| ENGG 5501: Foundations of Optimization | 2016–17 First Term |
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# Homework Set 1

Instructor: Anthony Man–Cho So — Due: September 30, 2016

**SOLVE THE FOLLOWING PROBLEMS:**

**Problem 1 (20pts).** Let $P = \left\{ x \in \mathbb{R}^n : a_i^T x \le b_i \text{ for } i = 1, \ldots, m \right\}$, where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. Recall that a ball with center $\bar{x} \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set $B(\bar{x}, r) = \{x \in \mathbb{R}^n : \|x - y\|_2 \le r\}$. We are interested in finding a ball with the largest possible radius, subject to the condition that it is entirely contained within the set $P$ (also known as the *largest inscribed ball* in $P$). Give a linear programming formulation of this problem.

**Problem 2 (30pts).** Let $S = \left\{ x \in \mathbb{R}^n : x^T A x + b^T x + c \le 0 \right\}$, where $A \in \mathcal{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given.

(a) **(15pts).** Show that $S$ is convex if $A \succeq \mathbf{0}$. Is the converse true? Explain.

(b) **(15pts).** Let $H = \left\{ x \in \mathbb{R}^n : g^T x + h = 0 \right\}$, where $g \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $h \in \mathbb{R}$. Show that $S \cap H$ is convex if $A + \lambda g g^T \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$.

**Problem 3 (50pts).** Let $S, T$ be closed convex sets in $\mathbb{R}^n$ such that $S \cap T \neq \emptyset$. A problem that arises frequently in optimization is that of finding a point $x \in S \cap T$. A natural algorithm is to start with an arbitrary $x_0 \in S$ and then alternately project onto $S$ and $T$; i.e., compute the sequence

$$y_k = \Pi_T(x_k), \quad x_{k+1} = \Pi_S(y_k) \qquad \text{for } k = 0, 1, \ldots.$$

Clearly, we have $x_k \in S$ and $y_k \in T$ for $k = 0, 1, \ldots$. Our goal is to prove that the sequences $\{x_k\}_{k \ge 0}$ and $\{y_k\}_{k \ge 0}$ both converge to a point $x^* \in S \cap T$.

(a) **(20pts).** Let $\bar{x} \in S \cap T$ be arbitrary. Show that

$$\|y_k - \bar{x}\|_2^2 \le \|x_k - \bar{x}\|_2^2 - \|x_k - y_k\|_2^2$$

and

$$\|x_{k+1} - \bar{x}\|_2^2 \le \|y_k - \bar{x}\|_2^2 - \|x_{k+1} - y_k\|_2^2.$$

Hence, or otherwise, show that the sequences $\{x_k\}_{k \ge 0}$ and $\{y_k\}_{k \ge 0}$ are bounded.

(b) **(15pts).** Using the results in (a), show that the sequences $\{\|x_k - y_k\|_2\}_{k \ge 0}$ and $\{\|x_{k+1} - y_k\|_2\}_{k \ge 0}$ both converge to 0.

(c) **(15pts).** Let $x^*$ be a limit point of the sequence $\{x_k\}_{k \ge 0}$, which exists because of the boundedness of $\{x_k\}_{k \ge 0}$. Using the result in (b), show that $x^* \in S \cap T$. Hence, or otherwise, show that $x_k \to x^*$ and $y_k \to x^*$. (*Hint: Note that $x^*$ is defined as a limit point of $\{x_k\}_{k \ge 0}$, which means that there is a* subsequence *of $\{x_k\}_{k \ge 0}$ converging to $x^*$. Here, you are asked to show that in fact the* entire sequence *$\{x_k\}_{k \ge 0}$ converges to $x^*$.*)