ENGG 5501: Foundations of Optimization	**2016–17 First Term**

Homework Set 2

Instructor: Anthony Man–Cho So — Due: October 14, 2016

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (15pts). Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be the function given by

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } \|(x_1, x_2)\|_2 < 1, \\ \in [0, +\infty] & \text{if } \|(x_1, x_2)\|_2 = 1, \\ +\infty & \text{if } \|(x_1, x_2)\|_2 > 1. \end{cases}$$

Show that f is convex. Hence, or otherwise, give an example of a convex function whose epigraph is not closed.

Problem 2 (20pts). Let $C \in \mathcal{S}_+^n$ be given. Show that the function $f : \mathcal{S}_{++}^n \to \mathbb{R}_+$ given by $f(X) = \mathrm{tr}(CX^{-1})$ is convex.

Problem 3 (25pts). Given a set $C \subset \mathbb{R}^n$, define the indicator function $i_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of C by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

Compute i_C^*, the conjugate of i_C, for the following sets. Show your calculations.

(a) **(15pts).** $C = \left\{ x \in \mathbb{R}^n : a^T x \le b \right\}$, where $a \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $b \in \mathbb{R}$ are given.

(b) **(10pts).** $C = \left\{ x \in \mathbb{R}_+^n : \|x\|_2 \le 1 \right\}$.

Problem 4 (20pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Consider the following systems:

$$\begin{array}{ll} \text{(I)} & Ax = b. \\ \text{(II)} & A^T y = \mathbf{0},\ b^T y = 1. \end{array}$$

(a) **(5pts).** Show that (I) and (II) cannot both have solutions.

(b) **(15pts).** Suppose that (I) has no solution. Show, without using any theorems of alternatives, that (II) has a solution.

Problem 5 (20pts). Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax = b,\ x \ge \mathbf{0}\}$. Suppose that $P \ne \emptyset$. We say that P contains a *recession direction* $d \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ if for any $x_0 \in P$, we have $\{x \in \mathbb{R}^n : x = x_0 + \lambda d,\ \lambda \ge 0\} \subset P$. Show that the following statements are equivalent:

(i) P contains a recession direction $d \in \mathbb{R}^n$.

(ii) There exists a vector $d \in \mathbb{R}^n$ satisfying

$$Ad = \mathbf{0},\ \ d \ge \mathbf{0},\ \ d \ne \mathbf{0}.$$

(iii) P is unbounded.