Time-Changed Ornstein-Uhlenbeck Processes And Their Applications In Commodity Derivative Models*

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Abstract

This paper studies *subordinate Ornstein-Uhlenbeck (OU) processes*, i.e., OU diffusions time changed by Lévy subordinators. We construct their sample path decomposition, show that they possess mean-reverting jumps, study their equivalent measure transformations, and the spectral representation of their transition semigroups in terms of Hermite expansions. As an application, we propose a new class of commodity models with mean-reverting jumps based on subordinate OU process. Further time changing by the integral of a CIR process plus a deterministic function of time, we induce stochastic volatility and time inhomogeneity, such as seasonality, in the models. We obtain analytical solutions for commodity futures options in terms of Hermite expansions. The models are consistent with the initial futures curve, exhibit Samuelson's maturity effect, and are flexible enough to capture a variety of implied volatility smile patterns observed in commodities futures options.

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1 Introduction

The contribution of this paper is two-fold. The first part studies *subordinate Ornstein-Uhlenbeck (SubOU) processes*. A SubOU process can be constructed by time changing an OU diffusion by a Lévy subordinator. SubOU processes are Markov semimartingales with mean-reverting jumps. SubOU transition semigroups possess spectral representations in terms of Hermite expansions. As an application, the second part of the paper develops a new class of analytically tractable commodity models with mean-reverting jumps by modeling the commodity spot price as the (scaled and compensated) exponential of a SubOU process. To model stochastic volatility and time inhomogeneity, such as seasonality, we further time change SubOU processes by the integral of the sum of an independent CIR diffusion and a deterministic function of time. The resulting models have the following features: (1) mean-reverting jumps, (2) stochastic volatility, (3) time inhomogeneity, (4) analytical solutions for futures options in terms of Hermite expansions, (5) consistency with the initial futures curve, (6) Samuelson's maturity effect, and (6) flexibility to capture a variety of implied volatility smile patterns observed in commodity futures options.

The mathematical part of the paper contains a self-contained presentation of SubOU processes. Section 2.1 defines SubOU semigroups as Bochner's subordinates of OU semigroups and gives explicit expressions for their infinitesimal generators based on the application of R.S. Phillips' theorem. This material is classical (see Schilling et al. (2010) for an excellent recent survey of Bochner's subordination and Albeverio and Rudiger (2003), (2005) for the treatment of SubOU semigroups in particular). Section 2.2 defines a class of SubOU Markov semimartingales, gives their local characteristics, proves uniqueness of the associated martingale problem, and proves the mean reversion property of their jumps. While the material in this section follows from the general semimartingale theory (our presentation follows Jacod and Shiryaev (2003)), it has not been presented in the literature in this form. Section 2.3 presents results on equivalent measure transformations for SubOU processes. In particular, a class of locally equivalent measure changes that transform one SubOU process into another SubOU process is characterized, along with a detailed treatment of some special cases important in applications. This section presents original results that, to the best of our knowledge, have not previously appeared in the literature. It serves as the basis for financial applications, characterizing equivalent martingale measures (EMMs) for this class of models. Section 2.4 presents the spectral decomposition of the SubOU semigroup in $L^2(\mathbb{R}, \mathfrak{m})$, where $\mathfrak{m}$ is the Gaussian measure, in terms of Hermite expansions. The L^2 spectral theory of SubOU semigroups has been previously given by Albeverio and Rudiger (2003), (2005). We supplement it with pointwise convergence results and truncation error bounds for the expansion that are important for options pricing.

The second part of the paper provides the development of our commodity futures model. Section 3.1 defines the model for the commodity spot price as the exponential of a SubOU process scaled and compensated so that, under $\mathbb{Q}$, the mean spot price evolves along the fixed initial futures curve. We then explicitly solve for the futures dynamics under $\mathbb{Q}$ in the form of a martingale expansion with basis martingales associated with Hermite polynomials. Section 3.2 demonstrates Samuelson's maturity effect in commodity futures in this class of models. Section 3.3 derives explicit analytical solutions for futures options in terms of Hermite expansions. In section 4 we further time change SubOU processes to induce stochastic volatility and time inhomogeneity and study the resulting commodity futures models. In particular, we derive the futures price process, demonstrate Samuelson's maturity effect, and obtain solutions for futures options. In section 5 we discuss efficient model implementation based on recursions for Hermite polynomials and present model calibration examples to futures options on a variety of commodities, including metals, energies and agriculturals. Appendix A contains a number of results on the CIR process needed in the development of models with stochastic volatility. Proofs are collected in Appendix B.

In the rest of this introduction we discuss relationships of models developed in this paper to the literature. We start with a brief survey of the commodity derivatives modeling literature. Mean reversion and jumps are two of the salient features of commodities prices (see monographs Eydeland and Wolyniec (2003), Geman (2005), and Geman (2008) for introduction to commodity and energy derivatives markets and modeling). Mean reversion in commodities markets is well documented in numerous empirical studies in the literature (e.g., Bessembinder et al. (1995), Pindyck (2001), Casassus and Collin-Dufresne (2005)). To capture the mean reversion property, the classical commodity models are based on OU diffusions. The simplest such model is the exponential OU model of Schwartz (1997). In this model the commodity spot price is assumed to follow the exponential of an OU process with constant long-run mean level, rate of mean reversion, and volatility. While the OU process itself lives on the whole real line, taking the exponential leads to the positive process for the commodity spot price. The geometric OU model plays the same role in commodity markets that the geometric Brownian motion model plays in the equity markets, serving as the simplest analytically tractable commodity derivatives pricing model. Being inherently the spot price model, the futures curve is derived endogenously in this model and, hence, does not generally match the futures curve observed in the market. This situation is similar to the Vasicek (1977) model of the short interest rate, where the yield curve is derived endogenously in the model and does not generally match the market yield curve. Similar to how the Vasicek model is extended to match an arbitrary market yield curve by making the long-run mean level of the short rate time-dependent (e.g., Hull and White (1993)), the exponential OU model can be extended to match an arbitrary market-observed futures curve (e.g., Clelow and Strickland (1999)). In this model futures prices of all maturities follow continuous martingales under $\mathbb{Q}$.

Along with mean reversion, discontinuous price movements (jumps) are another salient feature of commodity and energy markets. While jumps are a ubiquitous feature of all asset prices and financial variables, from equities to foreign exchange to interest rates, commodity and energy prices exhibit particularly large and frequent jumps, perhaps more so than other asset classes (see, e.g., Hilliard and Reis (1999), Deng (1999), Geman and Roncoroni (2005) for empirical evidence of jumps in commodity and energy prices). The question then arises as to how to extend commodity models based on mean-reverting OU diffusions to jumps. The first line of attack is to add a jump component to the diffusive mean-reverting component to form a jump-diffusion process similar to Merton (1976) classical jump-diffusion model widely used in equities. A variety of jump-diffusion models along these lines have been introduced in commodity markets (e.g., Hilliard and Reis (1998), Hilliard and Reis (1999), Deng (1999), Yan (2002), Benth and Šaltytė Benth (2004), Geman and Roncoroni (2005), Andersen (2008) and Crosby (2008)). Virtually all of the jump-diffusion models in the literature, with the exception of Geman and Roncoroni (2005) and Andersen (2008), add state-independent jumps to the mean-reverting diffusion. The resulting models exhibit mean reversion due to the OU drift, but do not have mean reversion in their jump measure that remains state-independent. That is, upon arrival, the direction of the jump and the probability distribution of its amplitude are independent of the current state of the process. The drift acting upon the process between the jumps is forced to account for all of the mean reversion in these models. A model with mean reverting jumps would, in contrast, feature state-dependent mean reverting jumps with the jump direction and the jump amplitude dependent on the current state of the process.

In contrast to jump-diffusion models with state-independent jumps, Geman and Roncoroni (2005) propose a jump-diffusion model with Poisson jumps independent of the diffusion component but with jump direction dependent on the pre-jump state of the process. They show that such models capture some of the empirical properties of electricity price data. However, analytical solutions for futures options have not been obtained in their model. Andersen (2008) considers jump-diffusion processes with jumps driven by a continuous time Markov chain whose states are interpreted as different market regimes. Jumps in this framework are dependent on the regimes and, hence, are state dependent. However, option pricing in this regime-switching framework is generally highly non-trivial unless some simplifying assumptions are made.

In this paper we take an alternative approach to the previous literature on commodity and energy models with jumps. Instead of adding state-independent jumps to the mean-reverting diffusion process, we time change the mean-reverting OU diffusion with a Lévy subordinator to yield a pure jump or a jump-diffusion process (depending on whether or not the subordinator has a positive drift) with state-dependent and mean reverting jumps. As such, our models can be viewed as a commodity markets counterpart of the time-changed Lévy process-based models in equity markets by Madan et al. (1998), Barndorff-Nielsen (1998), Geman et al. (2001), Carr et al. (2003), and Carr and Wu (2004). However, since mean reversion is the crucial feature of commodity markets, instead of time changing Brownian motion as in those references, we time change OU diffusions and obtain pure jump or jump-diffusion Markov semimartingales with state-dependent mean reverting jumps. Similar to Lévy-based models in equity markets, our models based on SubOU processes calibrate well to a variety of implied volatility smiles in commodity markets when the maturity is fixed.

To induce stochastic volatility (the need for stochastic volatility in energy markets has been advocated by Eydeland and Geman (1998)), we further time change these jump processes with the integral of an activity rate (stochastic volatility) that follows a CIR process. This is similar to the approach of Carr et al. (2003) and Carr and Wu (2004), but in contrast to those references we time change Markov jump processes that are generally not Lévy processes. This yields pure jump or jump-diffusion models with stochastic volatility modulating jump amplitudes. To additionally introduce explicit time dependence to capture the term structure of at-the-money (ATM) volatilities observed in commodity futures options markets (e.g., the seasonality effects in volatility, as well as the sharply declining term structure of ATM volatility often seen in some commodity futures options), we add a purely deterministic function of time to the CIR activity rate (turning it into the so-called CIR++ process, e.g., Brigo and Mercurio (2006)). Such models with mean-reverting jumps, stochastic volatility, and time dependence can be calibrated to the entire volatility surface across both the strike and maturity dimensions.

To conclude this introduction, we discuss analytical and computational aspects. While the time changed Lévy models of Carr et al. (2003) lead to fast and efficient option pricing by means of Fourier analysis (see Carr and Madan (1999) for the Fast Fourier Transform methodology and Feng and Linetsky (2008) and Feng and Linetsky (2009) for the closely related Hilbert transform methodology), our time changed OU models also lead to analytical option pricing, but by different mathematical means. While in the context of Lévy processes one exploits the explicit knowledge of the characteristic function, in the context of OU processes we exploit the explicit knowledge of the eigenfunction expansion of the SubOU transition semigroup. The eigenfunction expansion method is a powerful tool for pricing contingent claims written on symmetric Markov processes (see Linetsky (2004) and Linetsky (2007) for surveys). It is particularly well suited to time changes since the time variable enters the eigenfunction expansion of the transition semigroup only through the exponentials $e^{-\lambda_n t}$ and, after the time change with a Lévy subordinator, the eigenfunction expansion has the same form as for the original process, but with $e^{-\lambda_n t}$ replaced with $e^{-\phi(\lambda_n)t}$, where $\phi(\lambda)$ is the Laplace exponent of the subordinator. We note that the seminal paper by Bochner (1949) already contained this observation (see Eq.(11) in Bochner (1949); further see Albeverio and Rudiger (2003), (2005) for the mathematical development of subordination of symmetric Markov processes). In Mathematical Finance, this observation has been previously exploited by Albanese and Kuznetsov (2004) in the context of volatility smile modeling for equities, by Boyarchenko and Levendorskiĭ (2007) in the context of interest rate modeling, and by Mendoza et al. (2010) in the context of unified credit-equity modeling.

2 Subordinate Ornstein-Uhlenbeck Processes

2.1 SubOU Semigroups

We start with an OU semigroup $(\mathcal{P}_t)_{t\geq 0}$ defined on $\mathfrak{B}_b(\mathbb{R})$ (the space of bounded Borel measurable functions), where $\mathcal{P}_t f(x) = \int_{\mathbb{R}} f(y)p(t,x,y)dy$ with the OU transition kernel:

$$p(t,x,y) = \frac{1}{\sqrt{\frac{\pi\sigma^2}{\kappa}(1-e^{-2\kappa t})}} \exp\Big\{ -\frac{\big(y-x+(x-\theta)(1-e^{-\kappa t})\big)^2}{\frac{\sigma^2}{\kappa}(1-e^{-2\kappa t})}\Big\}. \tag{2.1}$$

$p(t,x,y)$ is the transition density of an OU diffusion with the rate of mean reversion $\kappa > 0$, long-run level $\theta \in \mathbb{R}$, and volatility $\sigma > 0$. $(\mathcal{P}_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $\mathfrak{B}_b(\mathbb{R})$. Restricted to $C_0(\mathbb{R})$ (the space of continuous functions vanishing at infinity), it is a Feller semigroup, and $C_c^\infty(\mathbb{R})$ is a core of the domain $D(\mathcal{G})$ of its infinitesimal generator $\mathcal{G}$ acting on $C_c^2(\mathbb{R})$ (the subscript c stands for functions with compact support) by $\mathcal{G}f(x) = \kappa(\theta - x)f'(x) + \frac{1}{2}\sigma^2 f''(x)$ (c.f. Duffie et al. (2003) Theorem 2.7).

Consider a vaguely continuous convolution semigroup $(q_t)_{t\geq 0}$ of probability measures on $\mathbb{R}_+$ (c.f. Schilling et al. (2010) Definition 5.1). For each t, $q_t([0,\infty)) = 1$ (we consider only conservative case in this paper), and its Laplace transform is given by the Lévy-Khintchine formula with the Laplace exponent $\phi(\lambda)$ defined for all $\lambda \geq 0$:

$$\int_{[0,\infty)} e^{-\lambda s} q_t(ds) = e^{-t\phi(\lambda)}, \quad \phi(\lambda) = \gamma\lambda + \int_{[0,\infty)} (1-e^{-\lambda s})\nu(ds)$$

with drift $\gamma \geqslant 0$ and Lévy measure ν satisfying the integrability condition $\int_{[0,\infty)}(s\wedge 1)\nu(ds) < \infty$. $(q_t)_{t\geq 0}$ is the family of transition probabilities of a subordinator, i.e., a non-negative Lévy process starting at the origin (c.f. Bertoin (1996) or Schilling et al. (2010)).

We define a subordinate semigroup $(\mathcal{P}_t^\phi)_{t\geq 0}$ on $\mathfrak{B}_b(\mathbb{R})$ as the Bochner integral:

$$\mathcal{P}_t^\phi f(x) := \int_{[0,\infty)} \mathcal{P}_s f(x) q_t(ds).$$

This procedure is called Bochner's subordination (c.f. Schilling et al. (2010) Definition 12.2). From Schilling et al. (2010) Proposition 12.1, the subordinate semigroup $(\mathcal{P}_t^\phi)_{t\geq 0}$ is also a strongly continuous contraction semigroup on $\mathfrak{B}_b(\mathbb{R})$. We call it the *SubOU semigroup* with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$. The superscript ϕ in $(\mathcal{P}_t^\phi)_{t\geq 0}$ signifies that it is constructed by subordinating the semigroup $(\mathcal{P}_t)_{t\geq 0}$ with the convolution semigroup of a subordinator with the Laplace exponent ϕ.

From Jacob (2001) Corollary 4.3.4, a Feller semigroup remains a Feller semigroup after subordination. It implies that $(\mathcal{P}_t^\phi)_{t\geq 0}$ restricted to $C_0(\mathbb{R})$ is Feller. Its infinitesimal generator is given by Phillips' Theorem (Sato (1999) Theorem 32.1). The assertion on its core comes from Sato (1999) Proposition 32.5 (ii) and the fact that $C_c^\infty(\mathbb{R})$ is a core of $D(\mathcal{G})$. We summarize these results in the following.

Theorem 2.1. *(i) A SubOU semigroup with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$ is a Feller semigroup. (ii)Let $\mathcal{G}^{\phi}$ be its infinitesimal generator. Then $C_c^{\infty}(\mathbb{R})$ is a core of $D(\mathcal{G}^{\phi})$, $C_c^2(\mathbb{R}) \subseteq D(\mathcal{G}^{\phi})$, and for any $f \in C_c^2(\mathbb{R})$,*

$$\mathcal{G}^{\phi} f(x) = \frac{1}{2}\gamma\sigma^2 f''(x) + b(x)f'(x) + \int_{y\neq 0} \left(f(x+y) - f(x) - y1_{\{|y|\leqslant 1\}}f'(x)\right)\Pi(x, dy),$$

with the state-dependent Lévy measure $\Pi(x, dy) = \pi(x, y)dy$ with density defined for all $y \neq 0$

$$\pi(x, y) = \int_{[0,\infty)} p(s, x, x+y)\nu(ds), \tag{2.2}$$

where $p(t, x, y)$ is the OU transition density (2.1). The drift with respect to the truncation function $y1_{\{|y|\leqslant 1\}}$ is

$$b(x) = \gamma\kappa(\theta - x) + \int_{[0,\infty)} \left(\int_{\{|y|\leqslant 1\}} yp(s, x, x+y)dy\right)\nu(ds).$$

Remark 2.1. (i) On $C_c^{\infty}(\mathbb{R})$, $\mathcal{G}^{\phi}$ can be represented as a pseudo-differential operator (PDO) (see for example Schnurr (2009) Corollary 1.21) $\mathcal{G}^{\phi} f(x) = -p(x, D)f(x) = -\int_{\mathbb{R}} p(x, \xi)\hat{f}(\xi)e^{ix\xi}d\xi$, where $\hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}f(x)dx$ is the Fourier transform of $f(x)$, and $p(x, \xi)$ is called the symbol of the PDO and is expressed as $p(x, \xi) = \frac{1}{2}\gamma\sigma^2\xi^2 - ib(x)\xi - \int_{y\neq 0}\left(e^{i\xi y} - 1 - i\xi y1_{\{|y|\leqslant 1\}}\right)\Pi(x, dy)$. Note that $p(x, \xi)$ is a continuous negative definite function (CNDF) for each x (c.f. Jacob (2001) Definition 3.6.5).
(ii) $\pi(x, y)$ satisfies the condition $\int_{y\neq 0}(y^2 \wedge 1)\pi(x, y)dy < \infty$ for each x. This is a direct result from the representation theorem for CNDF. See Jacob (2001) Theorem 3.7.7.
(iii) The Lévy measure of the SubOU semigroup has finite activity if and only if the Lévy measure of the subordinator has finite activity, which is justified by interchanging the order of integration in $\int_{y\neq 0}\int_{[0,\infty)} p(s, x, x+y)\nu(ds)dy$ by Tonelli's Theorem.
(iv) In general, it is not true that we can interchange the order of integration in $\int_{[0,\infty)}\int_{|y|\leq 1} yp(s, x, x+y)dy\nu(ds)$. However, if the Lévy density satisfies the integrability condition $\int_{|y|\leqslant 1} |y|\pi(x, y)dy < \infty$, then the interchange is valid and the truncation is not needed. It can be shown that this integrability condition is equivalent to $\int_0^1 \sqrt{s}\nu(ds) < \infty$ (if $x \neq \theta$) and $\int_0^1 \nu(ds) < \infty$ (if $x = \theta$). In this case the generator takes the simpler form on $C_c^2(\mathbb{R})$:

$$\mathcal{G}^{\phi} f(x) = \frac{1}{2}\gamma\sigma^2 f''(x) + \gamma\kappa(\theta - x)f'(x) + \int_{\mathbb{R}} (f(x+y) - f(x))\,\pi(x, y)dy.$$

2.2 SubOU Processes as Markov Semimaringales

Definition 2.1. *A time-homogeneous Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, X, \mathbb{P}^x)_{x\in\mathbb{R}}$ with state space $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is called a subordinate OU (SubOU) process with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$ if its semigroup is a SubOU semigroup with the same generating tuple.*

Since a SubOU semigroup is Feller, a SubOU process is a Feller process. Every Feller process has a càdlàg modification (c.f. Jacob (2005) Theorem 3.4.9 or Revuz and Yor (1999) Theorem III.2.7), so immediately we have the following

Corollary 2.1. *Every SubOU process admits a càdlàg modification.*

We will always consider càdlàg SubOU processes in this paper. From now on, without explicit mention, we will assume that $(X, (\mathbb{P}^x)_{x\in\mathbb{R}})$ is the canonical realization of a given SubOU semigroup defined on $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0})$, where $\Omega = \mathbb{D}(\mathbb{R})$ (the Skorohod space of càdlàg functions with values in $\mathbb{R}$, c.f. Jacod and Shiryaev (2003) Definition VI.1.1), $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$, and $\mathcal{F}^0 = \bigvee_{t\geq 0} \mathcal{F}_t^0$.

Schnurr (2009) gives an excellent discussion on the connection between càdlàg Feller processes and semimartingales. The Feller property of the SubOU process together with $C_c^\infty(\mathbb{R}) \subseteq D(\mathcal{G}^\phi)$ allows us to conclude that it is a semimartingale w.r.t. every $\mathbb{P}^x$ with $x \in \mathbb{R}$ (c.f. Schnurr (2009) Theorem 3.1). From Schnurr (2009) Theorem 3.14, the pseudo-differential operator representation of the infinitesimal generator $\mathcal{G}^\phi$ gives us the triplet (B, C, Π) of semimartingale characteristics of the SubOU process. For the definition of semimartingale characteristics see Jacod and Shiryaev (2003) Chapter II.

Theorem 2.2. *(i) The SubOU process $(\Omega, \mathcal{F}^0, (\mathcal{F}_t^0)_{t\geq 0}, X, \mathbb{P}^x)_{x\in\mathbb{R}}$ with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$ is a semimartingale w.r.t. every $\mathbb{P}^x$ and admits semimartingale characteristics (B, C, Π) w.r.t to the truncation function $h(x) = x1_{\{|x|\leq 1\}}$, where*

$$B_t(\omega) = \int_0^t \Big[\gamma\kappa(\theta - X_{s-}(\omega)) + \int_0^\infty \int_{\{|y|\leq 1\}} yp(u; X_{s-}(\omega), X_{s-}(\omega) + y)dy\nu(du)\Big]ds,$$

$$C_t(\omega) = \gamma\sigma^2 t, \quad \Pi(\omega, dt, dy) = \pi(X_{t-}(\omega), y)dtdy,$$

where $\pi(x, y)$ is given in Theorem 2.1.
(ii) Denote by μ^X the integer-valued random measure associated with the jumps of X (c.f. Jacod and Shiryaev (2003) Proposition II.1.16) and X^c the continuous local martingale part of X. Then X has the following sample path decomposition (under the starting point x):

$$X_t(\omega) = x + B_t(\omega) + X_t^c(\omega) + h(x) * (\mu^X - \Pi)_t(\omega) + (x - h(x)) * \mu_t^X(\omega) \tag{2.3}$$

with the quadratic variation of the continuous part $[X^c, X^c]_t(\omega) = C_t(\omega) = \gamma\sigma^2 t$ ($*$ denotes integration w.r.t. a random measure).

(iii) If X' is an $\mathbb{R}$-valued semimartingale defined on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t\geq 0}, \mathbb{P}')$ with $\mathbb{P}'(X_0' = x) = 1$ and with the semimartingale characteristics (B', C', Π') given in (1), where X is replaced by X', then $\mathbb{P}' \circ X'^{-1} = \mathbb{P}^x$.

The proof of Part (iii) of Theorem 2.2 is given in Appendix B. It essentially says the solution to the Martingale Problem in the canonical space setting as defined in Jacod and Shiryaev (2003) Definition III.2.4 is unique. This is a key result for the study of locally equivalent measure changes for SubOU processes in section 2.3.

Remark 2.2. (i) It is clear that the SubOU process is a jump-diffusion process if $\gamma > 0$ and a pure jump process if $\gamma = 0$.
(ii) If $\int_0^1 \sqrt{s}\nu(ds) < \infty$, then $\int_{|y|\leqslant 1} |y|\pi(x, y)dy < \infty$ for all $x \neq \theta$. Hence $|h(x)| * \Pi \in \mathscr{A}_{loc}^+$, which implies $h(x) * (\mu^X - \Pi) = h(x) * \mu^X - h(x) * \Pi$ (c.f. Jacod and Shiryaev (2003) Proposition II.1.28) and

$$X_t(\omega) = x + \int_0^t \gamma\kappa(\theta - X_{s-}(\omega))ds + X_t^c(\omega) + x * \mu_t^X(\omega).$$

Hence, in this case, the jump part of the SubOU process is of finite variation.

A SubOU process is a process with mean-reverting jumps. The mean reversion property of the state-dependent SubOU Lévy measure $\Pi(x, \cdot)$ is characterized in the following.

Theorem 2.3. *For any $y > 0$, we have*

(i) If $x > \theta$, then $\pi(x, -y) > \pi(x, y)$, and $\Pi(x, (-\infty, -y)) > \Pi(x, (y, \infty))$.

(ii) If $x < \theta$, then $\pi(x, -y) < \pi(x, y)$, and $\Pi(x, (-\infty, -y)) < \Pi(x, (y, \infty))$.

(iii) If $x = \theta$, then $\pi(x, -y) = \pi(x, y)$, and $\Pi(x, (-\infty, -y)) = \Pi(x, (y, \infty))$.

This theorem tells us that when the current state x is above (below) the long-run level θ, a downward (upward) jump is more likely to occur. When $x = \theta$, the intensity of downward and upward jumps are equal. This mean-reverting nature of jumps makes SubOU processes a natural candidate for modeling mean-reverting prices and other financial variables. If $\gamma = 0$, a SubOU process is a pure jump process with mean-reverting jumps. If $\gamma > 0$, it is a jump-diffusion process with mean-reverting diffusion drift and mean-reverting jumps.

Figure 1 plots SubOU Lévy densities when ν are Lévy measures of a compound Poisson process with exponential jump sizes and an inverse Gaussian (IG) process.

Remark 2.3. **Time Change Interpretation of Bochner's Subordination** The semigroup $(\mathcal{P}_t)_{t\geq 0}$ gives rise to an OU diffusion process X. The vaguely continuous convolution semigroup of probability measures $(q_t)_{t\geq 0}$ gives rise to a subordinator T. Assume that both X and T are defined on the same probability space and are independent. Then the time changed or subordinate process $X_t^\phi := X_{T_t}$ is again a Markov process. By independence of X and T, the associated operator semigroup is given by

$$\mathcal{P}_t^\phi f(x) = \mathbb{E}[f(X_{T_t})] = \int_{[0,\infty)} \mathbb{E}_x[f(X_s)] q_t(ds) = \int_{[0,\infty)} \mathcal{P}_s f(x) q_t(ds).$$

That is, X_t^ϕ is a SubOU process according to our definition, and Bochner's subordination can be interpreted as a stochastic time change with respect to an independent subordinator (cf. Schilling, Song and Vondracek (2010) p.141).

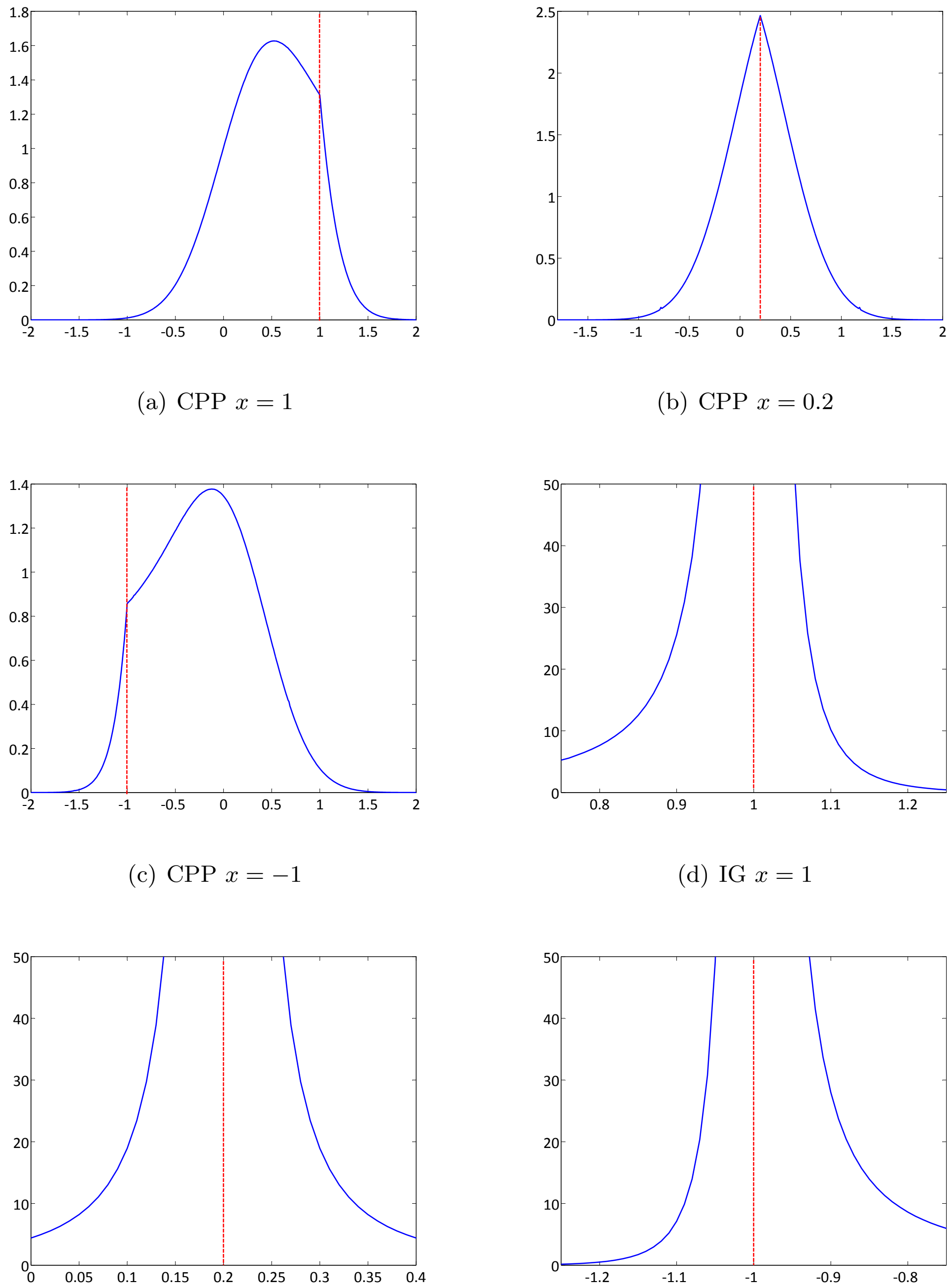
Remark 2.4. SubOU Markov semimartingales admit a representation in terms of a Brownian motion and an independent Poisson random measure. Explicit expressions follow from Cinlar and Jacod (1981) Theorem 3.13 and are omitted due to space constraints.

2.3 Equivalent Measure Transformations for SubOU Processes

For building financial models based on SubOU processes, we are interested in locally equivalent measure changes[1] that transform a SubOU process with a given generating tuple into another SubOU process with another generating tuple. We can then build financial models with SubOU processes under both the physical and the risk-neutral measures, and determine how the generating tuple of the SubOU process changes under the measure change.

As before, Ω is the space of all càdlàg functions taking values in $\mathbb{R}$. In this section we follow Jacod and Shiryaev (2003). In order to use their results, we use the right-continuous version of the filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ with $\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+}^0$ and $\widetilde{\mathcal{F}} = \bigvee_{t\geq 0} \widetilde{\mathcal{F}}_t = \mathcal{F}^0$. Let X be the canonical process. It is clear that if X is a SubOU process, it is also Markov and a SubOU process w.r.t. $(\widetilde{\mathcal{F}})_{t\geq 0}$. We fix the truncation function $h(x) = x1_{\{x\leq 1\}}$. Let $\mathbb{P}_0$ be a probability measure on $(\Omega, \widetilde{\mathcal{F}}_0)$ taken to be the initial distribution. Following Jacod and Shiryaev (2003) Definition III.2.4, we call a probability measure $\mathbb{P}$ on $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0})$ a solution to the martingale problem associated with $(\widetilde{\mathcal{F}}_0, X)$ and $(\mathbb{P}_0; B, C, \nu)$, where (B, C, ν) are given semimartingale characteristics, if the

[1]Two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ are said to be locally equivalent, if $\mathbb{P}_1|_{\mathcal{F}_t} \sim \mathbb{P}_2|_{\mathcal{F}_t}$ for each $t \geq 0$, where $\mathbb{P}|_{\mathcal{F}_t}$ is the restriction of measure $\mathbb{P}$ on the σ-filed $\mathcal{F}_t$.



(a) CPP $x = 1$

(b) CPP $x = 0.2$

(c) CPP $x = -1$

(d) IG $x = 1$

(e) IG $x = 0.2$

(f) IG $x = -1$

Figure 1: State-dependent Lévy densities of SubOU processes with $\theta = 0.2$, $\kappa = 1$, and $\sigma = 0.6$ when ν is the Lévy measure of a compound Poisson process with exponential jumps (arrival rate $\alpha = 2$, reciprocal of mean jump size $\eta = 1$) and an inverse Gaussian process (mean rate $\mu = 1$, variance rate $\nu = 1$) with $x = -1$, 0.2, 1. To emphasize the value of the current state, the horizontal axis plots the post jump state (not the jump size).

following hold: (i) the restriction $\mathbb{P}|_{\widetilde{\mathcal{F}}_0} = \mathbb{P}_0$; (ii) X is a semimartingale on the stochastic basis $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \mathbb{P})$ with characteristics (B, C, ν) relative to the truncation function h. The following proposition is crucial in proving the necessary and sufficient conditions for locally equivalent measure change.

Proposition 2.1. *Let (B, C, Π) be the SubOU semimartingale characteristics defined in Theorem 2.2. The solution to the martingale problem $(\sigma(X_0), X|\mathbb{P}_0; B, C, \Pi)$ exists and is unique. Moreover, local uniqueness holds.*

See Jacod and Shiryaev (2003) Definition III.2.35 for the definition of local uniqueness. The existence of the solution is quite obvious. Given a SubOU semigroup with generating tuple corresponding to the given SubOU semimartingale characteristics (B, C, ν), we can construct a time-homogeneous universal Markov process on the space of càdlàg functions taking values in $\mathbb{R}$. Such a process is a semimartingale with characteristics (B, C, ν) by Theorem 2.2 under every $\mathbb{P}^x$, and set $\mathbb{P}(A) = \int \mathbb{P}^x(A)\mathbb{P}_0(dx)$ for any $A \in \mathcal{F}$. The uniqueness follows from part (iii) of Theorem 2.2. The local uniqueness is a result of uniqueness and the Markov property of the process by Jacod and Shiryaev (2003) Theorem III.2.40. We then have the following.

Theorem 2.4. *Let $\mathbb{P}$ and $\mathbb{P}'$ be two probability measures on $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0})$ such that the canonical process is a SubOU process with generating tuples $(\kappa, \theta, \sigma, \gamma, \nu)$ and $(\kappa', \theta', \sigma', \gamma', \nu')$, respectively, and with initial distributions $\mathbb{P}_0$ and $\mathbb{P}'_0$, respectively. Then the following two statements are equivalent.*

(1) $\mathbb{P}$ and $\mathbb{P}'$ are locally equivalent, i.e., $\mathbb{P}|_{\widetilde{\mathcal{F}}_t} \sim \mathbb{P}'|_{\widetilde{\mathcal{F}}_t}$ for every $t \geq 0$.

(2) The following conditions are satisfied:
(i) $\mathbb{P}_0 \sim \mathbb{P}'_0$; (ii) $\gamma\sigma^2 = \gamma'\sigma'^2$;
(iii) For every $x \in \mathbb{R}$, the Hellinger condition $\int_{y\neq 0} \left(\sqrt{\pi'(x,y)} - \sqrt{\pi(x,y)}\right)^2 dy < \infty$ holds, where $\pi(x,\cdot)$ and $\pi'(x,\cdot)$ are defined as in Theorem 2.1.

(3) Furthermore, suppose these conditions are satisfied. Define

$$\beta_t(\omega) := \frac{(\gamma'\kappa'\theta' - \gamma\kappa\theta) - (\gamma'\kappa' - \gamma\kappa)X_{t-}(\omega)}{\gamma\sigma^2} 1_{\{\gamma\neq 0\}}, \quad \text{and} \quad Y(\omega, t, y) := \frac{\pi'(X_{t-}(\omega), y)}{\pi(X_{t-}(\omega), y)}.$$

Let X^c and μ^X denote the continuous martingale part of X and the jump measure associated with X. Then $N = \beta \cdot X^c + (Y - 1) * (\mu^X - \Pi)$ is a $\mathbb{P}$-local martingale, and the Radon-Nikodym density process D of $\mathbb{P}'$ w.r.t. $\mathbb{P}$ equals to the Doleans-Dade stochastic exponential $\mathscr{E}(N)$ of N.

Remark 2.5. Define $\varphi(x,y) := \ln(\pi'(x,y)/\pi(x,y))$. The Hellinger condition $\int_{y\neq 0} \left(\sqrt{\pi'(x,y)} - \sqrt{\pi(x,y)}\right)^2 dy < \infty$ is equivalent to the following (similar to Remark 33.3 of Sato (1999)):

$$\int_{\{y:|\varphi(x,y)|\leq 1\}} \varphi(x,y)^2\pi(x,y)dy < \infty, \int_{\{y:\varphi(x,y)>1\}} \pi'(x,y)dy < \infty, \int_{\{y:\varphi(x,y)<-1\}} \pi(x,y)dy < \infty.$$

Intuitively, when π and π' both have infinite activity, the Hellinger condition says that the region where large perturbations of the jump density occurs should not be arbitrarily close to the origin.

Remark 2.6. The limiting case $\kappa = 0$ corresponds to the subordinate Brownian motion without drift. Theorem 2.4 still holds when $\kappa = 0$ and/or $\kappa' = 0$. When $\kappa > 0$ and $\kappa' = 0$, it characterizes locally equivalent measure transformations of SubOU processes into subordinate Brownian motions without drift. When $\kappa = \kappa' = 0$, Theorem 2.4 reduce to the special case of Theorem 33.1 of Sato (1999) for Lévy processes specialized to the case of subordinate Brownian motions.

The general Hellinger condition is difficult to check. We wish to derive restrictions it places on SubOU generating tuples that can be transformed into each other under locally equivalent measure changes. It can be easily shown that: (i) If both ν and ν' are Lévy measures of finite activity subordinators, then the Hellinger condition is automatically satisfied. (ii) If ν is a Lévy measure of a finite activity subordinator and ν' is a Lévy measure of an infinite activity subordinator (or vice versa), then the Hellinger condition is not satisfied. Thus, equivalent measure changes cannot transform a SubOU process with a finite activity subordinator into a SubOU process with an infinite activity subordinator, and vice versa.

We now investigate the case when ν and ν' are Lévy measures of infinite activity subordinators. To verify the Hellinger condition in this case, we need to study the asymptotic behavior of the SubOU Lévy density $\pi(x, y)$ given in Eq.(2.2) as $y \to 0$. The following proposition shows that it is equivalent to the asymptotic behavior of the Lévy density of some subordinated Brownian motion.

Proposition 2.2. *Let $\pi(x, y)$ be the Lévy density of a SubOU process with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$. Suppose $\pi(x, y) \to \infty$ as $y \to 0$. For each fixed $x \in \mathbb{R}$, let $\overline{\pi}_x(y)$ be the Lévy density of a subordinate Brownian motion starting at 0 with drift $\kappa(\theta - x)$, volatility σ, and the same γ and ν. Then $\lim_{y\to 0} \pi(x, y)/\overline{\pi}_x(y) = 1$.*

We can further show that the asymptotics of the Lévy density of subordinate Brownian motion does not depend on drift. We then have.

Proposition 2.3. *The asymptotics of $\pi(x, y)$ as $y \to 0$ does not depend on κ, θ, and x.*

Hence, κ and θ can be freely changed by locally equivalent measure changes. In particular, $\kappa > 0$ can be changed to $\kappa = 0$ by a locally equivalent measure change. The problem of investigating the Hellinger condition now reduces to finding the asymptotics of the Lévy density of a subordinate Brownian motion. Song and Vondraček (2009) is an excellent reference on the potential theory of subordinate Brownian motions and provides many examples of subordinators and asymptotics of the Lévy densities of subordinate Brownian motions. If the Lévy measure ν of the subordinator has a density $\nu(s)$, then, in general, we have Proposition 2.4 to compute the asymptotics of the Lévy density of the subordinate Brownian motion

$$\overline{\pi}(y) := \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{y^2}{2\sigma^2 s}} \nu(s) ds \tag{2.4}$$

as $y \to 0$. Proposition 2.4 gives the asymptotics under two different types of sufficient conditions. The first sufficient condition is based on Lemma 3.3 of Song and Vondraček (2009). The applicability of their Lemma 3.3 is not restricted to Lévy densities of subordinators. However, in this case some of their conditions are not necessary. Below we give a more general result for this case. The second sufficient condition is a restriction of the Lévy density of the subordinator to the class of completely monotone functions[2] (see, for example Schilling et al. (2010) for its characterization and properties). This result is proved in Theorem 2.6 in Kim et al. (2010).

Proposition 2.4. *Let $\nu(s)$ be the Lévy density of a subordinator. Suppose there exist constants $c_0 > 0$ and $\frac{1}{2} < \beta < 2$ and a function $\ell : (0, \infty) \to (0, \infty)$ slowly varying at infinity[3] such that*

$$\nu(s) \sim \frac{c_0}{s^\beta \ell(\frac{1}{s})} \quad \text{as } s \to 0. \tag{2.5}$$

Let $\overline{\pi}(y)$ be defined as in (2.4). Suppose one of the following two conditions is satisfied:

[2] A completely monotone function $f : (0, \infty) \mapsto \mathbb{R}$ is a C^∞ function such that $(-1)^n f^{(n)}(x) \geq 0$ for $n = 0, 1, 2, \cdots$.

[3] A function ℓ defined in a neighborhood of infinity is called slowly varying at infinity if $\lim_{x\to\infty} \frac{\ell(ax)}{\ell(x)} = 1$ for all $a > 0$.

(1) Let $g:(0,\infty)\to(0,\infty)$ be a function such that $\int_0^\infty s^{\beta-\frac{3}{2}}e^{-s}g(s)ds<\infty$. Assume there is also some $\xi>0$ such that $f_{\ell,\xi}(y,s)\le g(s)$ for all $y,s>0$, where the auxiliary function $f_{\ell,\xi}(y,s)$ is defined by $f_{\ell,\xi}(y,s):=\frac{\ell(\frac{1}{y})}{\ell(\frac{2\sigma^2 s}{y})}$ if $y<\frac{s}{\xi}$ and 0 otherwise for any function ℓ slowly varying at infinity and any $\xi>0$.

(2) $\nu(s)$ is a completely monotone function.

Then

$$\overline{\pi}(y)\sim\frac{c_0\Gamma(\beta-\frac{1}{2})}{\sqrt{\pi}(2\sigma^2)^{1-\beta}}\frac{1}{|y|^{2\beta-1}\ell(\frac{1}{y^2})}\quad as\;\; y\to 0.$$

Remark 2.7. For slowly varying functions and regularly varying functions see Bingham et al. (1987). Every regularly varying function[4] at zero can be written in the form $\frac{1}{x^\beta\ell(\frac{1}{x})}$ for some real number β and ℓ slowly varying at infinity (c.f. Bingham et al. (1987) Theorem 1.4.1). Hence, the assumption on the asymptotics (2.5) is very general. Also note that from Bingham et al. (1987) Proposition 1.3.6, $\frac{1}{s^\beta\ell(\frac{1}{s})}\to\infty$ as $s\to 0$, so we are dealing with subordinators whose Lévy density tends to infinity at 0.

For a subordinator with Lévy density, if (2.5) is satisfied, there is a close connection between the Blumenthal-Getoor (BG) index and the parameter β in Proposition 2.4 when $\beta\ge 1$. For any subordinator with Lévy measure $\nu(ds)$ its BG index is defined by $p:=\inf\{\alpha>0:\int_{|s|\le 1}s^\alpha\nu(ds)<\infty\}$.

Proposition 2.5. *(1) Suppose* (2.5) *holds with $\beta\ge 1$. Then the BG index is equal to $\beta-1$. (2) Suppose the conditions in Proposition 2.4 are satisfied for two subordinators with $\beta\ge 1$ and $\beta'\ge 1$. Then the Hellinger condition implies their BG indexes are equal.*

We now apply Proposition 2.4 to the key example important in financial applications.

Example 2.1. ***Tempered Stable Subordinators.*** *Consider the tempered stable family of Lévy measures $\nu(s)=Cs^{-1-p}e^{-\eta s}$, where $C>0$, $p<1$, $\eta>0$. The limiting stable family has $\eta=0$ and $p\in(0,1)$. The tempered stable cases with $p\ge 0$ ($p<0$) give rise to subordinators with infinite activity (finite activity). Important special cases are the Gamma subordinator with $p=0$ (Madan et al. (1998)), the Inverse Gaussian (IG) subordinator with $p=\frac{1}{2}$ (Barndorff-Nielsen (1998)), and the compound Poisson subordinator with exponential jumps with $p=-1$ and $\eta>0$. For this family, the Laplace exponent is given by the following.*

$$\phi(\lambda)=\begin{cases}\gamma\lambda-C\Gamma(-p)[(\lambda+\eta)^p-\eta^p], & p\neq 0\\ \gamma\lambda+C\ln(1+\lambda/\eta), & p=0\end{cases},\tag{2.6}$$

where $\Gamma(\cdot)$ is the Gamma function.

For tempered stable subordinators with $p>-\frac{1}{2}$, it is clear that Proposition 2.4 condition (1) holds with $c_0=C$, $\beta=1+p$, $\ell(x)=1$, $g(s)=1$, and ξ chosen arbitrarily. Condition (2) also holds because the Lévy density of the subordinator is completely monotone. Hence we have

$$\overline{\pi}(y)\sim\frac{C\Gamma(p+\frac{1}{2})(2\sigma^2)^p}{\sqrt{\pi}|y|^{2p+1}}\quad as\;\; y\to 0.$$

From Proposition 2.2, $\pi(x,y)$ has the same asymptotics. It is now straightforward to show that Theorem 2.4 reduces to the following result for SubOU processes with tempered stable subordinators with drift.

[4] A function f defined in a neighborhood of infinity is called regularly varying at infinity with index $\rho\in\mathbb{R}$ if $\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\lambda^\rho$ for all $\lambda>0$. It is called regularly varying at 0 if $f(\frac{1}{x})$ is regularly varying at ∞.

Corollary 2.2. *Consider the setting in Theorem 2.4. Suppose ν and ν' belong to the tempered stable family with parameters (C,p,η) and (C',p',η') with $p,p'\geq 0$, respectively. Then $\mathbb{P}|_{\widetilde{\mathcal{F}}_t}\sim \mathbb{P}'|_{\widetilde{\mathcal{F}}_t}$ for every $t\geq 0$ if and only if $\mathbb{P}_0\sim\mathbb{P}'_0$ and the following equalities hold:*

$$\gamma\sigma^2=\gamma'\sigma'^2,\quad p=p',\quad C\sigma^{2p}=C'\sigma'^{2p}.$$

Thus, if we have SubOU processes with tempered stable subordinators with drift under both the physical and the risk-neutral measure, the p parameter p must remain the same under both measures, C and C' are related by $C'=C(\sigma/\sigma')^{2p}$, the subordinator drifts and the OU volatilities are related by $\gamma'\sigma'^2=\gamma\sigma^2$, and the OU drift parameters θ and κ and θ' and κ' can be arbitrarily changed.

For other examples of Lévy densities, where, e.g., $\ell(x)=(\ln(1+x))^{\alpha}$, one can also use Proposition 2.4. See Song and Vondraček (2009) section 2 for examples of subordinators and section 3 for the asymptotics of the Lévy density of subordinate Brownian motions. Replace 4 in their formulas by $2\sigma^2$ to coincide with our notation. Once the asymptotics of the Lévy density is determined, the Hellinger condition can be reduced to a simple relationship for the parameters similar to Corollary 2.2 for SubOU processes with tempered stable Lévy densities.

We are also interested in the following question: if under some measure $\mathbb{P}$ the semimartingale X is a SubOU process with generating tuple $(\kappa,\theta,\sigma,\gamma,\nu)$, characterize all measures $\mathbb{P}'$ locally equivalent to $\mathbb{P}$. In particular, we are interested in conditions on the semimartingale characteristics of X under $\mathbb{P}'$. The following result answers this question.

Theorem 2.5. *Let $\mathbb{P}$ and $\mathbb{P}'$ be two probability measures on $(\Omega,\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0})$ with initial distributions $\mathbb{P}_0$ and $\mathbb{P}'_0$, respectively. Suppose under $\mathbb{P}$, the canonical process X is a SubOU process with generating tuple $(\kappa,\theta,\sigma,\gamma,\nu)$ and local characteristics (B,C,Π). Suppose under $\mathbb{P}'$, X is a semimartingale with local characteristics (B',C',Π'). If $\mathbb{P}'$ and $\mathbb{P}$ are locally equivalent, then there exists a nonnegative predictable function $Y(\omega,t,x)$ and a predictable process β such that:*

$$B'_t(\omega)=B_t(\omega)+\gamma\sigma^2\int_0^t\beta_s(\omega)ds+\int_{[0,t]\times\mathbb{R}}y1_{\{|y|\leq 1\}}(Y(s,\omega,y)-1)\pi(X_{s-}(\omega),y)dyds,$$
$$C'_t(\omega)=\gamma\sigma^2 t,\ \Pi'(\omega,ds,dy)=Y(\omega,s,y)\Pi(\omega,ds,dy),\tag{2.7}$$

$$\int_0^t|\beta_s(\omega)|ds<\infty,\ \int_0^t\beta_s^2(\omega)ds<\infty,\tag{2.8}$$

$$\int_0^t\int_{y\neq 0}|y1_{\{|y|\leq 1\}}(Y(s,\omega,y)-1)|\pi(X_{s-}(\omega),y)dyds<\infty,\tag{2.9}$$

$$\int_0^t\int_{y\neq 0}(\sqrt{Y(s,\omega,y)}-1)^2\pi(X_{s-}(\omega),y)dyds<\infty,\quad\textit{(Hellinger condition)}\tag{2.10}$$

$\mathbb{P}'$ and $\mathbb{P}$-a.s. for all $t\geq 0$. Define $N=\beta\cdot X^c+(Y-1)*(\mu^X-\Pi)$. Then the density process Z of $\mathbb{P}'$ w.r.t. $\mathbb{P}$ is the Doleans-Dade stochastic exponential $\mathscr{E}(N)$ of N.

2.4 The Spectral Representation of the SubOU Semigroup

The OU and SubOU processes are stationary with the Gaussian stationary density

$$\mathfrak{m}(x)=\sqrt{\frac{\kappa}{\pi\sigma^2}}e^{-\frac{\kappa(\theta-x)^2}{\sigma^2}}.$$

Consider the Hilbert space $L^2(\mathbb{R},\mathfrak{m})$ with the inner product $(f,g)=\int_{\mathbb{R}}f(x)g(x)\mathfrak{m}(x)dx$, and denote by $\|\cdot\|$ the L^2-norm. The OU and SubOU semigroups are both symmetric semigroups in $L^2(\mathbb{R},\mathfrak{m})$, i.e. $(\mathcal{P}_t f,g)=(f,\mathcal{P}_t g)$ and $(\mathcal{P}_t^\phi f,g)=(f,\mathcal{P}_t^\phi g)$ for any $f,g\in L^2(\mathbb{R},\mathfrak{m})$. Their spectral decompositions in $L^2(\mathbb{R},\mathfrak{m})$ are available in closed form.

Theorem 2.6. *(1) The OU semigroup has the following eigenfunction expansion in $L^2(\mathbb{R},\mathfrak{m})$:*

$$\mathcal{P}_t f(x)=\sum_{n=0}^{\infty} e^{-\kappa n t} f_n \varphi_n(x), \quad f\in L^2(\mathbb{R},\mathfrak{m}), \quad t\geq 0, \tag{2.11}$$

with the orthonormal eigenfunctions expressed in terms of Hermite polynomials (see, e.g., Lebedev (1965))

$$\varphi_n(x)=\frac{1}{\sqrt{2^n n!}} H_n\left(\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\right), \quad n=0,1,\cdots, \tag{2.12}$$

and expansion coefficients $f_n=(f,\varphi_n)$.
(2) The SubOU semigroup has the following eigenfunction expansion in $L^2(\mathbb{R},\mathfrak{m})$:

$$\mathcal{P}_t^\phi f(x)=\sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} f_n \varphi_n(x), \quad f\in L^2(\mathbb{R},\mathfrak{m}), \quad t\geq 0 \tag{2.13}$$

with the same eigenfunctions and expansion coefficients as the OU semigroup.

General results for the spectral representation of one-dimensional diffusions go back to the fundamental work of McKean (1956). For each t, the OU transition semigroup operator $\mathcal{P}_t$ has a purely discrete spectrum with eigenvalues $\{e^{-\kappa n t}, n=0,1,...\}$. The explicit form of the eigenfunction expansion of the OU semigroup in terms of Hermite polynomials is well known and can be found in many references, including Wong (1964), Karlin and Taylor (1981), Schoutens (2000), Bakry and Mazet (2004), Alberverio and Rüdiger (2003), Alberverio and Rüdiger (2005), and Gorovoi and Linetsky (2004) p.62. The general spectral representation of the transition semigroup of a symmetric Markov process can be found in Fukushima et al. (1994). Bochner subordination replaces the eigenvalues $e^{-\lambda_n t}$ with $e^{-\phi(\lambda_n)t}$, where ϕ is the Laplace exponent of the subordinator, while the eigenfunctions remain the same. Thus the eigenvalues of the SubOU semigroup operator $\mathcal{P}_t^\phi$ are $\{e^{-\phi(\kappa n)t}, n=0,1,...\}$ with the same eigenfunctions. The general spectral representation of the semigroup of a subordinate symmetric Markov process can be found in Okura (2002) and in Alberverio and Rüdiger (2003) and Alberverio and Rüdiger (2005), where subordinate OU processes and their semigroups are studied in the general setting of symmetric Markov processes. Applications in finance can be found in Linetsky (2007), Mendoza et al. (2010) and Mendoza and Linetsky (2010).

For $t\geq 0$ the eigenfunction expansions on the RHS of (2.11) and (2.13) for the OU and the SubOU semigroup converge to $\mathcal{P}_t f$ and $\mathcal{P}_t^\phi f$ in the L^2-norm for any $f\in L^2(\mathbb{R},\mathfrak{m})$. In financial applications, we are interested in pointwise convergence, as we need to compute values at specific levels of the underlying variable. For $t>0$ pointwise convergence results are available for OU and SubOU semigroups.

Theorem 2.7. *(1) The eigenfunction expansion (2.11) converges to $\mathcal{P}_t f(x)$ pointwise in x for each $t>0$ and each $f\in L^2(\mathbb{R},\mathfrak{m})$.*
(2) If either of the following condition is satisfied: (i) $f(x)=\sum_{n=0}^\infty f_n\varphi_n(x)$ converges absolutely for all $x\in\mathbb{R}$, or (ii) $\sum_{n=1}^\infty e^{-\phi(\kappa n)t}n^{-1/4}<\infty$ for all $t>0$, then the eigenfunction expansion (2.13) converges to $\mathcal{P}_t^\phi f(x)$ pointwise for all $x\in\mathbb{R}$ for each $t>0$ and each $f\in L^2(\mathbb{R},\mathfrak{m})$.

The eigenfunction expansion (2.11) for the OU semigroup converges pointwise without any further conditions for each $t>0$ and $f\in L^2(\mathbb{R},\mathfrak{m})$. The eigenfunction (2.13) for the SubOU semigroup converges pointwise for each $t>0$ and $f\in L^2(\mathbb{R},\mathfrak{m})$ under the mild sufficient condition on the Laplace exponent of the subordinator in (2) of Theorem 2.7. In practice this condition is satisfied for all subordinators with drift $\gamma > 0$ due to the factor $e^{-\gamma\kappa t}$. In the pure jump case $\gamma = 0$, it is satisfied for all tempered stable subordinators with $p > 0$. Furthermore, for subordinators for which it is not satisfied, while the eigenfunction expansion (2.13) is not guaranteed to converge pointwise for each $t > 0$ and each $f \in L^2(\mathbb{R}, \mathfrak{m})$, it may converge pointwise for some $t > 0$ and some functions f, depending on the rate of decay of the coefficients f_n as n increases.

We also have the following expansions for OU and SubOU transition densities.

Theorem 2.8. *(1) The OU transition density (2.1) has the eigenfunction expansion*

$$p(t, x, y) = \mathfrak{m}(y) \sum_{n=0}^{\infty} e^{-\kappa n t} \varphi_n(x) \varphi_n(y) \tag{2.14}$$

converging for all $t > 0$ uniformly in x, y on compacts.
(2) If the Laplace exponent of the subordinator satisfies $\sum_{n=1}^{\infty} e^{-\phi(\kappa n)t} n^{-\frac{1}{2}} < \infty$ for all $t > 0$, the SubOU transition density has the eigenfunction expansion

$$p^{\phi}(t, x, y) = \mathfrak{m}(y) \sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} \varphi_n(x) \varphi_n(y) \tag{2.15}$$

converging for all $t > 0$ uniformly in x, y on compacts.

In the numerical implementation one needs to truncate eigenfunction expansions after a finite number of terms. Truncation error bounds of the expansion (2.15) in the L^2 and the pointwise sense can be easily derived. Here we present the pointwise error bound, as it is of most interest in finance. L^2 bounds can be derived similarly.

Theorem 2.9. *Suppose that the Laplace exponent of the subordinator satisfies $\sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} < \infty$ for all $t > 0$. Then for any $f \in L^2(\mathbb{R}, \mathfrak{m})$, the truncation error has the following bound:*

$$\left| \sum_{n=M}^{\infty} e^{-\phi(\kappa n)t} f_n \varphi_n(x) \right| \leq 1.0864 \|f\| e^{\frac{\kappa(x-\theta)^2}{2\sigma^2}} \sum_{n=M}^{\infty} e^{-\phi(\kappa n)t}.$$

If $\gamma > 0$, we can derive a particularly simple pointwise truncation error estimate:

$$\left| \sum_{n=M}^{\infty} e^{-\phi(\kappa n)t} f_n \varphi_n(x) \right| \leq 1.0864 \|f\| e^{\frac{\kappa(x-\theta)^2}{2\sigma^2}} \frac{e^{-\gamma\kappa M t}}{1 - e^{-\gamma\kappa t}}.$$

From these estimates it is clear that the convergence rate is governed by the OU mean reversion rate κ and time to maturity t, as well as the Laplace exponent of the subordinator. The greater the κ and the longer the time to maturity, the faster the convergence. In particular, if $\gamma > 0$, the convergence is exponential. In the pure jump case $\gamma = 0$ with tempered stable subordinators with $p > 0$, the truncation error can similarly be shown to be $O(e^{tC\Gamma(-p)(\kappa M)^p})$ with $\Gamma(-p) < 0$ and $C > 0$. We note that these error bounds are conservative since they rely on the estimate $|f_n| \leq \|f\|$. Depending on the properties of f, the coefficients f_n may converge to zero at a fast rate, resulting in faster convergence than is implied by these estimates.

3 Commodity Models With Mean-Reverting Jumps

3.1 Futures Dynamics

We start with $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0})$ as in section 2.3 endowed with a probability measure $\mathbb{Q}$ and assume that, under $\mathbb{Q}$, the canonical process X is a SubOU process with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$ and starting point $X_0 = x_0 \in \mathbb{R}$. Let $\{F(0,t), t \geq 0\}$ be the initial futures curve (a given deterministic function of time). We take $\mathbb{Q}$ to be the risk-neutral pricing measure chosen by the market and model the commodity spot price S_t under $\mathbb{Q}$ as the (scaled) exponential of the SubOU process X:

$$S_t = F(0,t)e^{X_t - G(t)}. \tag{3.1}$$

The function $G(t)$ is selected so that the expectation of the spot price under $\mathbb{Q}$ is equal to the initial futures price, $\mathbb{E}^{\mathbb{Q}}[S_t] = F(0,t)$, which implies $G(t) = \ln \mathbb{E}^{\mathbb{Q}}[e^{X_t}]$.

To compute futures price dynamics, we need the following.

Lemma 3.1. *The expansion of the exponential function in the eigenfunction basis (2.13) reads:*

$$e^x = \sum_{n=0}^{\infty} f_n \varphi_n(x), \quad f_n = e^{\theta + \frac{\sigma^2}{4\kappa}} \frac{1}{\sqrt{n!}} \left(\frac{\sigma}{\sqrt{2\kappa}}\right)^n. \tag{3.2}$$

The expansion converges absolutely for each $x \in \mathbb{R}$.

We now compute the futures price process $\{F(s,t) = \mathbb{E}^{\mathbb{Q}}[S_t|\widetilde{\mathcal{F}}_s], s \in [0,t]\}$ for each fixed maturity $t \geq 0$ using Lemma 3.1.

Theorem 3.1. *(1) The function $G(t)$ in the model (3.1) is given by:*

$$e^{G(t)} = \mathbb{E}^{\mathbb{Q}}[e^{X_t}] = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} f_n \varphi_n(x_0), \tag{3.3}$$

where f_n are given in (3.2), and the expansion converges absolutely for each $x_0 \in \mathbb{R}$, all $t \geq 0$ and any Laplace exponent ϕ.
(2) For each fixed maturity time $t > 0$, the futures price $F(s,t)$ is a martingale on $[0,t]$ given by:

$$F(s,t) = F(0,t)e^{-G(t)} \sum_{n=0}^{\infty} e^{-\phi(\kappa n)(t-s)} f_n \varphi_n(X_s), \quad s \in [0,t]. \tag{3.4}$$

At time zero, $s = 0$, (3.4) reduces to the identity $F(0,t) = F(0,t)$. At maturity, $s = t$, the futures price is equal to the spot price and (3.4) reduces to (3.1) due to Eq.(3.2). Eq.(3.4) gives a martingale expansion for the futures price. Note that for each n the process $\{e^{\phi(\kappa n)s}\varphi_n(X_s), s \geq 0\}$ is a martingale due to the eigenfunction property:

$$\mathbb{E}^{\mathbb{Q}}[\varphi_n(X_s)|X_t] = e^{-\phi(\kappa n)(s-t)}\varphi_n(X_t).$$

Thus, Eq.(3.4) represents the futures price process as an expansion in martingales associated with the eigenfunctions of the SubOU semigroup.

Since the process X can be expressed in terms of the spot price process S and the initial futures curve by inverting (3.1),

$$X_s = \ln\left(S_s/F(0,s)\right)) + G(s), \tag{3.5}$$

Eq.(3.4) expresses the dynamics of the futures price in terms of the spot price dynamics and the initial futures curve. Alternatively, we can view Eq.(3.4) as the process for the futures price driven by the SubOU process X without any reference to the spot price S. In this interpretation, our model can be viewed as the model for the evolution of the futures curve, rather than the spot price model. Eq.(3.4) directly defines the martingale futures dynamics. The spot dynamics (3.1) then follows as the limiting case.

Remark 3.1. **The Case without Time Change.** When X_t is an OU rather than SubOU process, our model reduces to the standard exponential OU model:

$$S_t = F(0,t)e^{X_t - x_0 e^{-\kappa t} - \theta(1-e^{-\kappa t}) - \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa t})}.$$

By applying Itô's formula, we obtain the spot price SDE: $dS_t = \kappa(\Theta(t) - \ln S_t)S_t dt + \sigma S_t dB_t$ with $\Theta(t) = \frac{1}{\kappa}\left(\frac{d}{dt}\ln F(0,t) + \frac{\sigma^2}{4\kappa}(1-e^{-2\kappa t})\right) + \ln F(0,t)$. This is essentially the same SDE as the Model 1 in Schwartz (1997) but with the long run level $\Theta(t)$ taken to be a deterministic function of time completely determined by the initial futures curve. Using the generating function of Hermite polynomials (Lebedev (1972) p.60), $\sum_{n=0}^{\infty}\frac{w^n}{n!}H_n(z) = e^{2zw-w^2}$, when X_t is an OU process (i.e., $\phi(\lambda)=\lambda$), Eq.(3.4) reduces to:

$$\begin{aligned} F(s,t) &= F(0,t)\exp\left\{X_s e^{-\kappa(t-s)} - x_0 e^{-\kappa t} - \theta(e^{-\kappa(t-s)} - e^{-\kappa t}) - \frac{\sigma^2}{4\kappa}(e^{-2\kappa(t-s)} - e^{-2\kappa t})\right\} \\ &= F(0,t)\left(\frac{S_s}{F(0,s)}\right)^{\exp\{-\kappa(t-s)\}} \exp\left\{-\frac{\sigma^2}{4\kappa}e^{-\kappa t}(e^{2\kappa s}-1)(e^{-\kappa t} - e^{-\kappa s})\right\}. \end{aligned}$$

This expression for the futures price dynamics in terms of the initial futures curve and the spot price dynamics in the OU model can be found in Clewlow and Strickland (1999), Eq.(2.5). Using Itô's formula, one can show that

$$dF(s,t) = \sigma e^{-\kappa(t-s)}F(s,t)dB_s, \quad s \in [0,t]. \tag{3.6}$$

We now discuss futures dynamics under the physical measure $\mathbb{P}$. The form for the futures process is still given by (3.4). However, the law of X changes under an equivalent measure change. Let $(\overline{B}^P, \overline{C}^P, \overline{\Pi}^P)$ be the semimartingale characteristics of X under $\mathbb{P}$. Theorem 2.5 gives the general conditions on the semimartingale characteristics of $(\overline{B}^P, \overline{C}^P, \overline{\Pi}^P)$. Any semimartingale satisfying these conditions can be chosen as a candidate driver for the commodity model under $\mathbb{P}$ that leads to the model driven by the given SubOU process with generating tuple $(\kappa,\theta,\sigma,\gamma,\nu)$ under $\mathbb{Q}$. In order to retain analytical tractability under $\mathbb{P}$, we are interested in equivalent measure transformations that transform a given SubOU process into another SubOU process plus possibly a deterministic function of time. Using Theorem 2.4 and Theorem 2.5, we obtain the following result.

Theorem 3.2. *Consider the canonical process X on $(\Omega, \widetilde{F}, (\widetilde{F}_t)_{t\geq 0})$. Suppose under measure $\mathbb{Q}$ with $\mathbb{Q}(X_0 = x_0) = 1$ the canonical process X is a SubOU process with generating tuple $(\kappa,\theta,\sigma,\gamma,\nu)$ and under measure $\mathbb{P}$ with $\mathbb{P}(X_0 = x_0) = 1$ it is a SubOU process with generating tuple $(\kappa_P,\theta_P,\sigma_P,\gamma_P,\nu_P)$ plus a deterministic function $H(t)$. Then $\mathbb{Q}$ and $\mathbb{P}$ are locally equivalent if and only if:*

(1) H is absolutely continuous with $H(0)=0$ if $\gamma>0$, and $H(t)=0$ for all t if $\gamma=0$.

(2) $\gamma_P\sigma_P^2 = \gamma\sigma^2$.

(3) the Hellinger condition $\int_{y\neq 0}\left(\sqrt{\pi^P(x,y)} - \sqrt{\pi(x,y)}\right)^2 dy < \infty$ is satisfied.

The Hellinger condition (3) can be simplified using Proposition 2.4. For example, in the case where the Lévy measures ν^P and ν^Q are both those of tempered stable subordinators, the Hellinger condition (3) reduces to the conditions presented in Corollary 2.2.

If X under $\mathbb{P}$ is specified to be a SubOU process plus some deterministic drift given by the function $H(t)$, the model parameters can be estimated from the time series of futures prices by filtering methods. In this case the transition density of the underlying SubOU process is known explicitly and given by (2.15). The pure OU diffusion based model has been estimated by Schwartz (1997). In that case, the noise term is Gaussian and the standard Kalman filter can be used. In our SubOU case, the noise term for the transition equation is not Gaussian, and the particle filter algorithm (or the extended particle filter or the unscented particle filter) can be used since we know the transition density of X in closed form (see Haykin (2001) and Javaheri et al. (2003)).

3.2 The Maturity Effect

The *maturity effect* (also known as the *Samuelson hypothesis*, see Samuelson (1965)) in the commodities futures markets is the well-known increase in commodity futures price volatility as the futures contract approaches maturity. The maturity effect implies that long term futures are less volatile than short term futures, and is well documented in the empirical literature (see Bessembinder et al. (1995), Kalev and Duong (2008) and references therein). The maturity effect is obviously present in the pure OU model (3.6), where futures volatility $\sigma e^{-\kappa\tau}$ decays exponentially as time to maturity $\tau = t - s$ increases, with mean reversion rate controlling the rate of decay. Here we investigate the maturity effect in our SubOU model.

We start with characterizing futures volatililty in the general semimartingale setting. For a futures contract with maturity time t, define $r^t_s = \ln \frac{F(s,t)}{F(0,t)}$, $s \in [0,t]$, the cumulative continuously compounded return process over the time interval $[0,s]$ with $s \le t$. Since $F(s,t)$ is a semimartingale, r^t is also a semimartingale. We measure volatility of the futures return process r^t experienced over the time interval $[0,s]$ by its quadratic variation (QV) $[r^t, r^t]_s$ (the square-bracket process). This definition of volatility has been widely used in the econometric literature (see Andersen et al. (2009)). With this definition, the maturity effect can be mathematically defined as follows.

Definition 3.1. *A futures model is said to exhibit the maturity effect almost surely if*

$$\mathbb{P}([r^{t_1}, r^{t_1}]_s > [r^{t_2}, r^{t_2}]_s) = 1 \quad \text{for any} \quad 0 < s < t_1 < t_2.$$

Remark 3.2. If $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent, then the QV of a semimartingale under $\mathbb{P}$ is a version of the QV under $\mathbb{Q}$ (Jacod and Shiryaev (2003) Theorem III.3.13). Hence, if the maturity effect is present in the futures dynamics under the physical measure, it is also present under the risk-neutral measure. We will compute $[r^t, r^t]_s$ under $\mathbb{Q}$.

Remark 3.3. In the pure OU model (3.6) the QV of futures return process is a deterministic function $[r^t, r^t]_s = \frac{\sigma^2}{2\kappa} e^{-2\kappa t}(e^{2\kappa s} - 1)$ decreasing in t for each fixed s, $0 < s < t$.

Note that $[r^t, r^t]_s = [r^{tc}, r^{tc}]_s + \sum_{u \le s} (\Delta r^t_u)^2$, where r^{tc} denotes the continuous martingale part of the process r^t. From Eq.(3.4), $r^t_s = -G(t) + \theta + \frac{\sigma^2}{4\kappa} + \ln g(X_s, s, t)$, where the function g is:

$$g(x,s,t) := \sum_{n=0}^{\infty} e^{-\phi(\kappa n)(t-s)} \Big(\frac{\sigma}{2\sqrt{\kappa}}\Big)^n \frac{1}{n!} H_n\Big(\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\Big).$$

Since we know the semimartingale characteristics of the SubOU process X, from Kallsen (2006) Proposition 2.5 we know that $[r^{tc}, r^{tc}]_s = \gamma\sigma^2 \int_0^s \left(\frac{\partial \ln g(X_{u-},u,t)}{\partial x}\right)^2 du = \gamma\sigma^2 \int_0^s \left(\frac{g_x(X_{u-},u,t)}{g(X_{u-},u,t)}\right)^2 du$

and $(\Delta r_u^t)^2 = (r_u^t - r_{u-}^t)^2 = (\ln g(X_u, u, t) - \ln g(X_{u-}, u-, t))^2 = \left(\int_{X_u \wedge X_{u-}}^{X_u \vee X_{u-}} \frac{g_x(x,u,t)}{g(x,u,t)} dx\right)^2$. Therefore, $[r^t, r^t]_s = \gamma\sigma^2 \int_0^s \left(\frac{g_x(X_{u-},u,t)}{g(X_{u-},u,t)}\right)^2 du + \sum_{u \le s} \left(\int_{X_u \wedge X_{u-}}^{X_u \vee X_{u-}} \frac{g_x(x,u,t)}{g(x,u,t)} dx\right)^2$. Note that $g(x,u,t) > 0$ and $g_x(x,u,t) > 0$ for each x, and $g(x,u,t)$ depends only on $t-u$. We thus have the following result.

Theorem 3.3. *If $g_x(x,0,t)/g(x,0,t)$ is decreasing in t for each x, then the maturity effect holds in the SubOU model.*

While the condition in Theorem 3.3 is hard to check analytically since the function g is given by the Hermite expansion, it can be easily checked numerically. We carried out extensive numerical testing for a wide range of parameter scenarios in pure jump ($\gamma = 0$) and jump-diffusion ($\gamma > 0$) cases and verified that it was indeed satisfied in all the cases. We thus conjecture that the condition in Theorem 3.3 is satisfied, and the maturity effect holds for our SubOU models.

Figure 2 illustrates the maturity effect as follows. We simulated 10,000 sample paths on the time interval $[0, 1/2]$ of pure jump ($\gamma = 0$) SubOU processes X with parameters $\theta = 0$, $\sigma = 0.5$, with the Inverse Gaussian subordinator with mean rate $\mu = 1$ and variance rate $\nu = 1$, and with $\kappa = 0.01, 0.1$, and 1. We then constructed 10,000 sample paths of the futures price processes with maturities 1/2, 1, 2, 3, 4 and 5 years for each of the underlying SubOU processes using the model relationship (3.4) under $\mathbb{Q}$, estimated realized quadratic variations of futures returns on each sample path (the quadratic variation is the same under $\mathbb{P}$ and under $\mathbb{Q}$), and verified that $[r^{t_1}, r^{t_1}]_{0.5} > [r^{t_2}, r^{t_2}]_{0.5}$ for $t_1 < t_2$ on each sample path. Figure 2 plots the estimated mean of the quadratic variation of futures returns as functions of futures contract maturity for the three values of the rate of mean reversion $\kappa = 0.01, 0.1$, and 1. The maturity effect is clearly seen in the plot. As in the pure diffusion OU model, κ controls the maturity effect in pure jump and jump-diffusion SubOU models.

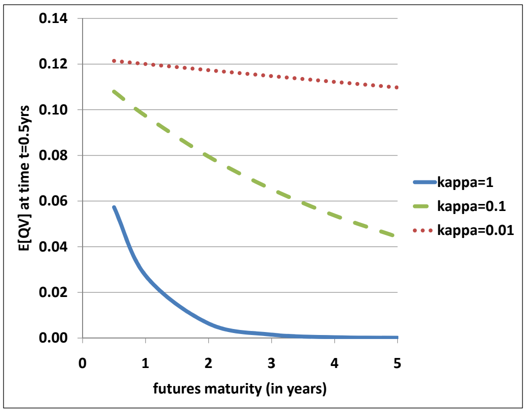
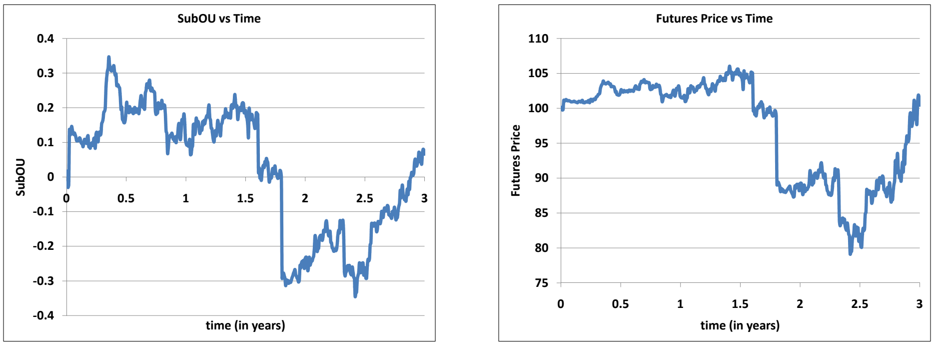


Figure 2: $\mathbb{E}^{\mathbb{Q}}\{[r^t, r^t]_{0.5}\}$ as a function of futures maturity t for a SubOU Process with Inverse Gaussian subordinator (with parameters $\kappa = 0.01, 0.1, 1$, $\theta = 0$, $\sigma = 0.5$, $\gamma = 0$, mean rate $\mu = 1$, variance rate $\nu = 1$).

To further illustrate, Figure 3 plots a sample path of the driving SubOU process in the jump-diffusion case and the corresponding futures price process with 3 years to maturity at time zero. The maturity effect is clearly seen in the sample path dynamics, as the futures price experiences low realized volatility far away from maturity, and the realized volatility substantially increases as the futures contract approaches maturity.



(a) the SubOU process (b) the futures process

Figure 3: A Sample Path of a SubOU Process and the corresponding futures price process with the Inverse Gaussian subordinator (with parameters $\kappa = 1$, $\theta = 0$, $\sigma = 0.5$, $\gamma = 0.1$, mean rate $\mu = 1$, variance rate $\nu = 1$).

Remark 3.4. It is important to note that the rate of mean reversion κ that enters the expression for the diffusion volatility in the pure diffusion OU case (3.6) and in the quadratic variation process through the functional form (3.4) of the dependence of the futures price on the SubOU process in the SubOU model is the *rate of mean reversion under the risk-neutral pricing measure* $\mathbb{Q}$. It is the *risk-neutral rate of mean reversion* that controls the maturity effect. That is, the presence of the maturity effect in the futures time series under the physical measure $\mathbb{P}$ is governed by the rate of mean reversion under the pricing measure. If there is no mean reversion under the pricing measure $\mathbb{Q}$, i.e., X is taken to be a subordinate Brownian motion under $\mathbb{Q}$ rather than a subordinate OU process, there is no maturity effect under $\mathbb{P}$. Thus, the presence of the maturity effect under $\mathbb{P}$ requires X to be a SubOU process under $\mathbb{Q}$, as futures models built on subordinate Brownian motions (Lévy processes) do not possess the maturity effect. In contrast, SubOU models are capable of modeling the maturity effect.

3.3 Futures Options Pricing

We consider pricing European put and call options on a futures contract. Suppose the strike price is K. The underlying futures contract matures at time t^* and the option expires at $t < t^*$. The time $\tau = t^* - t$ varies across commodities, ranging from several days for natural gas to one month for gold.

Here we only consider pricing the put option. The call option price is given by the put-call parity. Alternatively, a similar eigenfunction expansion can be obtained for the call pricing function, and the put-call parity can be verified directly. The put payoff at expiration t is $(K - F(t, t^*))^+$, where $F(t, t^*)$ is the t^*-maturity futures price at time t. In our model it is related to X_t by (3.4). It is convenient to write the payoff function as follows:

$$(K - F(x, t, t^*))^+ = (K - F(x, t, t^*))\mathbf{1}_{\{x < x^*\}},$$

where x^* is the unique solution of the equation $F(x, t, t^*) = K$, and $F(x, t, t^*)$ is the t^*-maturity futures price at time t as a function of the state variable $X_t = x$ given by (3.4). Since $F(x, t, t^*)$ is a strictly increasing function of x, the solution to this equation is unique and can be easily computed numerically using bisection or any other root bracketing algorithm. To price the put option at time zero, we thus need to first find x^* corresponding to the strike price K and then compute the expectation in:

$$P(t,t^*,F(0,t^*),K) = B(0,t)\mathbb{E}^{\mathbb{Q}}\Big[(K - F(X_t,t,t^*))\mathbf{1}_{\{X_t<x^*\}}\Big],$$

where $B(0,t)$ is the risk-free discount factor from the option expiration t to time zero.

Theorem 3.4. *Let x^* be the unique solution of the equation $F(x,t,t^*) = K$ and define $w^* := \frac{\sqrt{\kappa}}{\sigma}(x^* - \theta)$, $\tau := t^* - t$, $\alpha := \frac{\sigma}{2\sqrt{\kappa}}$ and $F := F(0,t^*)$. Suppose the Laplace exponent of the subordinator satisfies $\sum_{n=1}^{\infty} e^{-\phi(\kappa n)t} n^{-\frac{1}{4}} < \infty$. Then the put price has the absolutely convergent eigenfunction expansion:*

$$P(t,t^*,K,F) = B(0,t)\sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} p_n(t,t^*,w^*,F)\varphi_n(x_0), \tag{3.7}$$

$$p_n(t,t^*,w^*,F) = \frac{1}{\sqrt{\pi 2^n n!}}\Big\{Kb_n(w^*) - Fe^{\theta+\frac{\sigma^2}{4\kappa}-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)\tau}\frac{\alpha^m}{m!}a_{n,m}(w^*)\Big\}, \tag{3.8}$$

$$b_n(w) = \int_{-\infty}^{w} H_n(x)e^{-x^2}dx = \begin{cases} \sqrt{\pi}\Phi(\sqrt{2}w), & n = 0, \\ -H_{n-1}(w)e^{-w^2}, & n = 1,2,\cdots \end{cases}, \tag{3.9}$$

$$a_{n,m}(w) = \int_{-\infty}^{w} H_m(x)H_n(x)e^{-x^2}dx = \sum_{k=0}^{min(n,m)} \binom{m}{k}\binom{n}{k} 2^k k! b_{n+m-2k}(w). \tag{3.10}$$

The call price is given by the put-call parity $C(t,t^*,K,F) = B(0,t)(F-K) + P(t,t^*,K,F)$.

Remark 3.5. The option written on the spot price is obtained by setting $t = t^*$ in (3.8).

Remark 3.6. **The Case Without Time Change.** In the pure diffusion OU model, the option pricing formulas collapse to the Black-Scholes type formulas for the exponential OU diffusion model obtained by Clelow and Strickland (1999):

$$P(t,t^*,K,F) = B(0,t)\left[K\Phi(-d_-) - F\Phi(-d_+)\right], \; C(t,t^*,K,F) = B(0,t)\left[F\Phi(d_+) - K\Phi(d_-)\right],$$

$$d_- = \frac{\ln\left(\frac{F}{K}\right) - \frac{\sigma^2}{4\kappa}e^{-2\kappa\tau}(1-e^{-2\kappa t})}{\frac{\sigma}{\sqrt{2\kappa}}e^{-\kappa\tau}\sqrt{1-e^{-2\kappa t}}}, \quad d_+ = d_- + \frac{\sigma}{\sqrt{2\kappa}}e^{-\kappa\tau}\sqrt{1-e^{-2\kappa t}}.$$

4 Stochastic Volatility and Time Inhomogeneity

Models based on SubOU processes described in the previous section can be calibrated to fit a variety of volatility smile patterns observed in commodity options markets. However, they are generally not flexible enough in order to fit the entire volatility surface across different maturities. In this section we study a further extension of SubOU models to introduce stochastic volatility and time inhomogeneity, such as seasonality in options' implied volatility typical for some commodities, such as natural gas.

We consider absolutely continuous time changes of the form

$$T_t = \int_0^t \Big(a(u) + Z_u\Big)du, \tag{4.1}$$

where $a(t) \geq 0$ is a deterministic function of time and Z is a CIR diffusion solving the SDE

$$dZ_t = \kappa_Z(\theta_Z - Z_t)dt + \sigma_Z\sqrt{Z_t}dB_t, \; Z_0 = z_0,$$

with parameters assumed to satisfy the Feller condition, $2\theta_Z\kappa_Z/\sigma_Z^2 \geq 1$ to ensure that zero is an inaccessible boundary.

The *activity rate* process $a(t) + Z_t$ has the form of the so-called CIR++ process well known in the interest rate modeling literature (e.g., Brigo and Mercurio (2006)). The advantage of the CIR process is in its analytical tractability. Its transition probability density, the Laplace transform of its integral, and the Laplace transform conditional on the terminal state of the process are all known in closed form. The relevant results are collected in Appendix A.

Define the process S to be the inverse of T, $S_t := \inf\{u \geq 0 : T_u > t\}$. Since T is a strictly increasing continuous process, so is S. It is also clear that $T_t = \inf\{u \geq 0 : S_u > t\}$.

Assume that on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have a càdlàg SubOU process X with generating tuple $(\kappa, \theta, \sigma, \gamma, \nu)$, $X_0 = x_0$ and an independent absolutely continuous time change T of the form in (4.1). Let $(\mathcal{F}_t)_{t\geq 0}$ be the smallest right-continuous complete filtration generated by the processes X_t, Z_{S_t} and S_t. Then T_t is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ for every t, and we can define the time changed filtration $\mathcal{G}_t := \mathcal{F}_{T_t}$. It is clear that T and Z are adapted to $(\mathcal{G}_t)_{t\geq 0}$. Define a new process Y by $Y_t := X_{T_t}$, with $Y_0 = y_0 = x_0$. From Jacod (1979) Corollary 10.12, Y is a $(\mathcal{G}_t)$-semimartingale, and from Kallsen and Shiryaev (2002) Lemma 5 it admits the following local characteristics $(\overline{B}, \overline{C}, \overline{\Pi})$:

$$\overline{B}_t(\omega) = \int_0^t (a(s) + Z_s(\omega))\Big[\gamma\kappa(\theta - Y_{s-}(\omega)) + \int_0^\infty \int_{|x|\leq 1} xp(u; Y_{s-}(\omega), Y_{s-}(\omega) + x)dx\nu(du)\Big]ds,$$

$$\overline{C}_t(\omega) = \gamma\sigma^2 \int_0^t (a(s) + Z_s(\omega))ds, \quad \overline{\Pi}(\omega, dt, dx) = (a(t) + Z_t(\omega))\pi(Y_{t-}(\omega), x)dx,$$

where $\pi(\cdot, \cdot)$ is defined in Theorem 2.1 and here x is interpreted as the jump size. From these expressions we see that the role of the absolutely continuous time change is to scale all the local characteristics of the SubOU process with the stochastic activity rate or stochastic volatility. The bivariate process (Y, Z) is also a $(\mathcal{G}_t)$-semimartingale. We have the following result on its cross-variation process.

Proposition 4.1. *The cross-variation process* $[Y^c, Z^c]_t = 0$, *where* Y^c *and* Z^c *are the continuous local martingale parts of* Y *and* Z *respectively.*

It is clear that (Y, Z) is also a Markov process w.r.t. the filtration $(\mathcal{G}_t)_{t\geq 0}$. Given $\mathcal{G}_t$, the distribution of Y_{t+s} depends only on $T_{t+s} - T_t$ and Y_t, and $T_{t+s} - T_t$ depends only on Z_t. The distribution of Z_{t+s} depends only on Z_t. Thus, conditional expectations of the form $\mathbb{E}[f(Y_t)|\mathcal{G}_s]$ reduce to $\mathbb{E}[f(Y_t)|Y_s, Z_s]$ by the Markov property. Using conditioning and the spectral representation of the SubOU semigroup, such expectations can be computed in terms of eigenfunction expansions.

Theorem 4.1. *For* $f \in L^2(\mathbb{R}, \mathfrak{m})$, *suppose one of the following two conditions is satisfied:*

(1) The eigenfunction expansion $f(x) = \sum_{n=0}^\infty f_n\varphi_n(x)$, *where* $f_n = (f, \varphi_n)$, *converges absolutely for each* x.

(2) $\sum_{n=0}^\infty e^{-\phi(\kappa n)\int_s^t a(u)du}\mathcal{L}_{CIR}\left(t - s, \phi(\kappa n)\,|z\right) n^{-\frac{1}{4}} < \infty$ *for some* $z > 0$ *(and hence for all* z; *it is straightforward to show this using* (A.4)*), where the Laplace transform* $\mathcal{L}_{CIR}\left(t, \cdot\,|z\right)$ *is given in Appendix A.*

Then $\mathbb{E}[f(Y_t)|Y_s, Z_s] = \sum_{n=0}^\infty e^{-\phi(\kappa n)\int_s^t a(u)du}\mathcal{L}_{CIR}\Big(t - s, \phi(\kappa n)\Big|Z_s\Big)f_n\varphi_n(Y_s)$.

We can now introduce stochastic volatility and time inhomogeneity in commodity models. Let $Y_t = X_{T_t}$ be the time changed SubOU process as above. Under the risk-neutral pricing measure $\mathbb{Q}$ chosen by the market, we model the spot price as follows:

$$S_t = F(0,t)e^{Y_t - G(t)}, \tag{4.2}$$

where the function $G(t)$ is selected so that $e^{G(t)} = \mathbb{E}^{\mathbb{Q}}[e^{Y_t}]$. Applying Theorem 4.1 to the exponential function, we obtain the futures price process.

Theorem 4.2. *(1)* $e^{G(t)} = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_0^t a(u)du}\mathcal{L}_{CIR}\Big(t, \phi(\kappa n)\Big| z_0\Big) f_n \varphi_n(y_0)$*, where* f_n *are given in Lemma 3.1. The expansion converges absolutely for all* $z_0 > 0$*,* $y_0 \in \mathbb{R}$*, and any Laplace exponent* ϕ*.*
(2) For each $t > 0$*, the futures price is a martingale on* $[0,t]$ *given by:*

$$F(s,t) = F(0,t)e^{-G(t)} \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_s^t a(u)du}\mathcal{L}_{CIR}\Big(t-s, \phi(\kappa n)\Big| Z_s\Big) f_n \varphi_n(Y_s), \quad s \in [0,t]. \tag{4.3}$$

To investigate the maturity effect, we need to compute the QV process $[r^t, r^t]_s$, which is more involved in this case due to the extra state variable Z. From (4.3), $r_s^t = -G(t) + \theta + \frac{\sigma^2}{4\kappa} + \ln g(Y_s, Z_s, s, t)$, where

$$g(y,z,s,t) = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_s^t a(u)du}\mathcal{L}_{CIR}\Big(t-s, \phi(\kappa n)\Big| z\Big)\Big(\frac{\sigma}{2\sqrt{\kappa}}\Big)^n \frac{1}{n!} H_n\Big(\frac{\sqrt{\kappa}}{\sigma}(y-\theta)\Big).$$

Again we use Kallsen (2006) Proposition 2.5 to compute $[r^{tc}, r^{tc}]_s$ from the local characteristics of the semimartingale (Y, Z). Since the cross-variation is zero by Proposition 4.1, we do not have cross derivative terms and obtain:

$$[r^t, r^t]_s = [r^{tc}, r^{tc}]_s + \gamma\sigma^2 \int_0^s (a(u)+Z_u)\Big(\frac{g_y(Y_{u-}, Z_u, u, t)}{g(Y_{u-}, Z_u, u, t)}\Big)^2 du + \sigma_Z^2 \int_0^s Z_u \Big(\frac{g_z(Y_{u-}, Z_u, u, t)}{g(Y_{u-}, Z_u, u, t)}\Big)^2 du$$
$$+ \sum_{u \le s}\Big(\int_{Y_u \wedge Y_{u-}}^{Y_u \vee Y_{u-}} \frac{g_y(y,z,u,t)}{g(y,z,u,t)} dy\Big)^2.$$

Note that g and g_y are positive, but g_z is not necessarily so, and $g(y,z,u,t)$ depends on u and t only through $t-u$. It is thus clear that we have the following:

Theorem 4.3. *If* $\frac{g_y(y,z,0,t)}{g(y,z,0,t)}$ *and* $\Big(\frac{g_z(y,z,0,t)}{g(y,z,0,t)}\Big)^2$ *are decreasing in* t *for any* (y,z)*, then the maturity effect holds.*

As in the SubOU case in section 3.3, this condition is hard to check analytically, but can be easily verified numerically. We have conducted extensive numerical experiments and verified this condition for all parameter specifications we have tested.

For the model with stochastic volatility the option pricing formula is more involved since the futures price $F(t,t^*)$ at expiration of the option t is now determined by the values of two state variables Y_t and Z_t at that time, $F(t,t^*) = F(Y_t, Z_t, t, t^*)$. We condition on the state of the CIR process Z_t at time t and reduce the problem to the SubOU case. One then has to use the conditional Laplace transform (A.5) instead of (A.3), since we have conditioned on Z_t. Hence, the pricing formula is expressed as an integral with respect to the transition density of the CIR process (A.2). An additional subtlety is that y^* now depends on z. Namely, for each fixed $z > 0$, there exists a unique $y^* = y^*(z)$ such that $F(y^*, z, t, t^*) = K$. Then the put payoff function can be rewritten as $(K - F(y,z,t,t^*))^+ = (K - F(y,z,t,t^*))\mathbf{1}_{\{y < y^*(z)\}}$.

Theorem 4.4. *For each fixed $z > 0$, let $y^*(z)$ denote the unique solution of the equation $F(y, z, t, t^*) = K$, where $F(y, z, t, t^*)$ is the futures pricing function* (4.3)*. Define $w^*(z) := \frac{\sqrt{\kappa}}{\sigma}(y^*(z) - \theta)$, $\tau := t^* - t$, $\alpha := \frac{\sigma}{2\sqrt{\kappa}}$ and $F := F(0, t^*)$. Suppose condition (2) of Theorem 4.1 and the following condition are satisfied:*

$$\sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_0^t a(u)du} \mathcal{L}_{CIR}\left(t, \phi(\lambda)\,|z_0, z\right) n^{-\frac{1}{4}} < \infty \text{ for some } z \text{ and hence for all } z. \tag{4.4}$$

(It is easy to show this using (A.6)*.) Then the put price is given by:*

$$P(t, t^*, K, F) = B(0, t) \tag{4.5}$$
$$\times \int_0^{\infty} \left\{ \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_0^t a(u)du} \mathcal{L}_{CIR}(t, \phi(\kappa n)|z_0, z_t) p_n(t, t^*, w^*(z_t), F) \varphi_n(y_0) \right\} p_{CIR}(t, z_0, z_t) dz_t,$$

where $p_{CIR}(t, z_0, z_t)$ is the CIR transition density (A.2) *and*

$$p_n(t, t^*, w^*(z), F) = \frac{1}{\sqrt{\pi 2^n n!}} \tag{4.6}$$
$$\times \left\{ K b_n(w^*(z)) - F e^{\theta + \frac{\sigma^2}{4\kappa} - G(t^*)} \sum_{m=0}^{\infty} e^{-\phi(\kappa m)\int_t^{t^*} a(u)du} \mathcal{L}_{CIR}(\tau, \phi(\kappa m)|z_t) \frac{\alpha^m}{m!} a_{n,m}(w^*(z)) \right\},$$

where $b_n(w)$ and $a_{n,m}(w)$ are given by (3.9) *and* (3.10)*. The call price is given by the put-call parity.*

Remark 4.1. For options written on the spot price, in contrast to futures options, we only need the Laplace transform of the time change instead of the conditional Laplace transform. Furthermore, in this case $y^* = \ln\left(\frac{K}{F(0,t)}\right) + G(t)$ is independent of Z_t. By setting $t = t^*$ and using $\int_0^\infty \mathcal{L}_{CIR}(t, \phi(\kappa n)|z_0, z_t) p_{CIR}(t, z_0, z_t) dz_t = \mathcal{L}_{CIR}(t, \phi(\kappa n)|z_0)$, the put price becomes

$$P(t, K, F) = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_0^t a(u)du} \mathcal{L}_{CIR}(t, \phi(\kappa n)|z_0) p_n(t, t, w^*, F) \varphi_n(y_0).$$

5 Model Implementation and Calibration Examples

The models introduced in this paper were implemented in C++ on a PC. Hermite expansions can be efficiently computed using the following classical recursion for Hermite polynomials (Lebedev (1972) p.61):

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \quad n \geqslant 2.$$

To compute the option pricing formula (3.7), we need to evaluate the coefficients b_n and $a_{n,m}$. From the equation (3.9), it is easy to see b_n can be computed recursively using the recursion for Hermite polynomials. Equation (3.10) is a closed-form formula for $a_{n,m}$, but it is not convenient to use from a computational perspective. We have the following computationally efficient approach for $a_{n,m}$.

Proposition 5.1. *The coefficients $a_{n,m}$ satisfy the following:*

$$a_{0,0}(x) = \sqrt{\pi}\Phi(\sqrt{2}x), \quad a_{n,n}(x) = 2n a_{n-1,n-1}(x) - H_{n-1}(x) H_n(x) e^{-x^2}, \quad n \geq 1, \tag{5.1}$$

$$a_{n,m}(x) = \frac{H_n(x)H_{m+1}(x) - H_m(x)H_{n+1}(x)}{2(m-n)} e^{-x^2}, \quad n \neq m, n \geq 0, m \geq 0. \tag{5.2}$$

To evaluate the option pricing formula (4.5) for the model with stochastic volatility, we first truncate the integral in z_t at some level M large enough that the probability of the CIR process to exceed M at time t is less than the desired error tolerance. We then use the Simpson rule to discretize the integral on the interval $[0, M]$. The CIR transition density at each node $z_t(k)$ is computed by (A.2), at each integration node $z_t(k)$ the value of $y^*(k)$ is found by the bisection algorithm, and the integrand is computed similar to the option pricing formula (4.5) in the SubOU case (with the distinction that under the time changed SubOU the conditional Laplace transform (A.5) enters the expression in place of the Laplace transform (A.3) in the SubOU case).

CPU times generally depend on time to maturity t and the model parameters. For short maturities (say, less than two weeks to expiration), one may have to use infinite-precision arithmetics to achieve required accuracy in summing up the series. To compute short maturity option prices we used the GNU MP Bignum library. For longer maturity options double precision is sufficient. In our numerical experiments on a PC running Linux (Intel Core 2 Duo CPU at 2.53GHz with 2.00GB RAM), CPU times ranged from several milliseconds up to hundreds of milliseconds per option for the SubOU model, depending on the combination of parameters, and from hundreds of milliseconds up to several seconds per option for the SubOU model with stochastic volatility.

We now present calibration examples of the SubOU model with the IG subordinator to implied volatility smile curves extracted from market prices of options on six commodity futures. We have also calibrated for other commodities, and the results are similar to what are displayed here. However, due to space constraints, only six of them are shown. The commodities included two metals (copper, gold), two energies (crude oil, natural gas), and two agriculturals (corn and wheat). Market data on implied volatilities for this study were provided by Morgan Stanley's Commodity Stategies Group and were extracted from commodity futures options market prices on July 2nd 2009. All options had approximately six months to expiration. The moneyness defined as the ratio of the option strike price to the futures price ranged from 0.6 to 1.8 for all commodities. To calibrate the model to market implied volatilities, we minimized the sum of squared differences between the market and the model implied volatilities. There are a total of six parameters in the SubOU model: three parameters of the background OU process and three parameters of the inverse Gaussian subordinator with drift. Without loss of generality, the starting SubOU state x_0 was set to zero (it can always be set to zero by changing θ to $\theta - x_0$ without affecting the option price). Our calibration results are presented in Figure 4. In these instances the SubOU model with the IG subordinator provides an excellent fit to volatility smiles for all eight commodities (well within the bid/ask spread for each option).

While the SubOU model calibrates well to commodity volatility smiles for a fixed maturity, it may generally lack flexibility to capture the entire *volatility surface* across both the maturity dimension and the strike (moneyness) dimension. The time changed SubOU model with stochastic volatility and possible time inhomogeneity has additional flexibility to capture time dependence in the shape and steepness of the volatility smile and time dependence in the at-the-money volatility term structure. In Figure 5 we calibrate the SubOU model with the inverse Gaussian subordinator time changed with the integral of the CIR process to the implied volatility surfaces for zinc. We used four maturities (6 months, 1 year, 1.5 years and 2 years) in our calibration. The deterministic activity rate component was taken to be a piecewise constant function (constant between adjacent futures maturity dates). The time changed SubOU model provided an excellent fit to this volatility surface (well within the bid/ask spreads for all options). The deterministic activity rate allowed us to capture the sharp decay in the ATM implied volatilities, the IG subordinator allowed us to capture steep smiles for shorter-dated maturities, and the CIR stochastic volatility supported the longer-dated smiles. In contrast,

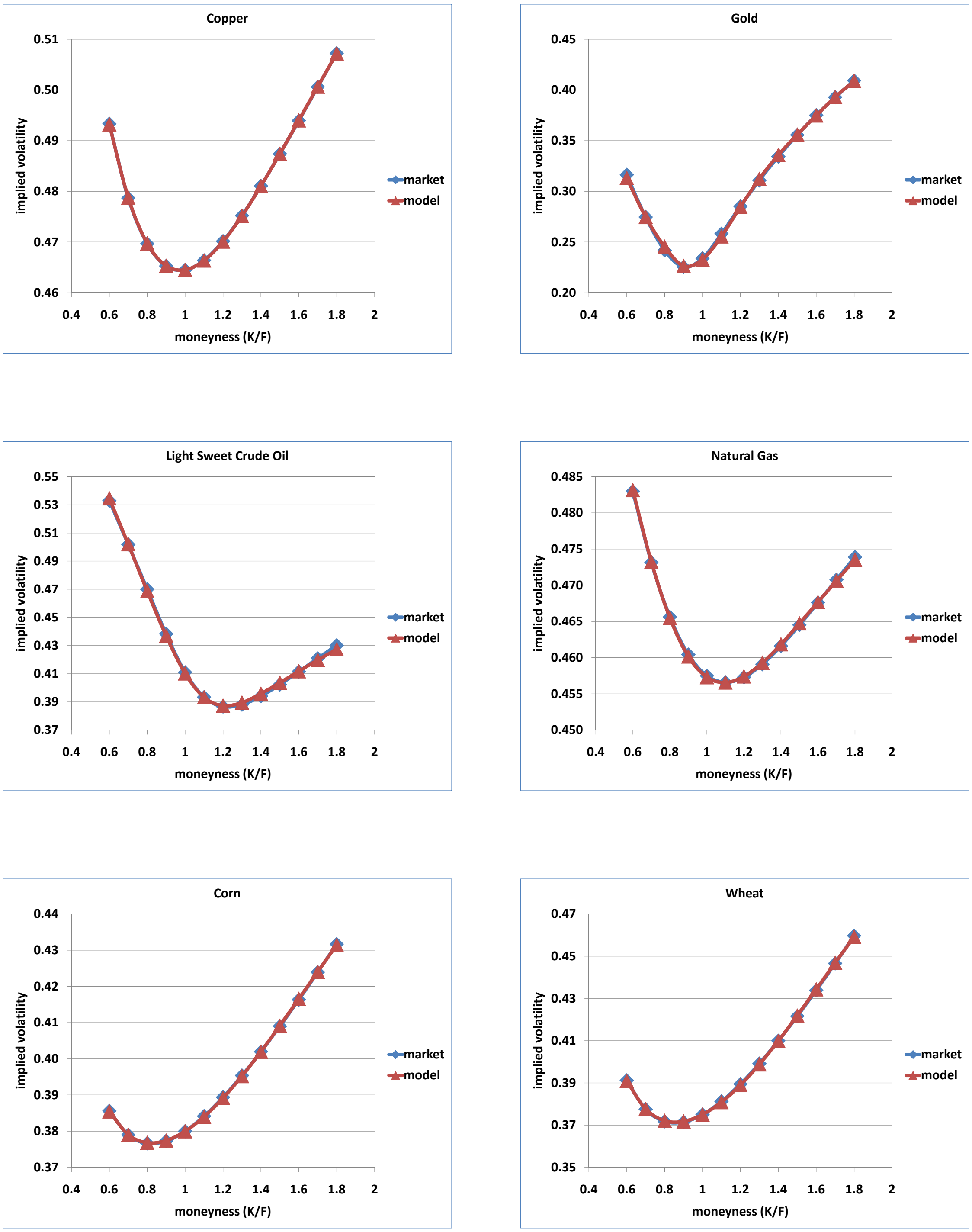


Figure 4: SubOU Model Calibration Results to Implied Volatility Smiles for Commodities

SubOU models without stochastic volatility exhibit faster flattening of the volatility smile as we go further out in maturity.

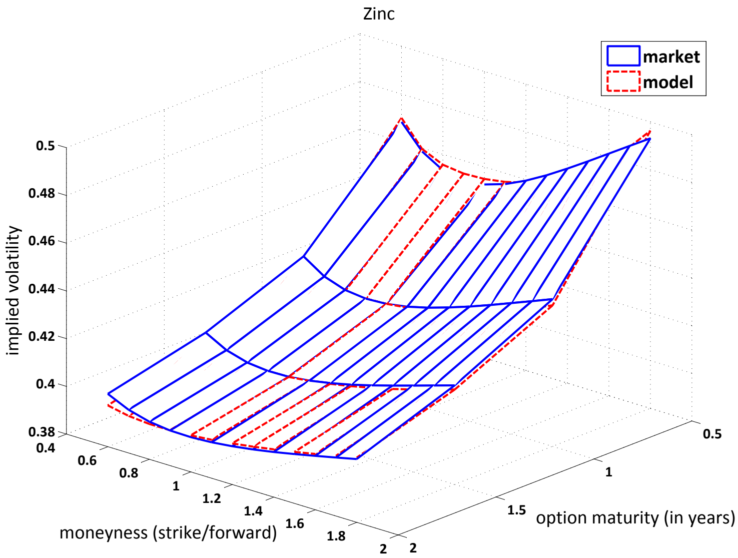


Figure 5: Time Changed SubOU Volatility Surface Calibration Results For Zinc

6 Conclusion

This paper studied a class of subordinate OU processes, their sample path properties, equivalent measure transformations, and the spectral representation of their transition semigroup. As an application, we constructed a new class of commodity models with mean-reverting jumps based on subordinate OU process. Further time changing by the integral of a CIR process plus a deterministic function of time, we induced stochastic volatility and time inhomogeneity in the models. We obtained analytical solutions for commodity futures options in terms of Hermite expansions and showed that the models exhibit the maturity effect and are flexible enough to capture a wide variety of implied volatility smile patterns observed in energy, metals, and agricultural commodities futures options.

We are currently developing computational methods for American-style futures options in these models. It turns out that the eigenfunction expansion approach to pricing European options followed in this paper can be extended to Bermudan-style options with a finite number of exercise opportunities. Richardson extrapolation can then be used to obtain solutions for American-style options.

In future work we plan to extend this class of models to multi-commodity products, such as spread options, and to path-dependent options such as Asian-style options. An extension to American-style options is developed in Li and Linetsky (2011). We also anticipate that subordinate OU processes studied in this paper will find other applications beyond commodities, such as in interest rate modeling, volatility modeling, and real options.

A CIR Processes

Let $\{Z_t, t \geq 0\}$ be a CIR diffusion starting from $Z_0 = z > 0$ and solving the SDE

$$dZ_t = \kappa(\theta - Z_t)\,dt + \sigma\sqrt{Z_t}dB_t. \tag{A.1}$$

Assume the long run level θ, the rate of mean reversion κ, and the volatility parameter σ satisfy the Feller condition $d := \frac{2\theta\kappa}{\sigma^2} \geq 1$ to ensure that the process stays strictly positive (zero is an inaccessible boundary).

The CIR transition density $p_{CIR}(t, z_0, z)$ is given by

$$p_{CIR}(t, z_0, z) = \frac{2\kappa}{\sigma^2(1-e^{-\kappa t})} e^{-\frac{2\kappa(z_0 e^{-\kappa t}+z)}{\sigma^2(1-e^{-\kappa t})}} \left(\frac{z}{z_0 e^{-\kappa t}}\right)^{\frac{d-1}{2}} I_{d-1}\left(\frac{4\kappa\sqrt{z_0 z e^{-\kappa t}}}{\sigma^2(1-e^{-\kappa t})}\right), \tag{A.2}$$

where $I_{d-1}(\cdot)$ is the modified Bessel function of the first kind of order $d-1$.

The Laplace transform $\mathcal{L}_{CIR}(t, \lambda|z_0) := \mathbb{E}_{z_0}\left[e^{-\lambda\int_0^t Z_u du}\right]$ is given by the CIR bond pricing formula for the short rate process λZ_t:

$$\mathcal{L}_{CIR}(t, \lambda|z_0) = C(t, \lambda)e^{-B(t,\lambda)z_0}, \tag{A.3}$$

where $C(t,\lambda) = \left(\frac{2\gamma(\lambda)e^{(\gamma(\lambda)+\kappa)t/2}}{(\gamma(\lambda)+\kappa)(e^{\gamma(\lambda)t}-1)+2\gamma(\lambda)}\right)^d$, $B(t,\lambda) = \frac{2\lambda(e^{\gamma(\lambda)t}-1)}{(\gamma(\lambda)+\kappa)(e^{\gamma(\lambda)t}-1)+2\gamma(\lambda)}$, and $\gamma(\lambda) = \sqrt{\kappa^2 + 2\sigma^2\lambda}$. The function $\mathcal{L}_{CIR}(t, \lambda|z_0)$ has the following asymptotic behavior as $\lambda \to \infty$:

$$\mathcal{L}_{CIR}(t, \lambda|z_0) \sim \exp\left\{-\frac{\kappa\theta}{\sigma}\sqrt{2\lambda}t - \frac{z_0}{\sigma}\sqrt{2\lambda}\right\}. \tag{A.4}$$

The Laplace transform conditional on the state of the process at time t, $\mathcal{L}_{CIR}(t, \lambda|z_0, z_t) := \mathbb{E}_{z_0}\left[e^{-\lambda\int_0^t Z_u du} \mid Z_t = z_t\right]$, is also known in closed form (Broadie and Kaya (2006)):

$$\mathcal{L}_{CIR}(t, \lambda|z_0, z_t) = \frac{\gamma(\lambda)e^{-0.5(\gamma(\lambda)-\kappa)t}(1-e^{-\kappa t})}{\kappa(1-e^{-\gamma(\lambda)t})}$$

$$\times \exp\left\{\frac{z_0+z_t}{\sigma^2}\left(\frac{\kappa(1+e^{-\kappa t})}{1-e^{-\kappa t}} - \frac{\gamma(\lambda)(1+e^{-\gamma(\lambda)t})}{1-e^{-\gamma(\lambda)t}}\right)\right\} \frac{I_{d-1}\left(\frac{4\gamma(\lambda)\sqrt{z_0 z_t}}{\sigma^2}\frac{e^{-0.5\gamma(\lambda)t}}{1-e^{-\gamma(\lambda)t}}\right)}{I_{d-1}\left(\frac{4\kappa\sqrt{z_0 z_t}}{\sigma^2}\frac{e^{-0.5\kappa t}}{1-e^{-\kappa t}}\right)}. \tag{A.5}$$

The function $\mathcal{L}_{CIR}(t, \lambda|z_0, z_t)$ has the following asymptotic behavior as $\lambda \to \infty$:

$$\mathcal{L}_{CIR}(t, \lambda|z_0, z_t) \sim \lambda^{\frac{d}{2}} \exp\left\{-\frac{\kappa\theta}{\sigma}\sqrt{2\lambda}t - \frac{z_0+z_t}{\sigma}\sqrt{2\lambda}\right\}. \tag{A.6}$$

B Proofs

Part (2) of Theorem 2.2. Denote the RHS of $\mathcal{G}^\phi$ in Theorem 2.1 by $\mathcal{G}^\#$. If X' admits characteristics (B', C', Π'), then from Itô's Formula for semimartingales, for any $f \in C_c^2(\mathbb{R})$

$$M_t := f(X'_t) - f(x) - \int_0^t \mathcal{G}^\# f(X'_{s-})ds$$

is a local martingale. Since $\mathcal{G}^\# f \in C_0(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$ vanishing at infinity), $\mathcal{G}^\# f$ is bounded. $f(X'_t) - f(x)$ is also bounded for all t. Hence $\mathbb{E}[M_t^*] < \infty$

($M_t^* := \sup_{s\le t}|M_s|$) for all t, and M is a martingale by Protter (2005) Chapter 1 Theorem 51. Note that from Theorem 2.1, $C_c^2(\mathbb{R})$ is a core of $D(\mathcal{G}^\phi)$. Hence applying Ethier and Kurtz (1986) Chapter 4 Theorem 4.1 to the martingale problem $\Big((\mathcal{G}^\phi, C_c^2(\mathbb{R})), \mathbb{P}^x\Big)$ and Corollary 4.3, it follows that $\mathbb{P}'\circ X'^{-1} = \mathbb{P}^x$ on the Skorohod space $(\Omega, \mathcal{F}^0)$. □

Theorem 2.3. If $x>\theta$, then for any $y>0$, $|-y+(x-\theta)(1-e^{-\kappa t})| < |y+(x-\theta)(1-e^{-\kappa t})|$. From (2.1), this implies $p(t,x,x-y) > p(t,x,x+y)$ for any $t>0$. Hence from the definition of $\pi(x,\cdot)$, $\pi(x,-y) > \pi(x,y)$ for any $y>0$. By integrating $\pi(x,\cdot)$ on $(-\infty,-y)$ and (y,∞), we also get $\Pi(x,(-\infty,-y)) > \Pi(x,(y,\infty))$. The cases with $x<\theta$ and $x=\theta$ are proved similarly. □

Theorem 2.4. *Sufficiency.* By replacing the Lévy measure used in Remark 33.3 of Sato (1999) by our state-dependent Lévy measure, we can show that the Hellinger condition $\int_{y\neq 0}\big(\sqrt{\pi'(x,y)}-\sqrt{\pi(x,y)}\big)^2 dy < \infty$ implies that

$$\int_{|y|\le 1} |y|\cdot|\pi'(X_{s-}(\omega),y) - \pi(X_{s-}(\omega),y)|dy < \infty, \tag{B.1}$$

so $h(x)(Y-1)*\Pi$ is finite. We first show that

$$\int_{[0,\infty)}\int_{|y|\le 1} yp'(u;X_{s-}(\omega),X_{s-}(\omega)+y)dy\nu'(du) - \int_{[0,\infty)}\int_{|y|\le 1} yp(u;X_{s-}(\omega),X_{s-}(\omega)+y)dy\nu(du)$$
$$= \int_{|y|\le 1} y\Big[\pi'(X_{s-}(\omega),y) - \pi(X_{s-}(\omega),y)\Big]dy.$$

Note that

$$\int_{[0,\infty)}\int_{|y|\le 1} yp'(u,x,x+y)dy\nu'(du) - \int_{[0,\infty)}\int_{|y|\le 1} yp(u,x,x+y)dy\nu(du)$$
$$= \lim_{n\to\infty}\left(\int_{[0,\infty)}\int_{1/n\le|y|\le 1} yp'(u,x,x+y)dy\nu'(du) - \int_{[0,\infty)}\int_{1/n\le|y|\le 1} yp(u,x,x+y)dy\nu(du)\right)$$
$$= \lim_{n\to\infty}\left(\int_{1/n\le|y|\le 1} y(\pi'(x,y)-\pi(x,y))dy\right) = \int_{|y|\le 1}(\pi'(x,y)-\pi(x,y))dy,$$

where the last equality comes from the Dominated Convergence Theorem since we have (B.1). So for all ω we have $B' = B + \gamma\sigma^2\beta\cdot t + h(x)(Y-1)*\Pi$, $C'=C$, $\Pi' = Y\cdot\Pi$.

Since $\Pi(\omega,t,dx)=0$, it is clear that $\sigma_{JS}=\infty$, where σ_{JS} is defined in Jacod and Shiryaev (2003) (JS) III.5.6. The process H defined in JS III.5.7 becomes the following in our case:

$$H_t(\omega) = \int_0^t \gamma\sigma^2\beta_s(\omega)^2 ds + \int_0^t\int_{y\neq 0}\Big(\sqrt{\pi'(X_{s-}(\omega),y)} - \sqrt{\pi(X_{s-}(\omega),y)}\Big)^2 dyds$$

It is clear that the integrand in the above expression is càdlàg for every ω. This implies that $H_t(\omega)<\infty$ for every ω and t. Hence the process H does not jump to infinity as defined in JS III.5.8. This fact together with $\sigma_{JS}=\infty$ implies that Hypothesis III.5.29 of JS holds.

$\mathbb{P}'$ is the unique solution to the martingale problem $(\sigma(X_0), X|\mathbb{P}'_0, B', C', \Pi')$. As remarked before, local uniqueness also holds. Note that $\mathbb{P}'_0 \preccurlyeq \mathbb{P}_0$. Now all conditions stated in JS Theorem III.5.34 are satisfied, which implies $\mathbb{P}'\preccurlyeq\mathbb{P}$ locally. By interchanging the role of $\mathbb{P}'$ and $\mathbb{P}$ and similarly defining β' and H', we can prove $\mathbb{P}_0\preccurlyeq\mathbb{P}'_0$ implies $\mathbb{P}'\preccurlyeq\mathbb{P}$ locally. Hence $\mathbb{P}'\sim\mathbb{P}$ locally.

Necessity. If $\mathbb{P}'\sim\mathbb{P}$ locally, then (i) holds, and (ii) is implied by JS Theorem III.3.24. The uniqueness of the solution to the martingale problem $(\sigma(X_0), X|\mathbb{P}_0, B, C, \Pi)$ implies the $\mathbb{P}$-martingale representation property w.r.t. X (JS Theorem III.4.29), hence JS Theorem III.5.19

holds, which further implies JS IV.3.32. Now the conditions in JS Theorem IV.3.35 are satisfied, and this theorem implies that the Hellinger process of order $\frac{1}{2}$ (see JS Definition IV.1.24) is given by

$$h_t^{\frac{1}{2}}(\omega) = \frac{1}{8}\int_0^t \gamma\sigma^2\beta_s(\omega)^2 ds + \frac{1}{2}\int_0^t \int_{y\neq 0} \left(\sqrt{\pi'(X_{s-}(\omega),y)} - \sqrt{\pi(X_{s-}(\omega),y)}\right)^2 dyds.$$

JS Theorem IV.2.1 says that $h_t^{\frac{1}{2}}(\omega) < \infty$ both $\mathbb{P}$ and $\mathbb{P}'$-a.s., hence there exists $x_0 \in \mathbb{R}$ such that $\int_{y\neq 0}\left(\sqrt{\pi'(x_0,y)} - \sqrt{\pi(x_0,y)}\right)^2 dy < \infty$. But one can show that the tail behavior at $y = 0$ of $\sqrt{\pi'(x,y)} - \sqrt{\pi(x,y)}$ does not depend on x (see Proposition 2.3, whose proof does not depend on Theorem 2.4), so we have $\int_{y\neq 0}\left(\sqrt{\pi'(x,y)} - \sqrt{\pi(x,y)}\right)^2 dy < \infty$ for any x.

Therefore, the process H defined in the proof of the sufficiency part does not jump to infinity. This together with $\sigma_{JS} = \infty$ allows us to apply JS Corollary III.5.22 (ii) which gives the form of the density process. □

Proposition 2.2. Define $q(s,0,y) := \frac{1}{\sqrt{\pi\sigma^2 s}}\exp\Big\{ - \frac{(y-\kappa(\theta-x)s)^2}{2\sigma^2 s}\Big\}$, the transition density of Brownian motion starting at 0 with drift $\kappa(\theta - x)$ and volatility σ. It is easy to see that

$$\lim_{s\to 0}\frac{p(s,x,x+y)}{q(s,0,y)} = 1$$

uniformly for y on any compact interval. We wish to prove that

$$\lim_{y\to 0}\frac{\int_{[0,\infty)} p(s,x,x+y)\nu(ds)}{\int_{[0,\infty)} q(s,0,y)\nu(ds)} = 1. \tag{B.2}$$

Note that

$$\lim_{y\to 0}\frac{\int_{[0,\delta)} p(s,x,x+y)\nu(ds)}{\int_{[0,\infty)} p(s,x,x+y)\nu(ds)} = 1 \tag{B.3}$$

for any $\delta > 0$. This is because for $s > \delta$, $p(s,x,x+y)$ is bounded in s, and $\int_{[\delta,\infty)} \nu(ds) < \infty$, so applying the Dominated Convergence Theorem

$$\begin{aligned}\lim_{y\to 0}\int_{[\delta,\infty)} p(s,x,x+y)\nu(ds) &= \int_{[\delta,\infty)} \lim_{y\to 0} p(s,x,x+y)\nu(ds)\\ &= \int_{[\delta,\infty)} \frac{1}{\sqrt{\frac{\pi\sigma^2}{\kappa}(1-e^{-2\kappa s})}}\exp\Big\{-\frac{(\theta-x)^2(1-e^{-\kappa s})}{\frac{\sigma^2}{\kappa}(1+e^{-\kappa s})}\Big\}\nu(ds),\end{aligned}$$

which is finite, and hence $\lim_{y\to 0}\frac{\int_{[\delta,\infty)} p(s,x,x+y)\nu(ds)}{\int_{[0,\infty)} p(s,x,x+y)\nu(ds)} = 0$. (B.3) is also true when $p(s,x,x+y)$ is replaced by $q(s,0,y)$ for the same reason.

Fix an interval $[-M,M]$ for y. Then for any $\epsilon > 0$, there exists some $\delta > 0$, such that for any $y \in [-M,M]$, $1-\epsilon < \frac{p(s,x,x+y)}{q(s,0,y)} < 1+\epsilon$ if $s < \delta$. Hence

$$1-\epsilon < \frac{\int_{[0,\delta)} p(s,x,x+y)\nu(ds)}{\int_{[0,\delta)} q(s,0,y)\nu(ds)} < 1+\epsilon$$

for any $y \in [-M,M]$. Now letting $y \to 0$ we have

$$1-\epsilon \le \lim_{y\to 0}\frac{\int_{[0,\delta)} p(s,x,x+y)\nu(ds)}{\int_{[0,\delta)} q(s,0,y)\nu(ds)} \le 1+\epsilon.$$

Equation (B.3) and $\lim_{y\to 0}\frac{\int_{[0,\delta)} q(s,0,y)\nu(ds)}{\int_{[0,\infty)} q(s,0,y)\nu(ds)}=1$ imply that

$$\lim_{y\to 0}\frac{\int_{[0,\infty)} p(s,x,x+y)\nu(ds)}{\int_{[0,\infty)} q(s,0,y)\nu(ds)}=\lim_{y\to 0}\frac{\int_{[0,\delta)} p(s,x,x+y)\nu(ds)}{\int_{[0,\delta)} q(s,0,y)\nu(ds)}.$$

Hence

$$1-\epsilon\le\lim_{y\to 0}\frac{\int_{[0,\infty)} p(s,x,x+y)\nu(ds)}{\int_{[0,\infty)} q(s,0,y)\nu(ds)}\le 1+\epsilon$$

for any ϵ. Now letting $\epsilon\to 0$, (B.2) is proved. □

Proposition 2.3. Now we prove that the asymptotic of the Lévy density $\overline{\pi}(y)$ of a SubBM does not depend on the drift. Suppose the drift and diffusion coefficients are μ and σ respectively. Then we have

$$\overline{\pi}(y)=\int_{[0,\infty)}\frac{1}{\sqrt{2\pi\sigma^2 s}}\exp\Big\{-\frac{(y-\mu s)^2}{2\sigma^2 s}\Big\}\nu(ds).$$

Similar to the proof in Proposition 2.2, it is straightforward to show that

$$\lim_{y\to 0}\overline{\pi}(y)=\lim_{y\to 0}\int_{[0,\infty)}\frac{1}{\sqrt{2\pi\sigma^2 s}}\exp\Big\{-\frac{y^2}{2\sigma^2 s}\Big\}\nu(ds),$$

which does not depend on μ. □

Proposition 2.4. We prove the case with condition (1) here. The case with condition (2) is proved in Kim et al. (2010). We can write:

$$\overline{\pi}(y)=\int_0^{\frac{1}{2\sigma^2\xi}}\frac{1}{\sqrt{2\pi\sigma^2 s}}\exp\Big\{-\frac{y^2}{2\sigma^2 s}\Big\}\nu(s)ds+\int_{\frac{1}{2\sigma^2\xi}}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2 s}}\exp\Big\{-\frac{y^2}{2\sigma^2 s}\Big\}\nu(s)ds.$$

Similar to the proof in Proposition 2.2, the second integral on the RHS is finite as $y\to 0$, so we only need to be concerned with the first integral. The rest of the proof is similar to Song and Vondraček (2009). Let $u=y^2/(2\sigma^2 s)$. Then

$$\begin{aligned}\int_0^{\frac{1}{2\sigma^2\xi}}\frac{1}{\sqrt{2\pi\sigma^2 s}}\exp\Big\{-\frac{y^2}{2\sigma^2 s}\Big\}\nu(s)ds&=\frac{|y|}{2\sigma^2\sqrt{\pi}}\int_{\xi y^2}^{\infty}u^{-\frac{3}{2}}e^{-u}\nu(\frac{y^2}{2\sigma^2 u})du\\&=\frac{(2\sigma^2)^{\beta-1}}{\sqrt{\pi}|y|^{2\beta-1}\ell(\frac{1}{y^2})}\int_{\xi y^2}^{\infty}u^{\beta-\frac{3}{2}}e^{-u}\frac{\nu(\frac{y^2}{2\sigma^2 u})}{h(y,u)}\frac{\ell(\frac{1}{y^2})}{\ell(\frac{2\sigma^2 u}{y^2})}du,\end{aligned}$$

where $h(y,u):=\frac{1}{(\frac{y^2}{2\sigma^2 u})^{\beta}\ell(\frac{2\sigma^2 u}{y^2})}$. From assumption (2.5), there is a constant $c>0$ such that for all $u>\xi y^2$, we have $\frac{\nu(\frac{y^2}{2\sigma^2 u})}{h(y,u)}<c$. Note that $\ell(\frac{1}{y^2})/\ell(\frac{2\sigma^2 u}{y^2})=f_{\ell,\xi}(y^2,u)$ for $u>\xi y^2$. So it follows from the assumption that we have $u^{\beta-\frac{3}{2}}e^{-u}\frac{\nu(\frac{y^2}{2\sigma^2 u})}{h(y,u)}\frac{\ell(\frac{1}{y^2})}{\ell(\frac{2\sigma^2 u}{y^2})}\le cu^{\beta-\frac{3}{2}}e^{-u}g(u)$. By the Dominated Convergence Theorem,

$$\lim_{y\to 0}\int_{\xi y^2}^{\infty}u^{\beta-\frac{3}{2}}e^{-u}\frac{\nu(\frac{y^2}{2\sigma^2 u})}{h(y,u)}\frac{\ell(\frac{1}{y^2})}{\ell(\frac{2\sigma^2 u}{y^2})}du=c_0\int_0^{\infty}u^{\beta-\frac{3}{2}}e^{-u}du=c_0\Gamma(\beta-\frac{1}{2}).$$

So the claim in Proposition 2.4 follows. □

Proposition 2.5. Let $p = \beta - 1$. For any $0 < \alpha < p$ we have that $\beta - \alpha > 1$. $\frac{1}{s^{\beta-\alpha}\ell(\frac{1}{s})}$ is not integrable near zero because $\lim_{s\to 0} \frac{1}{s^{\beta-\alpha}\ell(\frac{1}{s})} \Big/ \frac{1}{s} = \lim_{s\to 0} \frac{1}{s^{\beta-\alpha-1}\ell(\frac{1}{s})} = \infty$ by Bingham et al. (1987) Proposition 1.3.6.

For any $\alpha > p$ we have that $\beta - \alpha < 1$. There is a value γ such that $\beta - \alpha < \gamma < 1$. So $\lim_{s\to 0} \frac{1}{s^{\beta-\alpha}\ell(\frac{1}{s})} \Big/ \frac{1}{s^{\gamma}} = \lim_{s\to 0} \frac{1}{s^{\beta-\alpha-\gamma}\ell(\frac{1}{s})} = 0$, again from Bingham et al. (1987) Proposition 1.3.6. Therefore $\frac{1}{s^{\beta-\alpha}\ell(\frac{1}{s})}$ is integrable near 0 for any $\alpha > p$.

Together we have the BG index is $\beta - 1$. The assertion for the second part follows from the asymptotic implied by Proposition 2.4. □

Theorem 2.5. The proof is entirely similar to the proof of the necessity part of Theorem 2.4. First, JS Theorem III.3.24 implies (2.7), (2.8) and (2.9). To prove the (2.10) and the form of the density process, replace $\int_0^t \int_{y\neq 0} \left(\sqrt{\pi'(X_{s-}(\omega), y)} - \sqrt{\pi(X_{s-}(\omega), y)}\right)^2 dyds$ by $\int_0^t \int_{y\neq 0} (\sqrt{Y(s,\omega,y)} - 1)^2 \pi(X_{s-}(\omega), y) dyds$ and the rest remains the same. □

Theorem 2.7. (1) First we notice two facts. (1) On any compact interval $I \in \mathbb{R}$, there exists a constant C depending on I, such that for $n \geq 1$ (c.f. Nikiforov and Uvarov (1988) p.54 Eq. (28a))

$$|\varphi_n(x)| \leq C n^{-1/4}, \quad x \in I. \tag{B.4}$$

(2) $|f_n| \leq \|f\|$ for all n by the Cauchy-Schwartz inequality.

For $t > 0$, on one hand, the RHS of (2.11) is bounded by $|f_0| + C\|f\| \sum_{n=1}^{\infty} e^{-\kappa n t} n^{-\frac{1}{4}}$, which is finite due to the rapid decay of $e^{-\kappa n t}$. This expansion converges absolutely for each x and uniformly in x on compacts, thus it defines a continuous function. On the other hand, the function $\mathcal{P}_t f(x)$ is infinitely differentiable in x. In fact, if x is replaced with a complex variable z, $\mathcal{P}_t f(z)$ is an entire function (see Theorem 3.1 in Thangavelu (2006)). The L^2-convergence implies convergence almost everywhere in this case. To be more precise, let $S(x)$ denotes the RHS of (2.11), and $S_n(x)$ its n-th partial sum. Convergence of $S_n(x)$ to $\mathcal{P}_t f(x)$ in L^2 implies that there is a subsequence $S_{k_n}(x)$ converging to $\mathcal{P}_t f(x)$ almost everywhere. But the limit of $S_{k_n}(x)$ is $S(x)$, so $S(x) = \mathcal{P}_t f(x)$ almost everywhere. Furthermore, since both sides of (2.11) are continuous functions, they must agree at every point. Therefore, for the OU semigroup, the eigenfunction expansion in (2.11) is valid pointwise for each $f \in L^2(\mathbb{R}, \mathfrak{m})$ and $t > 0$.

(2) For the SubOU semigroup, when the eigenfunction expansion on the RHS of (2.13) converges absolutely for each x, the spectral representation (2.13) for $\mathcal{P}_t^{\phi} f(x)$ is valid for each x, as the following calculation can be justified:

$$\begin{aligned}
\mathcal{P}_t^{\phi} f(x) &= \int_{[0,\infty)} \mathcal{P}_s f(x) q_t(ds) = \int_{[0,\infty)} \sum_{n=0}^{\infty} e^{-\kappa n s} f_n \varphi_n(x) q_t(ds) \\
&= \sum_{n=0}^{\infty} \int_{[0,\infty)} e^{-\kappa n s} q_t(ds) f_n \varphi_n(x) = \sum_{n=0}^{\infty} e^{-\phi(\kappa n) t} f_n \varphi_n(x).
\end{aligned}$$

In the above, we first use the definition of the SubOU semigroup, then represent $\mathcal{P}_s f(x)$ by the eigenfunction expansion which also converges pointwise, interchange the summation and expectation justified by the absolute convergence of the expansion and the dominated convergence theorem, and use the Laplace transform of the convolution semigroup.

For $t > 0$, either condition (i) or (ii) in Theorem 2.7 ensures the absolute convergence of the expansion for each x, and thus the eigenfunction expansion for the SubOU semigroup converges pointwise. □

Theorem 2.8. Convergence of the expansion in the RHS of (2.14) to the RHS of (2.1) follows from the well-known Mehler formula for Hermite polynomials (e.g., Thangavelu (2006)

Proposition 2.3). Part (2) follows from the estimate (B.4) for the eigenfunctions. □

Theorem 2.9. We first notice that for all n and all real x, $|\varphi_n(x)| \leq 1.0864 e^{\frac{\kappa(x-\theta)^2}{2\sigma^2}}$. This bound is given in Boyd (1984) and is shown there to be tight. Therefore we have $\left|\sum_{n=M}^{\infty} e^{-\phi(\kappa n)t} f_n \varphi_n(x)\right| \leq 1.0864 e^{\frac{\kappa(x-\theta)^2}{2\sigma^2}} \sum_{n=M}^{\infty} e^{-\phi(\kappa n)t} |f_n|$. Using $|f_n| \leq \|f\|$ and assuming that $\sum_{n=0}^{\infty} e^{-\phi(\kappa n)t} < \infty$ is satisfied for all $t > 0$, we obtain the estimate in Theorem 2.9. □

Lemma 3.1 and Theorem 3.1. First, note that the function $e^x \in L^2(\mathbb{R}, \mathfrak{m})$, so the spectral representation theorem applies and

$$\begin{aligned} f_n &= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\right) \sqrt{\frac{\kappa}{\pi\sigma^2}} e^{-\frac{\kappa(\theta-x)^2}{\sigma^2}} dx \\ &= e^{\theta + \frac{\sigma^2}{4\kappa}} \frac{1}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} e^{-(y - \frac{\sigma}{2\sqrt{\kappa}})^2} H_n(y) dy = e^{\theta + \frac{\sigma^2}{4\kappa}} \frac{1}{\sqrt{n!}} \left(\frac{\sigma}{\sqrt{2\kappa}}\right)^n, \end{aligned}$$

where we used the identity $\int_{-\infty}^{\infty} e^{-(y-z)^2} H_n(y) dy = \sqrt{\pi}(2z)^n$ (Prudnikov et al. (1986) p.488 No.17 of 2.20.3). It can be shown by using the estimate of the eigenfunctions (B.4) that the Hermite expansion of the exponential function is absolutely convergent for each x, hence condition (i) in Theorem 2.7 is satisfied. The results in Theorem 3.1 are obtained by applying (2.13) to e^x. □

Theorem 3.2. *Necessity.* Let (B^P, C^P, Π^P) be the semimartingale characteristics of the SubOU process with generating tuple $(\kappa_P, \theta_P, \sigma_P, \gamma_P, \nu_P)$. Then $(B^P + H, C^P, \Pi^P)$ is the set of characteristics for X under $\mathbb{P}$. Since $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent, Theorem 2.5 implies condition (2) and (3), and that there exists some deterministic function $\overline{\beta}$ such

$$\begin{aligned} B_t^P(\omega) + H(t) &= B_t(\omega) + \gamma\sigma^2 \int_0^t \left(\beta_s(\omega) + \overline{\beta}(s)\right) ds \\ &+ \int_{[0,t]\times\mathbb{R}} y 1_{\{|y|\leq 1\}} \left(\pi^P(X_{s-}(\omega), y) - \pi(X_{s-}(\omega), y)\right) dy ds, \end{aligned}$$

where $\beta_s(\omega) = \frac{(\gamma_P\kappa_P\theta_P - \gamma\kappa\theta) - (\gamma_P\kappa_P - \gamma\kappa)X_{s-}(\omega)}{\gamma\sigma^2} 1_{\{\gamma\neq 0\}}$. Thus if $\gamma > 0$, then H is an absolutely continuous function of time, and $H(0) = 0$. If $\gamma = 0$, then $H(t) = 0$ for all t.

Sufficiency. If $\gamma = 0$, then the conclusion is directly implied by Theorem 2.4. If $\gamma > 0$, then using Theorem 2.4, we can first find a measure $\widetilde{\mathbb{P}}$ locally equivalent to $\mathbb{Q}$, and under $\widetilde{\mathbb{P}}$, X is a SubOU process with generating tuple $(\kappa_P, \theta_P, \sigma_P, \gamma_P, \nu_P)$. Let X^c be the continuous local martingale part of X under $\widetilde{\mathbb{P}}$. Since H is absolutely continuous, we can define $\lambda(t) := \frac{1}{\gamma_P\sigma_P^2}\frac{dH(t)}{dt}$. Then define a measure $\mathbb{P}$ by $\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} = \mathscr{E}(\lambda \cdot X^c)$. This is a Radon-Nikodym density process because the Novikov condition is satisfied, so the stochastic exponential is a true martingale. Now $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are locally equivalent. Under $\mathbb{P}$, the first component of the semimartingale characteristics becomes $B_t^P + \int_0^t \lambda_s \gamma_P \sigma_P^2 ds = B_t^P + H(t)$. Thus, X is a SubOU process with the generating tuple $(\kappa_P, \theta_P, \sigma_P, \gamma_P, \nu_P)$ plus a deterministic function $H(t)$. □

Theorem 3.4. Since the put payoff is bounded and the measure is Gaussian, it belongs to $L^2(\mathbb{R}, \mathfrak{m})$. The expansion coefficients are computed as follows:

$$\int_{-\infty}^{\infty} (K - F(x,t,t^*))^+ \varphi_n(x)\mathfrak{m}(x) dx = \int_{-\infty}^{\infty} (K - F(x,t,t^*)) 1_{\{x<x^*\}} \varphi_n(x)\mathfrak{m}(x) dx.$$

$$\int_{-\infty}^{\infty} K 1_{\{x<x^*\}} \varphi_n(x)\mathfrak{m}(x) dx = \frac{K}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\frac{\sqrt{\kappa}}{\sigma}(x^*-\theta)} H_n(x) e^{-x^2} dx = \frac{K}{\sqrt{\pi 2^n n!}} b_n(w^*).$$

The integral in (3.9) is given in Prudnikov et al. (1986). For the second integral,

$$\int_{-\infty}^{\infty} F(x,t,t^*)1_{\{x<x^*\}}\varphi_n(x)\mathfrak{m}(x)dx = \int_{-\infty}^{x^*} Fe^{-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)\tau} f_m\varphi_m(x)\varphi_n(x)\mathfrak{m}(x)dx$$

$$= Fe^{-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)\tau} f_m \int_{-\infty}^{x^*}\varphi_m(x)\varphi_n(x)\mathfrak{m}(x)dx = \frac{1}{\sqrt{\pi 2^n n!}}Fe^{\theta+\frac{\sigma^2}{4\kappa}-G(t^*)}\sum_{m=0}^{\infty} e^{-\phi(\kappa m)\tau}\frac{\alpha^m}{m!}a_{n,m}(w^*).$$

The interchange of integration and summation is justified by the Dominated Convergence Theorem due to the estimate:

$$\left|\int_{-M}^{x^*}\varphi_m(x)\varphi_n(x)\mathfrak{m}(x)dx\right| \leqslant \int_{-\infty}^{\infty}|\varphi_m(x)\varphi_n(x)|\,\mathfrak{m}(x)dx \leqslant ||\varphi_m||\cdot||\varphi_n|| = 1,$$

and $\sum_{m=0}^{\infty} e^{-\phi(\kappa m)\tau} f_m < \infty$. With some further simplifications we obtain (3.7). The integral in (3.10) is calculated as follows. Consider the integral $J^L_{n,m}(x) := \int_{-\infty}^{x} H_n(z)H_m(z)e^{-z^2}dz$. By the identity $H_n(z)H_m(z) = \sum_{k=0}^{min(n,m)}\binom{m}{k}\binom{n}{k}2^k k! H_{n+m-2k}(z)$ (Prudnikov et al. (1986) p.640 No.11 of 4.5.1), we have

$$J^L_{n,m}(x) = \sum_{k=0}^{min(n,m)}\binom{m}{k}\binom{n}{k}2^k k!\int_{-\infty}^{x} H_{n+m-2k}(z)e^{-z^2}dz = \sum_{k=0}^{min(n,m)}\binom{m}{k}\binom{n}{k}2^k k! b_{n+m-2k}(x). \quad \square$$

Theorem 4.1.

$$\mathbb{E}[f(Y_t)|Y_s,Z_s] = \mathbb{E}\Big[\mathbb{E}[f(X_{T_t})|T_t-T_s,Y_s,Z_s]\Big|Y_s,Z_s\Big] = \mathbb{E}\Big[\sum_{n=0}^{\infty} e^{-\phi(\kappa n)(T_t-T_s)} f_n\varphi_n(Y_s)\Big|Z_s\Big]$$

$$= \sum_{n=0}^{\infty}\mathbb{E}[e^{-\phi(\kappa n)\int_s^t (a(u)+Z_u)du}|Z_s]f_n\varphi_n(Y_s) = \sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_s^t a(u)du}\mathcal{L}_{CIR}\Big(t-s,\phi(\lambda)\Big|Z_s\Big)f_n\varphi_n(Y_s),$$

where condition (1) or (2) in Theorem 4.1 justify the interchange of summation and expectation. $\square$

Proposition 4.1. Define $\tilde{Z}_t = Z^c_{S_t}$, where S is the inverse of T. Then $Z^c_t = \tilde{Z}_{T_t}$. Since the time change S is continuous, Z^c is adapted to S (see Jacod (1979) Definition X.13 for adaption to a time change), and by Jacod (1979) Theorem X.16, $\tilde{Z}$ is a continuous local martingale w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Now $[Y^c,Z^c]_t = [X^c_T,\tilde{Z}_T]_t = [X^c,\tilde{Z}]_{T_t}$, where the second equality is from Jacod (1979) Theorem X.17. Since X and Z are independent, X^c and $\tilde{Z}$ are independent. Because the cross-variation of two independent continuous local martingale is 0, we have $[X^c,\tilde{Z}]_t = 0$ for all t, hence $[X^c,\tilde{Z}]_{T_t} = 0$, and the claim is proved. $\square$

Theorem 4.4. Conditioning on the terminal state Z_t of the CIR process, we have:

$$\mathbb{E}\left[(K-F(Y_t,Z_t,t,t^*))^+\right] = \int_0^{\infty}\mathbb{E}\left[(K-F(Y_t,z_t,t,t^*))^+|Z_t=z_t\right]p_{CIR}(t,z_0,z_t)dz_t$$

$$= \int_0^{\infty}\mathbb{E}\left[\sum_{n=0}^{\infty} e^{-\phi(\kappa n)T_t}p_n(t,t^*,w^*,F)\varphi_n(y_0)\right| Z_t = z_t\Bigg] p_{CIR}(t,z_0,z_t)dz_t$$

$$= \int_0^{\infty}\left\{\sum_{n=0}^{\infty}\mathbb{E}\left[e^{-\phi(\kappa n)T_t}\right| Z_t=z_t\Big] p_n(t,t^*,w^*,F)\varphi_n(y_0)\right\} p_{CIR}(t,z_0,z_t)dz_t$$

$$= \int_0^{\infty}\left\{\sum_{n=0}^{\infty} e^{-\phi(\kappa n)\int_0^t a(u)du}\mathcal{L}_{CIR}(t,\phi(\kappa n)|z_0,z_t)p_n(t,t^*,w^*,F)\varphi_n(y_0)\right\} p_{CIR}(t,z_0,z_t)dz_t.$$

The interchange of expectation and summation is justified by the assumption. □

Proposition 5.1. Using the recursion for Hermite polynomials, for $n \ge 1$, $m \ge 0$,

$$a_{n+1,m+1}(x) = \int_{-\infty}^{x} H_{n+1}(y)H_{m+1}(y)e^{-y^2}dy = \int_{-\infty}^{x}[2yH_n(y) - 2nH_{n-1}(y)]H_{m+1}(y)e^{-y^2}dy$$

$$= -\int_{-\infty}^{x} H_n(y)H_{m+1}(y)de^{-y^2} - 2na_{n-1,m+1}(x)$$

$$= -H_n(y)H_{m+1}(y)e^{-y^2}|_{-\infty}^{x} + \int_{-\infty}^{x}[H_n'(y)H_{m+1}(y) + H_n(y)H_{m+1}'(y)]e^{-y^2}dy - 2na_{n-1,m+1}(x)$$

$$= -H_n(y)H_{m+1}(y)e^{-y^2} + \int_{-\infty}^{x}[2nH_{n-1}(y)H_{m+1}(y) + 2(m+1)H_n(y)H_m(y)]e^{-y^2}dy - 2na_{n-1,m+1}(x)$$

$$= 2(m+1)a_{n,m}(x) - H_n(x)H_{m+1}(x)e^{-x^2}.$$

It is easy to verify that this recursion is also true for $n = 0$. Therefore we have

$$a_{n+1,m+1}(x) = 2(m+1)a_{n,m}(x) - H_n(x)H_{m+1}(x)e^{-x^2}, \quad n \ge 0, m \ge 0. \tag{B.5}$$

In particular, $a_{n,n}(x) = 2na_{n-1,n-1}(x) - H_{n-1}(x)H_n(x)e^{-x^2}$, $n \ge 1$. Noting the symmetry $a_{n,m}(x) = a_{m,n}(x)$, we also obtain the following by exchanging the role of n and m in (B.5):

$$a_{m+1,n+1}(x) = 2(n+1)a_{n,m}(x) - H_m(x)H_{n+1}(x)e^{-x^2}, \quad m \ge 0, n \ge 0. \tag{B.6}$$

If $m \ne n$, subtracting (B.6) from (B.5), we obtain:

$$a_{n,m}(x) = e^{-x^2}(H_n(x)H_{m+1}(x) - H_m(x)H_{n+1}(x))/(2(m-n)) \quad (n \ne m, n \ge 0, m \ge 0). \quad \square$$

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