,i+1} = a$. Applying the result in [58, Eq.(11)], its eigenvalues and eigenvectors are given by (recall that W is the eigenvector matrix for $\overline{H}$)*

$$\Lambda_k = b + 2a\cos\frac{k\pi}{n+1}, W_{j,k} = \frac{2}{\sqrt{n+1}}\sin\frac{jk\pi}{n+1}, \ j,k = 1,2,\cdots,n.$$

Let $i = \sqrt{-1}$ be the imaginary unit. We can rewrite $W_{j,k}$ as

$$W_{j,k} = P_{j,k} - Q_{j,k}, \ P_{j,k} = \frac{-i}{\sqrt{n+1}}\exp\left(i\frac{jk\pi}{n+1}\right), \ Q_{j,k} = \frac{-i}{\sqrt{n+1}}\exp\left(-i\frac{jk\pi}{n+1}\right).$$

Now $u^\phi(t)$ can be written as

$$\begin{aligned} u_h^\phi(t) = {} & J^{-1/2} P e^{-\phi(-D)t} P^T J^{1/2} f_h - J^{-1/2} P e^{-\phi(-D)t} Q^T J^{1/2} f_h \\ & - J^{-1/2} Q e^{-\phi(-D)t} P^T J^{1/2} f_h + J^{-1/2} Q e^{-\phi(-D)t} Q^T J^{1/2} f_h. \end{aligned}$$

To evaluate the above formula, we go from right to left for each term. Note that multiplying a diagonal matrix with a vector has time complexity $O(n)$, and multiplying

P, Q with a vector can be accomplished by the Fractional Discrete Fourier transform with time complexity $O(n \log_2 n)$. Thus, in this special case, the overall complexity of our algorithm is only $O(n \log_2 n)$.

REMARK 6 (Comparison to existing numerical schemes for PIDEs). *We have already discussed that in general existing numerical schemes for PIDEs do not apply to general subordinate diffusions as their jump density is given by an unknown integral. There are cases that this integral can be obtained in closed form and hence existing PIDE schems can be applied without resorting to expensive numerical approximation of the integral. These schemes typically have time complexity $O(mn^2)$ when jumps are state-dependent and $O(mn \log_2 n)$ when jumps are state-independent (this is true when X is a BM), where m is the number of time steps and n is the number of grid points for the space variable. They require more computations when the time horizon increases as more time steps are needed. In contrast, our method does not discretize time and hence there is no time discretization error and option prices for all time horizons are calculated at the same cost. The time complexity is $O(n^2)$ for the case of state-dependent jumps and $O(n \log_2 n)$ when jumps are state-independent. Thus one would expect our method is more efficient than standard PIDE schemes when they are applicable for not-too-small maturities. Another feature of our method is that it does not have any approximation error from truncating large and small jumps, which is done in standard numerical methods for PIDEs. In Section 5, we compare our method to an existing PIDE scheme for the NIG equity model for different maturities.*

3.5. Analysis of Discretization Error. Consider the discretization error vector

$$\varepsilon_h(t) := (u_{h,i}(t) - u(t, x_i))_{0 \le i \le N}^T, \ \varepsilon_h^\phi(t) := (u_{h,i}^\phi(t) - u^\phi(t, x_i))_{0 \le i \le N}^T,$$

where $u_{h,i}(t)$ is the entry of the vector $u_h(t)$ that approximates $u(t, x_i)$ and the meaning of $u_{h,i}^\phi(t)$ is similar. When l and r are killing boundaries, we define $\varepsilon_h(t)$ and $\varepsilon_h^\phi(t)$ to be the vector containing errors at x_1 to x_{N-1}, as the error at the boundary point is zero. Our goal is to show the maximum discretization error $\|\varepsilon_h^\phi(t)\|_\infty$ converges to 0 as $h \to 0$ and find out its convergence rate. Throughout this section, we impose Assumption 1. We will first analyze the convergence rate for $\|\varepsilon_h^\phi(t)\|_\infty$ when the payoff function is smooth enough, based on which we can then prove the convergence for more general payoffs.

It is easy to see that

$$\|\varepsilon_h^\phi(t)\|_\infty \le \int_{[0,\infty)} \|\varepsilon_h(s)\|_\infty \, q_t(ds). \tag{3.11}$$

Hence we need to estimate $\|\varepsilon_h(s)\|_\infty$ for all $s \ge 0$, to do which estimates for the partial derivatives of the diffusion value function $u(t, x)$ are crucial. We summarize such estimates in the following two lemmas. $\mathcal{H}^{n+\epsilon}([l, r])$ denotes the set of functions $g : [l, r] \to \mathbb{R}$ that are n times continuously differentiable with the n-th order derivative ϵ-Hölder continuous.

LEMMA 3.7. *Suppose that Assumption 1 holds and l, r are killing. Assume that for some $\epsilon \in (0, 1)$, $k(x) \in \mathcal{H}^{2+\epsilon}([l, r])$, $f(x) \in \mathcal{H}^{4+\epsilon}([l, r])$ and $f(x) = 0$ for $x = l, r$. Then $\partial_{xxx} u(t, x)$ and $\partial_{xxxx} u(t, x)$ are uniformly bounded on $[0, \infty) \times [l, r]$.*

LEMMA 3.8. *Suppose that Assumption 1 holds and l, r are reflecting. Assume that for some $\epsilon \in (0, 1)$, $\mu(x), k(x) \in \mathcal{H}^{3+\epsilon}([l, r])$, $f(x) \in \mathcal{H}^{5+\epsilon}([l, r])$ and $f'(x) = 0$ for $x = l, r$. Then $\partial_{xxx} u(t, x)$, $\partial_{xxxx} u(t, x)$ and $\partial_t \partial_{xxx} u(t, x)$ are uniformly bounded on $[0, \infty) \times [l, r]$.*

Based on Lemma 3.7 and 3.8, we show that the discretization error is second order in h. The result is not surprising when both l and r are killing boundaries, where central difference is used everywhere on the grid to approximate the derivatives (except at the killing boundaries where no approximation of derivatives is needed) and it is second-order accurate. In the case where l and r are reflecting, the discretization error is still second order despite that the accuracy for derivative approximation is only first order at the boundary points.

PROPOSITION 3.9. *Suppose that either the assumptions of Lemma 3.7 or those of Lemma 3.8 hold, and* $\int_{(1,\infty)} s\nu(ds) < \infty$. *Then for some constant* $C > 0$,

$$\|\varepsilon_h(s)\|_\infty \leq \begin{cases} Csh^2, & l, r \text{ are killing,} \\ C(s+1)h^2, & l, r \text{ are reflecting.} \end{cases}$$

For any $t > 0$, *there exists a constant* $C_t > 0$ *(which only depends on* t*) such that* $\|\varepsilon_h^\phi(t)\|_\infty \leq C_t h^2$.

Proof. Let $\sigma_{h,i}(s)$ be the local truncation error at x_i, i.e., the difference between the finite difference approximation and the PDE at x_i, and $\sigma_h(s) := (\sigma_{h,i}(s))_{0\leq i\leq N}^T$. When l, r are killing boundaries, we only consider $\sigma_h(s) := (\sigma_{h,i}(s))_{1\leq i\leq N-1}^T$ as no finite difference approximation is used at the boundary. In the following, we treat the case of killing boundaries and the case of reflecting boundaries separately.
(i) Both l and r are killing: For $1 \leq i \leq N-1$,

$$\begin{aligned}
\sigma_{h,i}(s) &= \mu(x_i)\frac{u(s,x_i+h)-u(s,x_i-h)}{2h} - \mu(x_i)\partial_x u(s,x_i) \\
&\quad + \frac{1}{2}\sigma^2(x_i)\frac{u(s,x_i+h)-2u(s,x_i)+u(s,x_i-h)}{h^2} - \frac{1}{2}\sigma^2(x_i)\partial_{xx}u(s,x_i) \\
&= \frac{1}{12}\mu(x_i)h^2(\partial_{xxx}u(s,\xi_1)+\partial_{xxx}u(s,\xi_2)) \\
&\quad + \frac{1}{48}\sigma^2(x_i)h^2(\partial_{xxxx}u(s,\eta_1)+\partial_{xxxx}u(s,\eta_2)) \qquad (3.12)
\end{aligned}$$

for some $\xi_1, \xi_2, \eta_1, \eta_2 \in [x_i - h, x_i + h]$. Hence by Lemma 3.7, there exists $C_0 > 0$ independent of s such that $\|\sigma_h(s)\|_\infty \leq Ch^2$. From the diffusion PDE (3.1) and central difference approximation, we have

$$\varepsilon_h'(s) = H\varepsilon_h(s) + \sigma_h(s),\ \varepsilon_h(0) = 0. \qquad (3.13)$$

Here $\varepsilon_h'(s)$ means taking derivative of $\varepsilon_h(s)$ w.r.t. s elementwise. The solution to (3.13) is given by $\varepsilon_h(s) = \int_0^s e^{H(s-v)}\sigma_h(v)dv$, therefore

$$\|\varepsilon_h(s)\|_\infty \leq \int_0^s \|e^{H(s-v)}\sigma_h(v)\|_\infty dv \leq \int_0^s \|\sigma_h(v)\|_\infty dv \leq Csh^2. \qquad (3.14)$$

In the second inequality, we used the fact that for any $t > 0$, e^{Ht} is a sub-probability matrix (i.e., all its entries are nonnegative and the sum of each row is ≤ 1) as H is a tridiagonal matrix with positive off-diagonal entries and negative diagonal entries. This implies $\|e^{Ht}v\|_\infty \leq \|v\|_\infty$ for any vector v.

By [53, Theorem 25.3], the condition $\int_{(1,\infty)} s\nu(ds) < \infty$ is equivalent to $\int_{[0,\infty)} sq_t(ds)$ for any $t > 0$. Hence let $C_t = C\int_{[0,\infty)} sq_t(ds)$, from (3.11) and (3.14) we have $\|\varepsilon_h^\phi(t)\|_\infty \leq C_t h^2$.

(ii) Both l and r are reflecting: For $i = 1, \cdots, N-1$, $\sigma_{h,i}(s)$ is given by (3.12). For $i = 0$, for some $\xi \in [x_0, x_0 + h]$,

$$\sigma_{h,0}(s) = \sigma^2(x_0)\frac{u(s, x_0 + h) - u(s, x_0)}{h^2} - \frac{1}{2}\sigma^2(x_0)\partial_{xx}u(s, x_0) = \frac{1}{6}\sigma^2(x_0)h\partial_{xxx}u(s, \xi).$$

For $i = N$, for some $\eta \in [x_N - h, x_N]$,

$$\sigma_{h,N}(s) = \sigma^2(x_N)\frac{u(s, x_N - h) - u(s, x_N)}{h^2} - \frac{1}{2}\sigma^2(x_N)\partial_{xx}u(s, x_N) = -\frac{1}{6}\sigma^2(x_N)h\partial_{xxx}u(s, \eta).$$

Let $\eta_h(s) = (0, \sigma_{h,1}(s), \cdots, \sigma_{h,N-1}(s), 0)^T$ and $\xi_h(s)$ be the solution to the linear system $G\xi_h(s) = \sigma_h(s) - \eta_h(s)$, which is given recursively by

$$\begin{cases} \xi_{h,0}(s) = \frac{1}{3}h^3\partial_{xxx}u(s, \xi),\ \xi_{h,1}(s) = \frac{1}{2}h^3\partial_{xxx}u(s, \xi), \\ \xi_{h,i}(s) = -\frac{1}{G_{i-1,i}}\left(G_{i-1,i-2}\xi_{h,i-2}(s) + G_{i-1,i-1}\xi_{h,i-1}(s)\right),\ 2 \le i \le N-1, \\ \xi_{h,N}(s) = \frac{1}{G_{N,N}}\left(-\frac{1}{6}\sigma^2(x_N)h\partial_{xxx}u(s, \eta) - G_{N,N-1}\xi_{h,N-1}(s)\right). \end{cases}$$

Taking differentiation w.r.t. time gives us

$$\begin{cases} \xi'_{h,0}(s) = \frac{1}{3}h^3\partial_s\partial_{xxx}u(s, \xi),\ \xi'_{h,1}(s) = \frac{1}{2}h^3\partial_s\partial_{xxx}u(s, \xi), \\ \xi'_{h,i}(s) = -\frac{1}{G_{i-1,i}}\left(G_{i-1,i-2}\xi'_{h,i-2}(s) + G_{i-1,i-1}\xi'_{h,i-1}(s)\right),\ 2 \le i \le N-1, \\ \xi'_{h,N}(s) = \frac{1}{G_{N,N}}\left(-\frac{1}{6}\sigma^2(x_N)h\partial_s\partial_{xxx}u(s, \eta) - G_{N,N-1}\xi'_{h,N-1}(s)\right). \end{cases}$$

Recall that $G_{i-1,i-1} = -\frac{\sigma^2(x_{i-1})}{h^2} - k(x_{i-1})$, $G_{i-1,i} = \frac{\sigma^2(x_{i-1})}{2h^2} + \frac{\mu(x_{i-1})}{2h}$, $G_{i-1,i-2} = \frac{\sigma^2(x_{i-1})}{2h^2} - \frac{\mu(x_{i-1})}{2h}$, hence $\frac{G_{i-1,i-2}}{G_{i-1,i}}$, $\frac{G_{i-1,i-1}}{G_{i-1,i}}$ are bounded when $h \to 0$. By Lemma 3.8, there is a constant $C_1 > 0$ such that $|\xi_{h,0}(s)| \le C_1h^3$, $|\xi_{h,1}(s)| \le C_1h^3$, $|\xi'_{h,0}(s)| \le C_1h^3$, $|\xi'_{h,1}(s)| \le C_1h^3$. From the recursive relation, there is also a constant $C_2 > 0$ such that $|\xi_{h,i}(s)| \le C_2h^3$, $|\xi'_{h,i}(s)| \le C_2h^3$ for $2 \le i \le N-1$. Notice that $\frac{\sigma^2(x_N)h}{G_{N,N}} = \frac{\sigma^2(x_N)h}{-\sigma^2(x_N)/h^2 - k(x_N)}$ and $\frac{G_{N,N-1}}{G_{N,N}} = \frac{\sigma^2(x_N)/h^2}{-\sigma^2(x_N)/h^2 - k(x_N)}$ are bounded when $h \to 0$, so there is a constant $C_3 > 0$ such that $|\xi'_{h,N}(s)| \le C_3h^3$, $|\xi_{h,N}(s)| \le C_3h^3$. Together, there is a constant C_0 independent of s such that $\|\xi_h(s)\|_\infty \le C_0h^3$, $\|\xi'_h(s)\|_\infty \le C_0h^3$.

Let $\tilde{\varepsilon}_h(s) = \varepsilon_h(s) + \xi_h(s)$ and recall that $\varepsilon'_h(s) = G\varepsilon_h(s) + \sigma_h(s)$, then,

$$\begin{aligned} \tilde{\varepsilon}'_h(s) &= \varepsilon'_h(s) + \xi'_h(s) = G\varepsilon_h(s) + \sigma_h(s) + \xi'_h(s) \\ &= G\varepsilon_h(s) + G\xi_h(s) + \eta_h(s) + \xi'_h(s) = G\tilde{\varepsilon}_h(s) + \eta_h(s) + \xi'_h(s), \end{aligned}$$

with $\tilde{\varepsilon}_h(0) = \xi_h(0)$. Then,

$$\begin{aligned} \|\varepsilon_h(s)\|_\infty &\le \|\xi_h(s)\|_\infty + \|\tilde{\varepsilon}_h(s)\|_\infty \\ &\le \|\xi_h(s)\|_\infty + \|\xi_h(0)\|_\infty + \int_0^s \|e^{G(s-v)}(\xi'_h(v) + \eta_h(v))\|_\infty dv, \\ &\le 2C(s+1)h^2 \text{ (note that } \|\eta_h(v)\|_\infty \le \tilde{C}h^2 \text{ for some } \tilde{C} > 0) \end{aligned}$$

for some $C > 0$. Let $C_t = 2C\int_{[0,\infty)}(s+1)q_t(ds)$. Then we have $\|\varepsilon^\phi_h(t)\|_\infty \le C_th^2$. □

REMARK 7. *The condition $\int_{(1,\infty)} s\nu(ds) < \infty$ is equivalent to $E[T_t] < \infty$ for all $t > 0$, which is satisfied for all Lévy subordinators used in finance.*

Our scheme converges for more general payoffs and we will prove this based on Proposition 3.9. We define the diffusion transition operator $\mathcal{P}_t f(x) := E_x[f(X_t)1_{\{\zeta>t\}}]$. $\mathcal{P}_t$ is a contraction on the space of measurable and bounded functions, i.e., $\|\mathcal{P}_t f\|_\infty \leq \|f\|_\infty$. We also introduce $P_t v := e^{Ht}v$ for any vector $v \in \mathbb{R}^n$. P_t is also a contraction, i.e., $\|P_t v\|_\infty \leq \|v\|_\infty$, where $\|\cdot\|_\infty$ is now the maximum norm for a vector. For simplicity, we use $\|\cdot\|_\infty$ for both the maximum norm for a function and that for a vector. The exact meaning can be determined from the context.

PROPOSITION 3.10. *Suppose that Assumption 1 holds. For any measurable and bounded payoff f on $[l,r]$, $\|\varepsilon_h^\phi(t)\|_\infty \to 0$ when $h \to 0$ if*

$$\lim_{\delta\to 0} \|\mathcal{P}_\delta f - f\|_\infty = 0. \tag{3.15}$$

Proof. Let π_h be the operator that maps a function on $[l,r]$ to the grids excluding the killing boundaries, i.e., $(\pi_h g)_i = g(x_i)$. For any $\delta > 0$.

$$\begin{aligned}
&\|\varepsilon_h(s)\|_\infty = \|P_s\pi_h f - \pi_h\mathcal{P}_s f\|_\infty \\
&\leq \|P_s\pi_h f - P_s\pi_h\mathcal{P}_\delta f\|_\infty + \|P_s\pi_h\mathcal{P}_\delta f - \pi_h\mathcal{P}_s\mathcal{P}_\delta f\|_\infty + \|\pi_h\mathcal{P}_s\mathcal{P}_\delta f - \pi_h\mathcal{P}_s f\|_\infty \\
&\leq \|\pi_h f - \pi_h\mathcal{P}_\delta f\|_\infty + \|P_s\pi_h\mathcal{P}_\delta f - \pi_h\mathcal{P}_s\mathcal{P}_\delta f\|_\infty + \|\mathcal{P}_s\mathcal{P}_\delta f - \mathcal{P}_s f\|_\infty \\
&\leq 2\|\mathcal{P}_\delta f - f\|_\infty + \|P_s\pi_h\mathcal{P}_\delta f - \pi_h\mathcal{P}_s\mathcal{P}_\delta f\|_\infty.
\end{aligned}$$

For the second term, $\mathcal{P}_\delta f$ is the payoff function, which can be represented by the eigenfunction expansion

$$\mathcal{P}_\delta f(x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n \delta}\varphi_n(x).$$

From Lemma 3.4, we can show that $\mathcal{P}_\delta f \in C^\infty([l,r])$ (We only give bounds for up to the fourth order derivative of the eigenfunctions in Lemma 3.4, but following the arguments used there one can show that $\|\varphi_n^{(k)}\|_\infty \leq Cn^k$ for all integer $k \geq 0$). Furthermore, when l is killing, we have $\mathcal{P}_\delta f(l) = 0$ and when l is reflecting, $(\mathcal{P}_\delta f)'(l) = 0$ by Proposition 3.5. The same applies to r. Hence the conditions on the payoff function required in Proposition 3.9 are satisfied, applying which shows the second term $\leq C_\delta s h^2$ in the killing boundary case and $C_\delta(s+1)h^2$ in the reflecting boundary case for some positive constant C_δ, which depends on δ. Therefore,

$$\lim_{h\to 0} \|\varepsilon_h(s)\|_\infty \leq 2\|\mathcal{P}_\delta f - f\|_\infty,$$

for any $\delta > 0$. Letting $\delta \to 0$, (3.15) implies that $\lim_{h\to 0}\|\varepsilon_h(s)\|_\infty = 0$. From (3.11), by dominated convergence theorem, $\|\varepsilon_h^\phi(t)\|_\infty \to 0$. □

[29, Theorem 7.4.2] shows that the strong continuity condition (3.15) is satisfied for any continuous f on $[l,r]$ when l and r are reflecting (f is also bounded as $[l,r]$ is compact). When l and r are killing, the same theorem shows that for any continuous f such that $\lim_{x\downarrow l} f(x) = 0$ and $\lim_{x\uparrow r} f(x) = 0$, (3.15) is valid. However, for general continuous f, (3.15) fails to hold as for boundary point x, $\mathcal{P}_t f(x) = 0$ for any $t > 0$ while $f(x)$ is not necessarily zero. Furthermore, in finance, sometimes the payoff is even discontinuous (consider digital options). For such payoffs, in general one cannot expect the strong continuity condition to hold. Below we refine our analysis and prove

that our scheme actually converges for piecewise continuous and bounded payoffs on $[l,r]$ when the end-points are killing or reflecting. We need the following lemma.

LEMMA 3.11. *Suppose Assumption 1 holds and both l,r are killing boundaries or both are reflecting boundaries. Then for any $s>0$, there is a constant C independent of h such that*

$$\max_{i,j} P_{s,i,j} \le Ch, \quad \sup_{(x,y)\in(l,r)\times(l,r)} p(s,x,y) \le C.$$

Here $P_s = e^{Hs}$ and $P_{s,i,j}$ is the (i,j)-th element of P_s. $p(s,x,y)$ is the transition density of the background diffusion X.

PROPOSITION 3.12. *Suppose that Assumption 1 holds and both l,r are killing boundaries or both are reflecting boundaries. For any piecewise continuous and bounded payoff f on $[l,r]$ (i.e., it has a finite number of discontinuities on (l,r)), $\|\varepsilon_h^{\phi}(t)\|_\infty \to 0$ when $h\to 0$.*

Proof. Suppose the discontinuities on (l,r) are $y_1, y_2, \cdots, y_k$. Define function $f^\epsilon(x)$ as follows ($1\le i\le k$):

$$f^\epsilon(x) = \begin{cases} \frac{f(l+\epsilon)}{\epsilon}(x-l) & \text{if } x\in[l,l+\epsilon), \\ f(y_i-\epsilon/2) + \frac{f(y_i+\epsilon/2)-f(y_i-\epsilon/2)}{\epsilon}(x-y_i+\epsilon/2) & \text{if } x\in[y_i-\epsilon/2, y_i+\epsilon/2), \\ \frac{f(r-\epsilon)}{\epsilon}(r-x) & \text{if } x\in[r-\epsilon,r], \\ f(x) & \text{otherwise.} \end{cases}$$

where $\epsilon>0$ is small enough such that the intervals above are disjoint. f^ϵ is a continuous function which vanishes at l and r. Let $D = [l,l+\epsilon)\cup[y_1-\epsilon/2, y_1+\epsilon/2)\cup\cdots\cup[y_k-\epsilon/2, y_k+\epsilon/2)\cup[r-\epsilon,r]$. Its Lebesgue measure $|D| = (k+2)\epsilon$. For any $s>0$,

$$\begin{aligned}\|\varepsilon_h(s)\|_\infty &= \|P_s\pi_h f - \pi_h\mathcal{P}_s f\|_\infty \\ &\le \|P_s\pi_h f - P_s\pi_h f^\epsilon\|_\infty + \|P_s\pi_h f^\epsilon - \pi_h\mathcal{P}_s f^\epsilon\|_\infty + \|\mathcal{P}_s f - \mathcal{P}_s f^\epsilon\|_\infty.\end{aligned}$$

For the first term, by Lemma 3.11, there is a constant C independent of h and i,j such that $P_{s,i,j}\le Ch$. Hence (recall that $n = N+1-\#$ of killing boundaries)

$$\begin{aligned}&|(P_s\pi_h f - P_s\pi_h f^\epsilon)_i| \le \sum_{j=1}^n P_{s,i,j}|f(x_j)-f^\epsilon(x_j)| \\ &\le Ch\sum_{j=1}^n |f(x_j)-f^\epsilon(x_j)|1_D(x_j) \le Ch\times 2\|f\|_\infty \times \frac{|D|}{h} = 2(k+2)C\|f\|_\infty\epsilon.\end{aligned}$$

Since the bound is independent of i, $\|P_s\pi_h f - \pi_h\mathcal{P}_s f\|_\infty \le 2(k+2)C\|f\|_\infty\epsilon$.

For the third term, by Lemma 3.11, there is a constant C independent of x,y such that $p(s,x,y)\le C$. Hence

$$\begin{aligned}&|\mathcal{P}_s f(x) - \mathcal{P}_s f^\epsilon(x)| \le \int_l^r p(s,x,y)\,|f(y)-f^\epsilon(y)|\,dy \\ &\le C\int_D |f(y)-f^\epsilon(y)|\,dy \le C\times 2\|f\|_\infty\times|D| = 2(k+2)C\|f\|_\infty\epsilon.\end{aligned}$$

Since the bound is independent of x, $\|\mathcal{P}_s f - \mathcal{P}_s f^\epsilon\|_\infty \le 2(k+2)C\|f\|_\infty\epsilon$. Together

$$\lim_{h\to0}\|\varepsilon_h(s)\|_\infty \le 4(k+2)C\|f\|_\infty\epsilon + \lim_{h\to0}\|P_s\pi_h f^\epsilon - \pi_h\mathcal{P}_s f^\epsilon\|_\infty = 4(k+2)C\|f\|_\infty\epsilon$$

for any $\epsilon > 0$. The vanish of the second term follows from $\lim_{\delta\to 0} \|\mathcal{P}_s f^\epsilon - f^\epsilon\|_\infty = 0$ (f^ϵ is a continuous and bounded function on $[l, r]$ that vanishes at the end-points, so strong continuity holds when l, r are both killing or reflecting boundaries) and Proposition 3.10. Taking ϵ to 0, we have $\lim_{h\to 0} \|\varepsilon_h(s)\|_\infty = 0$. By dominated convergence theorem, $\lim_{h\to 0} \|\varepsilon_h^\phi(t)\|_\infty = 0$. □

In general, when the payoff function f is not smooth enough, to estimate the error order theoretically is a challenging task under finite difference. In fact, even for PDEs, such results are scarce. Under special cases, sharp estimates are available. See [27] for results under the Black-Scholes model which utilizes the Black-Scholes formula for the option price to derive error estimates and [55] for results under the heat equation on the whole real line. We anticipate that in our setting, error estimates can be developed under some special cases. However, we do not pursue such question here. Typical financial payoffs are not smooth enough. For example, the put payoff is not differentiable at the strike price. In Section 5, for financial payoffs, we will show how the discretization error converges through various numerical examples, and one can observe that the convergence is oscillatory but the order is still roughly two. We will apply a smoothing technique in Section 5 so that convergence becomes smooth with order equal to two.

3.6. The Case When Assumption 1 is Violated. We discuss how to extend our method to the setting when Assumption 1 is violated. This is true for some diffusions used in finance. An example is given by the CIR process, $dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$ with $X_0 = x > 0$, $\kappa, \theta, \sigma > 0$. $l = 0$ is an entrance boundary if $2\kappa\theta \geq \sigma^2$ and reflecting otherwise. After localizing the infinite boundary, at $x = 0$, $\sigma(x)$ vanishes, violating Assumption 1. Another example is given by the JDCEV process, whose drift $\mu(x) = (\theta + b + ca^2x^{2\beta})x$, diffusion coefficient $\sigma(x) = ax^{\beta+1}$, and killing rate $k(x) = b + ca^2x^{2\beta}$ where $\beta < 0$, $a > 0$, $b \geq 0$ and $c \geq 0$. The boundary behavior at $l = 0$ can be found in [42, p.263], where it could be inaccessible, exit and killing, depending on the value of c and β. After localizing the infinite boundary, $k(x)$ tends to ∞ near $x = 0$, and this is also true for $\mu(x)$ and $\sigma(x)$ when $\beta < -1/2$ and $\beta < -1$, respectively. Hence Assumption 1 does not hold in this case.

Now let's assume that the diffusion X under consideration lives on a finite interval with end-point l and r (if there is any infinite boundary we localize it first), and conditions in Assumption 1 hold at r but fail at l. We can pick an $\epsilon > 0$ and consider a new diffusion X^ϵ living on $[l+\epsilon, r]$ which has the same infinitesimal characteristics as X. At the boundary $l+\epsilon$, X^ϵ is either killed or reflected if l is a natural and entrance. If l is exit or killing, X^ϵ is killed at $l+\epsilon$, and if l is reflecting, X^ϵ is reflected at $l+\epsilon$. If $\mu(x), \sigma(x), k(x)$ are smooth enough on $[l+\epsilon, r]$, then Assumption 1 holds on $[l+\epsilon, r]$. It is not difficult to see that $u_\epsilon^\phi(t, x)$ (the value function for the subordinate process obtained from X^ϵ) converges to $u^\phi(t, x)$ as $\epsilon \to 0$, which can be proved along the lines in the proof of Lemma 3.1. We can apply the previously developed scheme under Assumption 1 to compute $u_\epsilon^\phi(t, x)$. By choosing a very small ϵ, the error of using $u_\epsilon^\phi(t, x)$ to approximate $u^\phi(t, x)$ is negligible compared to the discretization error.

Alternatively, one can try to deal with X directly and we need to specify the boundary condition at l so that the boundary behavior is correctly captured. In general, the term "boundary condition" is quite inappropriate. In some cases, for instance when l is an entrance boundary, an equation is not needed at the boundary to single out the solution that correctly represents the behavior of the diffusion on the boundary. Nevertheless, a boundary equation is needed from a numerical perspective. Here for simplicity, we will call such boundary equation simply as boundary condition.

When l is exit or killing, it is easy to see that the boundary condition is $u(t,l)=0$ for $t\geq 0$. However, when l is entrance or reflecting, we do not have results in general. In many cases, one can still expect the PDE to be valid at l. For example, for the CIR process (which has no killing rate, i.e., $k(x)=0$), we have

$$\partial_t u(t,l)=\mu(l)\partial_x u(t,l)-k(l)u(t,l), t>0, \tag{3.16}$$

which can be verified directly from the eigenfunction expansion for $u(t,x)$ where the eigenvalues and eigenfunctions are explicitly known (see [42, p.258] for the expression). For diffusions with $\sigma(l)=0$ and $\mu(l)>0$, [19, Theorem 2.3] provides sufficient conditions under which (3.16) is valid (also see [49] for other types of sufficient conditions).

If l is exit or killing, one can use the same discretization scheme developed in Section 4.3. When l is entrance or reflecting, in the following we consider the case where $\sigma(l)=0$, $\mu(l)>0$, and we assume (3.16) is valid. We approximate $\mu(l)\partial_x u(t,l)-k(l)u(t,l)$ as follows (the grid is the same as the one in Section 4.3):

$$\mu(x_0)\partial_x u(t,x_0)-k(x_0)u(t,x_0)\approx \mu(x_0)\frac{u(t,x_1)-u(t,x_0)}{h}-k(x_0)u(t,x_0),$$

At points inside (l,r), we approximate the derivatives by central difference. Now in the matrix G, $G_{0,0}=-\mu(x_0)/h-k(x_0)$, $G_{0,1}=\mu(x_0)/h$. We want the off-diagonal entries of G to be positive (see Proposition 3.6). However, if central difference is used, in some cases it is possible that some off-diagonal entries is negative no matter how small h is for some choice of $\mu(x)$. So, instead we use the up-wind scheme for the convection term $\mu(x_i)\partial_x u(t,x_i)$ for points where $G_{i,i\pm 1}<0$ under central difference. If $\mu(x_i)>0$, we approximate it by $\mu(x_i)(u(t,x_{i+1})-u(t,x_i))/h$ and if $\mu(x_i)<0$, we approximate it by $\mu(x_i)(u(t,x_i)-u(t,x_{i-1}))/h$. This guarantees all off-diagonal entries are positive for all $h>0$.

Once G is obtained, the remaining steps are the same as those in Section 4.4.

4. Numerical Examples. All computations in this section were performed on a laptop computer with Intel Core i5 at 2.5 GHz with 4 GB RAM and all codes were written in C++. We consider five representative models based on Lévy subordinate diffusions in our numerical examples. In all these models, we set the Lévy subordinator T_t to be the Inverse Gaussian process, which is a popular choice in finance (it corresponds to setting $p=1/2$ in the tempered stable family). Its drift is denoted as γ and its Lévy measure and Laplace exponent are given by

$$\nu(ds)=m\sqrt{\frac{m}{2\pi v}}s^{-\frac{3}{2}}e^{-\frac{m}{2v}s}ds,\ \phi(\lambda)=\frac{m^2}{v}\left[\sqrt{1+2\frac{v}{m}\lambda}-1\right].$$

Here m is the mean rate for the jump part (i.e., $m=E[T_1]-\gamma$) and v is the variance rate (i.e., $v=Var[T_1]$). The five models we consider are listed below. We choose them because they are representative of the various cases we have discussed and in these models, we can calculate a very accurate benchmark using alternative methods. Here r_f and q are the risk-free rate and the stock's dividend yield, respectively.

(1) Normal Inverse Gaussian (NIG) model for equity prices ([2]): X is a Brownian motion with drift, i.e., $\mu(x)=\theta,\sigma(x)=\sigma$. The risk neutral stock price is modelled as $S_t=S_0e^{\rho t+X_t^{\phi}}$, where $\rho=r_f-q+\phi(-\theta-\sigma^2/2)$. We assume $1-(2\theta+\sigma^2)v/m\geq 0$, which is equivalent to $\mathbb{E}[S_t]<\infty$ for $t\geq 0$. We set $\theta=0.1$, $\sigma=0.3$, $\gamma=0$, $m=1$, $v=1$, $r_f=0.05$ and $q=0$.

(2) Subordinate reflected Brownian motion (SubRBM) model for real asset prices in a highly competitive market: X is a drifted Brownian motion in $[l, r]$ with two finite reflecting boundaries l and r, $\mu(x) = \theta$, $\sigma(x) = \sigma$. The risk-neutral asset price is modelled as $S_t = S_0 \exp(X_t^\phi)$. A model based on geometric reflected Brownian motion is considered in [18] for real asset prices in highly competitive markets. Applying subordination introduces jumps into the model, making it more realistic, and the jump model also moves between l and r. We set $\theta = 0.1$, $\sigma = 0.2$, $l = -0.2$, $r = 0.2$, $\gamma = 0$, $m = 1$, $v = 1$ and $r_f = 0.05$.
(3) Subordinate OU (SubOU) model for commodity spot prices ([36]): X is an Ornstein-Uhlenbeck process with $\mu(x) = \kappa(\theta - x)$ and $\sigma(x) = \sigma$. The risk-neutral commodity spot price is modelled as $S_t = S_0 e^{X_t^\phi}$. To be simple, here we do not introduce time-dependent but deterministic compensator to match the initial futures curve, which is done in [36]. However, it can be easily incorporated in our numerical scheme. We set $\kappa = 0.5$, $\theta = 1.0$, $\sigma = 0.3$, $\gamma = 0$, $m = 1$, $v = 1$ and $r_f = 0.05$.
(4) Subordinate CIR (SubCIR) model for commodity spot prices ([32]): X is a CIR process with $\mu(x) = \kappa(\theta - x)$ and $\sigma(x) = \sigma\sqrt{x}$. The risk-neutral commodity spot price is modelled as $S_t = S_0 X_t^\phi$. As in the SubOU model, we do not introduce time-dependent but deterministic compensator to match the initial futures curve here to be simple. We set $\kappa = 0.3$, $\theta = 0.8$, $\sigma = 0.3$, $\gamma = 0$, $m = 1$, $v = 1$ and $r_f = 0.05$. Here $x = 0$ is a reflecting boundary.
(5) Subordinate JDCEV (SubJDCEV) model for equity prices with bankruptcy risk ([46]): X is a JDCEV process with $\mu(x) = (\theta + b + ca^2x^{2\beta})x$, $\sigma(x) = ax^{\beta+1}$, and $k(x) = b + ca^2x^{2\beta}$ where $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$, $\theta + b > 0$. Denote the lifetime of SubJDCEV by τ. The risk neutral equity price is modelled as $S_t = 1_{\{\tau > t\}} e^{\rho t} X_t^\phi$ with $\rho = r - q + \phi(-\theta)$. We set $a = 10$, $b = 0.01$, $c = 0.1$, $\theta = 0$, $\beta = -1$, $\gamma = 0$, $m = 1$, $v = 1/16$, $r_f = 0.05$ and $q = 0$. Here $x = 0$ is a killing boundary.

In our examples we consider two types of options: European put and digital call (which pays out 1 dollar if the asset price is above the strike and otherwise nothing). In all cases, the strike price $K = 100$ and the the time to maturity $t = 1$. The benchmark price is calculated by the FFT method in [11] for the NIG model and by eigenfunction expansion for the other models, as the eigenvalues and eigenfunctions are explicitly known in these cases (see e.g., [42]). To apply our method, we adopt the uniform grid in all cases. For the SubCIR and SubJDCEV model, we directly apply our discretization scheme on $[0, A]$ (A is the localized boundary) as discussed in Section 3.6.

4.1. Convergence of Localization Error. Proposition 3.3 and Remark 3 show that the localization error converges exponentially in the NIG model and the SubOU model, which is confirmed in our numerical experiment. In Figure 1, we plot the convergence of localization error in these two models for a European put. For each A, we calculate the option price under the localized problem using a large number of grid points so that the discretization error is negligible compared to the localization error, and we can regard the difference between the approximate price and the benchmark price as the localization error. As discussed in Section 4.1, the finite boundaries $-A$ and A can be specified as either killing or reflecting. It is observed in Figure 1 that in both specifications, the error converges exponentially, consistent with our theoretical result. Comparing the two specifications, using reflecting boundaries produces much more accurate results. This is expected, as for the original diffusion X, it is possible for it to go back to $(-A, A)$, which is impossible when $-A$ and A are killing boundaries. We also plot the convergence of localization error for a European put under the

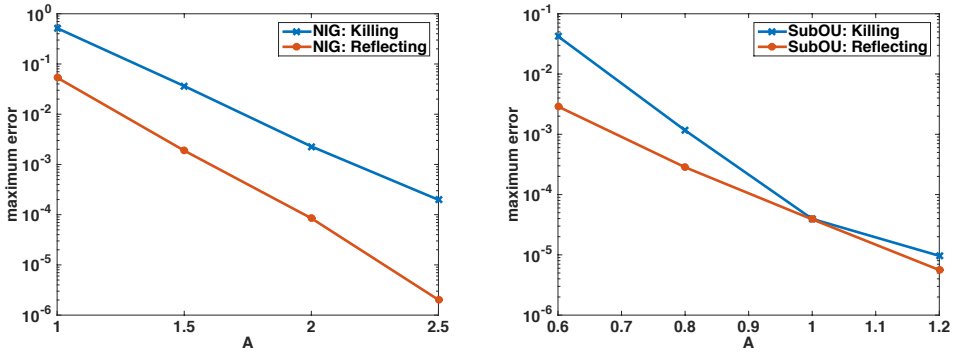

Fig. 1: Convergence of localization error for a European put under the NIG and SubOU model. We plot the maximum localization error of the option price for S_0 over $[80, 120]$ in log scale.

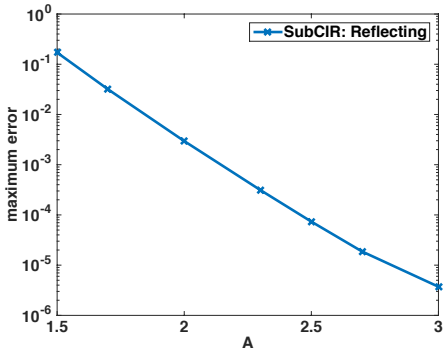

Fig. 2: Convergence of localization error for a European put under the SubCIR model. We plot the maximum localization error of the option price for S_0 over $[80, 120]$ in log scale.

SubCIR model in Figure 2. Here A is specified as reflecting. Although the SubCIR model does not satisfy the conditions in Proposition 3.3, the localization error still converges exponentially.

4.2. Smoothing the Payoff and Convergence of Discretization Error. In view of the result in Section 5.1, we specify the localized boundaries as reflecting in all models that require localization except for the NIG model, in which using killing boundaries allow us to apply fractional FFT which reduces the time complexity (see Remark 5). We set A to be large enough so that the localization error is negligible compared to the discretization error, and we can regard the difference between the approximate price and the benchmark price as the discretization error. In Figure 3, we plot the convergence of discretiztion error in various models for European put and digital call option. As the payoff of these options are not smooth enough, convergence of discretization error displays an oscillatory pattern, and hence we can not apply

Richardson extrapolation to speed up convergence. We propose to use the projection method in [51] to smooth the payoff (see their paper for detailed account of this approach). Figure 3 shows convergence of discretization error now becomes smooth and extrapolation becomes applicable. Furthermore, the slope of the smooth line indicates the convergence order is two. It is interesting to note that although projection introduces error to the initial data, it does not necessarily make the numerical solution less accurate. As Figure 3 shows, using projection leads to more accurate numerical solution for pricing European put options under the subordinate reflected Brownian motion and digital call options under the NIG model. In other cases, for a given grid, projection results in slightly larger error and in the worse case the error is three times of the error using the original payoff. The advantage of projection is that it makes extrapolation possible so that we can obtain highly accurate solution with small grid size, despite that for each grid, the solution under projection might be less accurate.

We next apply extrapolation to the price sequence for the projected payoff for various models and results for the ATM case ($S_0 = K$) are displayed in Table 1. Extrapolation works very well in all models. By extrapolating the price in the first two rows, we already achieve a relative error around 0.01% or 0.001% in a few milliseconds in all cases! This suffices for financial applications where typically a relative error of order 0.1% is good enough (see [8]). We have also tested the accuracy in the OTM ($S_0 > K$) and ITM ($S_0 < K$) case, and results are similar to the ATM case. We do not display such results here to save space.

4.3. Comparison to a PIDE Scheme for the NIG Model. For popular Lévy subordinate Brownian motion models, there already exist several PIDE schemes. Here we compare our method to a very efficient PIDE scheme developed in [16] (hereafter we call it DFV) for the NIG model. The DFV scheme is developed for jump processes with finite activity. To apply it, we approximate the small jump part by a Brownian motion as in [14] and large jumps are also truncated. As mentioned in Remark 6, the DFV schemes has time complexity $O(mn \log_2 n)$, where m is the number of time steps and n is the number of grid points for the space variable. In contrast, our method does not discretize time and the time complexity is $O(n \log_2 n)$. Furthermore, our method does not have approximation error for small and large jumps. Figure 4 compares the DFV scheme and our FDEIG method for European put option for two maturities. We plot the maximum error and the corresponding computation time. It is clear that for all maturities considered, the FDEIG method is faster for given levels of accuracy. Moreover, as expected, the difference becomes more pronounced as the maturity increases.

4.4. The Option Delta. The delta of an option is important for hedging. For a point x_i in the interior of the grid, we approximate $u_x^\phi(t, x_i)$ as

$$u_x^\phi(t, x_i) \approx \frac{u_{h,i+1}^\phi(t) - u_{h,i-1}^\phi(t)}{2h},$$

where $u_{h,i\pm1}^\phi(t)$ approximates $u^\phi(t, x_i \pm h)$. To calculate $u_h^\phi(t)$, we can use the original payoff or the projected payoff. Figure 5 plots the convergence of delta for both payoffs under three models. To obtain the benchmark, we use the transform approach under the NIG model and for the SubCIR and SubJDCEV model, we use an analytical expression for the delta derived from the eigenfunction expansion for the price via term-by-term differentiation (see [40, Eq.(19)]). Like the case for option price, the convergence is oscillatory under the original payoff but smooth under the projected

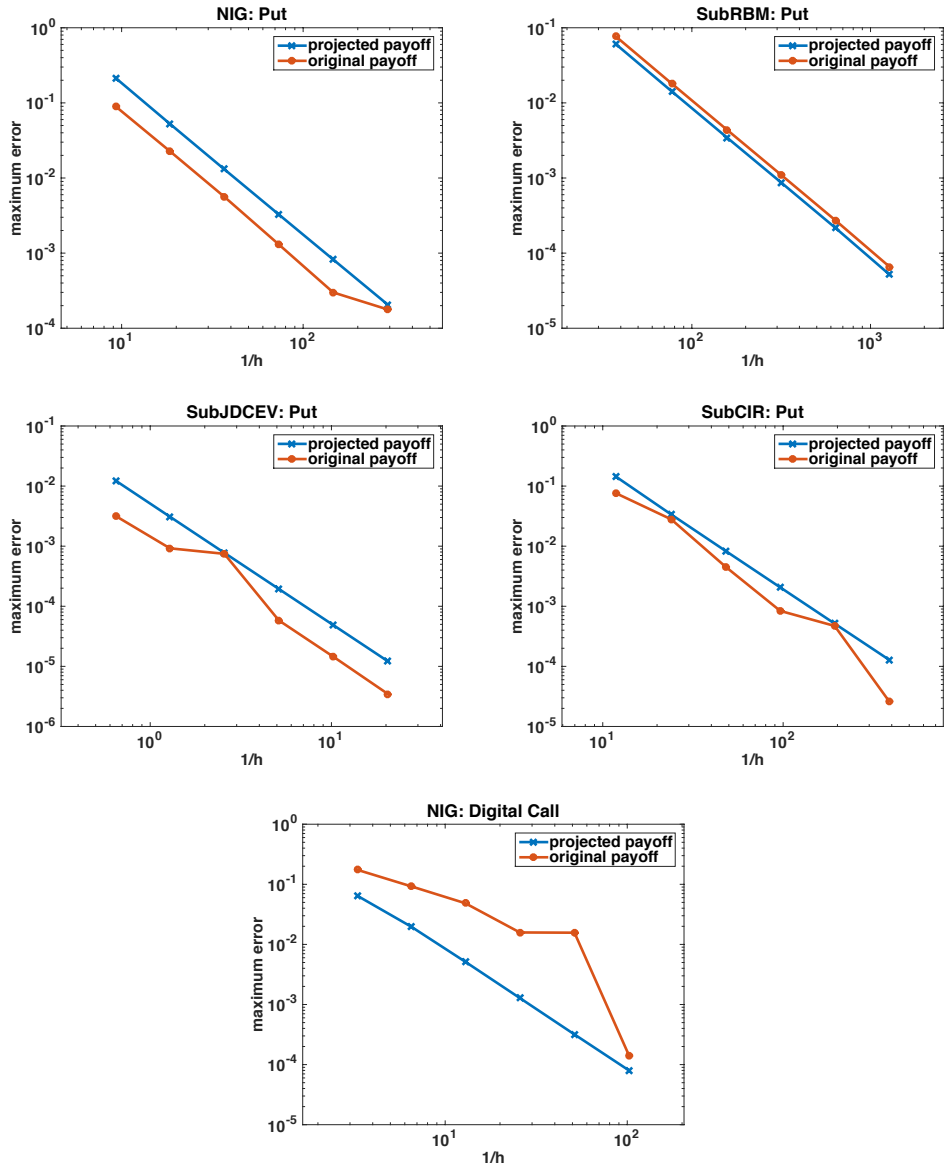

Fig. 3: Convergence of discretization error for various models (log-log scale). We plot the maximum error of the option price for S_0 over $[80, 120]$ for the original and projected payoff.

payoff. Furthermore, the convergence order is two in the projected payoff case. We apply extrapolation to speed up convergence and Table 2 shows the extrapolation results. We are able to achieve high level of accuracy for the delta in a few milliseconds.

5. Conclusions. This paper develops a novel and efficient method named as FDEIG for pricing European options in models based on one-dimensional subordi-

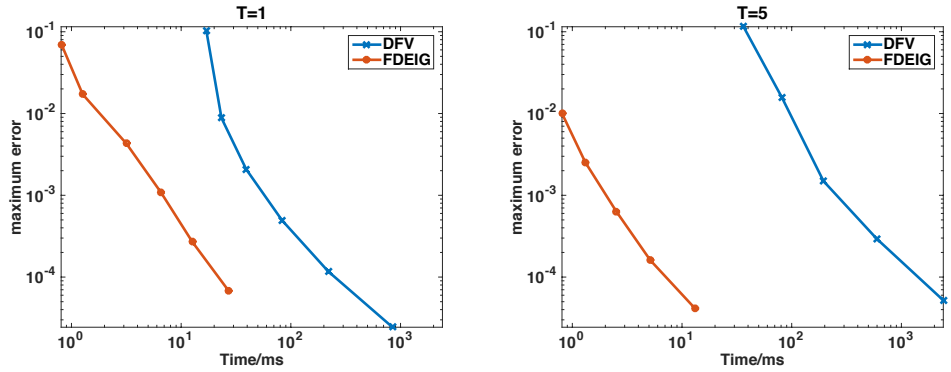

Fig. 4: Comparison of the DFV scheme and the FDEIG method for pricing European put under the NIG model. The localization interval is set to be $(-4, 4)$ in both methods. We plot the maximum error for $S_0 \in [80, 120]$ with the corresponding computation time.

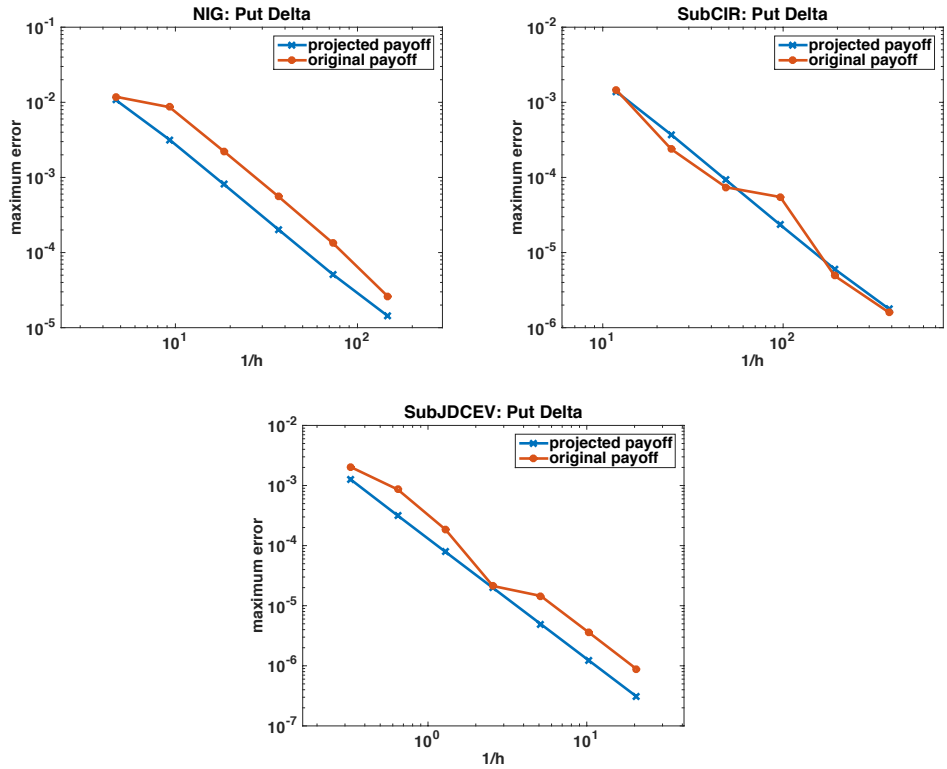

Fig. 5: Convergence of the delta of European put options under the NIG and SubCIR, SubJDCEV models (log-log scale). We plot the maximum error for the delta for $S_0 \in [80, 120]$.

NIG: Put						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	9.528940	-3.37E-02		0.6		
256	9.554093	-8.54E-03	1.98	1.8	9.562478	-1.54E-04
512	9.560486	-2.15E-03	1.99	2.4	9.562617	-1.49E-05
SubRBM: Put						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
16	2.436106	-9.77E-03		0.2		
32	2.443634	-2.24E-03	2.13	0.3	2.446143	2.70E-04
64	2.445333	-5.39E-04	2.05	0.8	2.445899	2.70E-05
SubCIR: Put						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	11.053394	-3.37E-02		2.2		
256	11.078793	-8.29E-03	2.02	6.6	11.087259	1.78E-04
512	11.085021	-2.06E-03	2.01	26.5	11.087097	1.54E-05
SubJDCEV: Put						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	1.655903	-9.71E-03		1.9		
256	1.663168	-2.44E-03	1.99	6.1	1.665590	-2.19E-05
512	1.664999	-6.13E-04	1.99	28.5	1.665609	-2.96E-06
NIG: Digital Call						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	0.412030	-4.97E-03		0.6		
256	0.415784	-1.21E-03	2.03	1.3	0.417035	3.85E-05
512	0.416701	-2.96E-04	2.03	2.5	0.417006	9.41E-06

Table 1: Extrapolation results for the option price when $S_0 = 100$. The "Projected" column shows the option price with projected payoff for various grid size N while the "Extrapolated" column shows the extrapolated value using the price from the "Projected" column in previous and current row. The first error column shows the error for unextrapolated price, while the second error column shows the error for the extrapolated price. The order is defined as $-\log_2(e_N/e_{N/2})$ where e_N is the error for a grid with size N. "ms" stands for milliseconds.

nate diffusions, which are obtained by time changing a one-dimensional diffusion with an independent Lévy or additive subordinator. The diffusion process under consideration is a regular one with drift $\mu(x)$, diffusion coefficient $\sigma(x)$ and killing rate $k(x)$ and it lives on an interval with end-point l and r ($-\infty \le l < r \le \infty$). The boundary behavior can be either natural, exit, entrance or regular specified as killing and reflecting. Our method is applicable if on any compact sub-interval $[a, b] \subset (l, r)$, $\sigma(x) > 0$, $\sigma(x) \in C^4([a, b])$, $\mu(x) \in C^3([a, b])$, $k(x) \in C^2([a, b])$ and the proposed numerical scheme converges if the payoff function is bounded and piecewise continuous (see Proposition 3.2 for the convergence of localization and Proposition 3.12 for the convergence of spatial discretization; the condition for the payoff can be weakened for the convergence of localization). Subject to further regularity conditions, we prove that the localization error converges exponentially (Proposition 3.3) and for smooth payoffs, the discretezation error converges in second order (Proposition 3.9). Since financial payoffs are typically not smooth, we apply a smoothing technique and use extrapolation to speed up convergence. The computation complexity of our method

NIG: Put Delta						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	-0.43645502	2.15E-03		0.7		
256	-0.43806513	5.39E-04	2.00	1.5	-0.43860184	1.97E-06
512	-0.43847061	1.33E-04	2.02	2.5	-0.43860577	-1.96E-06
SubCIR: Put Delta						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	-0.39393376	-7.52E-05		2.2		
256	-0.39387687	-1.83E-05	2.04	7.0	-0.39385790	7.10E-07
512	-0.39386314	-4.53E-06	2.01	24.1	-0.39385856	4.84E-08
SubJDCEV: Put Delta						
N	Projected	Error	Order	Time/ms	Extrapolated	Error
128	-0.27181061	2.99E-05		1.8		
256	-0.27183261	7.87E-06	1.92	6.2	-0.27183994	5.37E-07
512	-0.27183848	2.00E-06	1.98	28.8	-0.27184044	3.93E-08

Table 2: Extrapolation results for the option delta when $S_0 = 100$.

is $O(n^2)$ where n is the number of grid points (excluding killing boundaries) and the complexity can be reduced to $O(n \log_2 n)$ for subordinate Brownian motions. Compared to existing numerical PIDE schemes, the FDEIG method does not discretize time and hence there is no time discretization error. Furthermore, it does not truncate large and small jumps, which is done in existing PIDE schemes. Various numerical experiments confirm the computational efficiency and accuracy of the FDEIG method. We also compare it to a popular PIDE scheme for subordinate Brownian motions and show that it can have significant computational advantages over the existing PIDE scheme for these processes.

While we use finite difference to discretize the diffusion PDE, we remark that finite element can also be used, which leads us to solve some generalized matrix eigenvalue problem in the form $Hx = \lambda Mx$. Compared to the finite difference case, this problem cannot be solved in $O(n^2)$ in general, thus one would expect that using the same number of grid points, the finite difference discretization is faster than the finite element method. This is the primary reason why we choose to use finite difference in this paper. However, the finite element approach can be advantageous in that there is no need to smooth the payoff in order to apply extrapolation and error order can be found quite easily for general payoffs.

The current paper focuses on pricing European options. For path-dependent options, we expect that our method can be extended to price discretely monitored ones with backward induction, and for American options, we will extend our scheme to solve the corresponding variational inequality. In future research, we also plan to consider how to price multivariate options in asset price models based on multivariate subordinate diffusions that are constructed via multivariate subordination in [47].

Appendix A. Proof for Lemmas.

Proof. [Lemma 3.1] (i) A and $-A$ are killing: Let $\zeta' := \inf\{t \geq 0 : M_{t,x} \geq A\}$. Then the lifetime of X^A, $\zeta^A = \min(\zeta, \zeta')$, where ζ is the lifetime of X (X^A can be killed either by hitting A or $-A$, or by the killing rate, whichever occurs first). We

also have $X_s^A = \hat{X}_s 1_{\{\zeta^A > s\}} + \Delta 1_{\{\zeta^A \le s\}}$.

$$u^A(s,x) - u(s,x) = E_x[f(X_s^A)1_{\{\zeta^A>s\}}] - E_x[f(X_s)1_{\{\zeta>s\}}]$$
$$= E_x[f(\hat{X}_s)1_{\{\zeta>s\}}1_{\{\zeta'>s\}}] - E_x[f(\hat{X}_s)1_{\{\zeta>s\}}] = -E_x[f(\hat{X}_s)1_{\{\zeta>s\}}1_{\{\zeta'\le s\}}].$$

Therefore $|u^A(s,x) - u(s,x)| \le E_x[f(\hat{X}_s)1_{\{M_{s,x}\ge A\}}]$. Under the second assumption on f, we have $|u^A(s,x) - u(s,x)| \le \|f\|_\infty P_x(M_{s,x} \ge A)$.
(ii) A and $-A$ are reflecting: $\hat{X}_s^A = \hat{X}_s^A 1_{\{M_{s,x}\ge A\}} + \hat{X}_s 1_{\{M_{s,x}<A\}}$ for $s \ge 0$ and

$$u_A(s,x) = E_x[f(X_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}<A\}}] + E_x[f(X_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}\ge A\}}]$$
$$= E_x[f(\hat{X}_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}<A\}}] + E_x[f(X_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}\ge A\}}]$$
$$= E_x[f(\hat{X}_s)1_{\{\zeta>s\}}1_{\{M_{s,x}<A\}}] + E_x[f(X_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}\ge A\}}].$$

Hence under the first assumption, we have

$$|u_A(s,x) - u(s,x)| = |-E_x[f(\hat{X}_s)1_{\{\zeta>s\}}1_{\{M_{s,x}\ge A\}}] + E_x[f(X_s^A)1_{\{\zeta^A>s\}}1_{\{M_{s,x}\ge A\}}]|$$

$$\le E_x[f(\hat{X}_s)1_{\{M_{s,x}\ge A\}}] + E_x[f(M_{s,x})1_{\{M_{s,x}\ge A\}}] + E_x[f(-M_{s,x})1_{\{M_{s,x}\ge A\}}],$$

where we used monotonicity of f to conclude that $f(X_s^A)$ is bounded by $f(M_{s,x}) + f(-M_{s,x})$. Under the second assumption, $|u^A(s,x) - u(s,x)| \le 2\|f\|_\infty P_x(M_{s,x} \ge A)$. □

Proof. [Lemma 3.4] [25] proves the claim for λ_n and $\varphi_n^{(k)}$ for $k = 0, 1$ based on Volterra's integral equation. We can follow this idea by taking successive differentiation of the integral equation to prove the result for $\varphi_n^{(k)}$ $(k = 2, 3, 4)$. The detailed proof is omitted here. □

Proof. [Lemma 3.7] Our assumption implies that the conditions in Theorem 5.2 of [30] are satisfied, applying which shows that for any $T > 0$, $u \in \mathcal{H}^{\frac{4+\epsilon}{2},4+\epsilon}([0,T] \times [l,r])$. This implies that $\partial_{xxx}u(t,x)$, $\partial_{xxxx}u(t,x)$ are bounded on $[0,T] \times [l,r]$ for any $T > 0$. By the Cauchy-Schwartz inequality, $|f_n| \le \|f\|_2$ for all n, where $\|f\|_2 := (\int_{[l,r]} f^2(x)m(dx))^{1/2}$ is the L^2-norm of f. This fact together with Lemma 3.4 imply that there exists a constant $C > 0$ such that,

$$|\partial_{xxx}u(t,x)| \le \sum_{n=1}^{\infty} |f_n|\, e^{\lambda_n t}\, |\varphi_n'''(x)| \le C \sum_{n=1}^{\infty} n^3 e^{\lambda_n t},$$
$$|\partial_{xxxx}u(t,x)| \le \sum_{n=1}^{\infty} |f_n|\, e^{\lambda_n t}\, |\varphi_n''''(x)| \le C \sum_{n=1}^{\infty} n^4 e^{\lambda_n t}.$$

As $t \to \infty$, both infinite series on the RHS tend to zero. Therefore $\partial_{xxx}u(t,x)$, $\partial_{xxxx}u(t,x)$ are uniformly bounded on $[0,\infty) \times [l,r]$. □

Proof. [Lemma 3.7] Under our assumption, the conditions in Theorem 5.3 of [30] are satisfied, applying which shows that for any $T > 0$, $u \in \mathcal{H}^{\frac{5+\epsilon}{2},5+\epsilon}([0,T] \times [l,r])$. Hence $\partial_{xxx}u(t,x)$, $\partial_{xxxx}u(t,x)$ and $\partial_t\partial_{xxx}u(t,x)$ are bounded on $[0,T] \times [l,r]$. The uniform boundedness of $\partial_{xxx}u(t,x)$, $\partial_{xxxx}u(t,x)$ on $[0,\infty) \times [l,r]$ can be derived in the same way as Lemma 3.7. Now consider $\partial_t\partial_{xxx}u(t,x)$. Applying Cauchy-Schwartz inequality to f_n and Lemma 3.4, there exists $C > 0$ such that

$$|\partial_t\partial_{xxx}u(t,x)| \le \sum_{n=1}^{\infty} |f_n\lambda_n|\, e^{\lambda_n t}\, |\varphi_n'''(x)| \le C \sum_{n=1}^{\infty} n^3 |\lambda_n| e^{\lambda_n t},$$

As $t \to \infty$, the infinite series on the RHS of the inequality tend to zero. Hence $\partial_t \partial_{xxx} u(t,x)$ is uniformly bounded in $[0,\infty) \times [l,r]$. □

Proof. [Lemma 3.8] Under our assumption, the conditions in Theorem 5.3 of [30] are satisfied, applying which shows that for any $T > 0$, $u \in \mathcal{H}^{\frac{5+\epsilon}{2},5+\epsilon}([0,T] \times [l,r])$. Hence $\partial_{xxx} u(t,x)$, $\partial_{xxxx} u(t,x)$ and $\partial_t \partial_{xxx} u(t,x)$ are bounded on $[0,T] \times [l,r]$. The uniform boundedness of $\partial_{xxx} u(t,x)$, $\partial_{xxxx} u(t,x)$ on $[0,\infty) \times [l,r]$ can be derived in the same way as Lemma 3.7. Now consider $\partial_t \partial_{xxx} u(t,x)$. Applying Cauchy-Schwartz inequality to f_n and Lemma 3.4, there exists $C > 0$ such that

$$|\partial_t \partial_{xxx} u(t,x)| \leq \sum_{n=1}^{\infty} |f_n \lambda_n| \, e^{\lambda_n t} \, |\varphi_n'''(x)| \leq C \sum_{n=1}^{\infty} n^3 |\lambda_n| e^{\lambda_n t},$$

As $t \to \infty$, the infinite series on the RHS of the inequality tend to zero. Hence $\partial_t \partial_{xxx} u(t,x)$ is uniformly bounded in $[0,\infty) \times [l,r]$. □

Proof. [Lemma 3.11] We first note that there exists positive constant $C_l < C_u$ which are independent of h such that

$$C_l \leq \|J\|_\infty \leq C_u. \tag{A.1}$$

This can be proved following the argument in the proof of Lemma 1 in [26]. Recall that Λ_k is the k-th eigenvalue of H and let Φ_k be the k-th normalized eigenvector ($\Phi_k^T J \Phi_k = 1$). We obtain an expression for $P_{s,i,j}$. In (3.10), setting the function $\phi(\lambda) = \lambda$ (which corresponds to the no time change case) and f_h to be a vector whose j-th element is one and all others zero, we obtain $P_{s,i,j} = J_{j,j} \sum_{k=1}^{n} e^{\Lambda_k s} \Phi_{k,i} \Phi_{k,j}$ (recall that $n = N + 1 - \#$ of killing boundaries).

We will apply results from [26] and [9]. The boundary conditions for the differential equation in their set-up are homogeneous Dirichlet conditions (corresponding to killing boundaries) and the finite-difference scheme they analyze is a little bit different from ours. Nevertheless one can prove that their results still hold in our finite-difference scheme for both killing and reflecting boundaries by modifying the arguments in these references (we omit such lengthy discussions here). From the theorem in [26], there is a constant C_1 independent of h and k such that

$$\|\Phi_k / d_k\|_\infty \leq C_1 \sqrt{h}, \text{ where } d_k = \sqrt{\sum_{i=1}^{n} \Phi_{k,i}^2}.$$

Since $\sum_{i=1}^{n} \Phi_{k,i}^2 J_{i,i} = 1$, applying (A.1), we obtain $d_k \leq 1/\sqrt{C_l}$. Therefore $\|\Phi_k\|_\infty \leq (C_1/\sqrt{C_l})\sqrt{h}$. By [9, Lemma 3], there is a positive integer k_0, independent of n and constants K_1, K_2, independent of h such that for $k_0 \leq k \leq n$, $K_1 k^2 \pi^2 \leq \Lambda_k \leq K_2 k^2 \pi^2$ which implies that $\sum_{k=1}^{n} e^{\Lambda_k s} \leq (k_0 - 1) + \sum_{k=0}^{\infty} e^{-K_1 k^2 \pi^2 s} \leq C_2$ where C_2 is independent of h. Therefore

$$P_{s,i,j} \leq \max_j |J_{j,j}| \max_{k,i,j} |\Phi_{k,i} \Phi_{k,j}| \sum_{k=1}^{n} e^{\Lambda_k s} \leq C_u (C_1^2 / C_l) C_2 h.$$

By Lemma 3.4, for all k, $\|\varphi_k\|_\infty$ is bounded by a constant. We also have $\lambda_k \sim O(-k^2)$, so $\sum_{k=1}^{\infty} e^{\lambda_k s}$ is also bounded by some constant. From [45], $p(s,x,y) = m(y) \sum_{k=1}^{\infty} e^{\lambda_k s} \varphi_k(x) \varphi_k(y)$. Hence $\sup_{(x,y) \in (l,r) \times (l,r)} p(s,x,y) \leq \|m\|_\infty \|\varphi_k\|_\infty^2 \sum_{k=1}^{\infty} e^{\lambda_k s}$, which is bounded by some constant (the speed density m is continuous on $[l,r]$ and hence bounded). □

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