ENGG 5501: Foundations of Optimization	2016–17 First Term
Homework Set 3	
Instructor: Anthony Man–Cho So	Due: October 31, 2016

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (25pts). Consider a *production game* defined as follows. Let $\mathcal{N} = \{1, \ldots, n\}$ be the set of players, each of whom is given a vector $b^i = (b^i_1, \ldots, b^i_m)$ $(i = 1, \ldots, n)$ of resources. These resources can be used to produce goods, which in turn can be sold at a given market price. Specifically, we assume the following production model: For player $i \in \mathcal{N}$, a unit of the j–th good $(j = 1, \ldots, p)$ requires a^i_{kj} units of the k–th resource $(k = 1, \ldots, m)$ to produce. Furthermore, the j–th good can be sold at a price c_j, where $j = 1, \ldots, p$.

Now, let $S \subset \mathcal{N}$ be a coalition of players. Such a coalition will possess

$$b_k(S) = \sum_{i \in S} b^i_k$$

units of the k–th resource, where $k = 1, \ldots, m$. Using all of their resources, the coalition S can produce any vector $x = (x_1, \ldots, x_p) \in \mathbb{R}^p_+$ of goods that satisfies $A(S)x \le b(S)$, where

$$A(S)_{kj} = \min_{i \in S} \left\{ a^i_{kj} \right\} \quad \text{for } k = 1, \ldots, m;\ j = 1, \ldots, p \quad \text{and} \quad b(S) = (b_1(S), \ldots, b_m(S)).$$

Naturally, a coalition S would like to maximize its revenue. The optimization problem it faces can be formulated as the following LP:

$$\begin{array}{rcll} v(S) & = & \text{maximize} & c^T x \\ & & \text{subject to} & A(S)x \le b(S), \\ & & & x \ge \mathbf{0}. \end{array} \tag{1}$$

The function v will be the value function of this game. Recall that an allocation vector $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is in the core iff $\sum_{i \in \mathcal{N}} z_i = v(\mathcal{N})$ and $\sum_{i \in S} z_i \ge v(S)$ for all $S \subset \mathcal{N}$.

(a) **(10pts).** Consider the LP faced by the grand coalition $\mathcal{N}$. Write down its dual.

(b) **(15pts).** Suppose that the dual LP given in (a) is feasible. Let y^* be one of its optimal solutions. Show that the allocation vector

$$z^* = \left(\left(b^1\right)^T y^*, \left(b^2\right)^T y^*, \ldots, \left(b^n\right)^T y^* \right) \in \mathbb{R}^n$$

belongs to the core.

Problem 2 (15pts). Show that for $n \ge 2$, the Lorentz cone

$$\mathcal{Q}^{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \ge \|x\|_2\}$$

is non–polyhedral.

Problem 3 (45pts). Consider the set $\mathcal{C}^n = \{X \in \mathcal{S}^n : v^T X v \ge 0 \text{ for all } v \ge \mathbf{0}\}$.

(a) **(15pts).** Show that $\mathcal{C}^n$ is a non–empty closed convex cone.

(b) **(15pts).** Show that $\mathrm{conv}(\{vv^T : v \geq \mathbf{0}\})$ is a non–empty closed convex set.

(c) **(15pts).** Let

$$(\mathcal{C}^n)^* = \{Y \in \mathcal{S}^n : X \bullet Y \geq 0 \text{ for all } X \in \mathcal{C}^n\}$$

be the dual cone of $\mathcal{C}^n$. Show that $(\mathcal{C}^n)^* = \mathrm{conv}(\{vv^T : v \geq \mathbf{0}\})$.

Problem 4 (15pts). Let $Q \in \mathcal{S}^n_+$ be given. Give an equivalent SOCP formulation of the following problem:

$$\begin{array}{ll} \inf & t \\ \text{subject to} & x^T Q x \leq t. \end{array}$$

Justify your answer.