ENGG 5501: Foundations of Optimization	2015–16 First Term

Final Examination

Time Limit: 2 Hours — December 17, 2015

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (25pts). Let $\mathcal{E} \subset \{(i,j) : 1 \le i < j \le n\}$ with $\mathcal{E} \neq \emptyset$ and $d_{ij} > 0$ be given. Define $e = (1,1,\ldots,1) \in \mathbb{R}^n$ and $E_{ij} = (e_i - e_j)(e_i - e_j)^T \in \mathcal{S}_+^n$, where $e_i \in \mathbb{R}^n$ is the i–th basis vector. Consider the SDP

$$\begin{array}{ll} \inf & 0 \\ \text{subject to} & E_{ij} \bullet X = d_{ij} \quad \text{for } (i,j) \in \mathcal{E}, \\ & ee^T \bullet X = 0, \\ & X \succeq \mathbf{0}. \end{array} \tag{L}$$

(a) **(10pts).** Let y_{ij} and θ be the dual variables corresponding to the constraints $E_{ij} \bullet X = d_{ij}$ and $ee^T \bullet X = 0$, respectively. Write down the dual of (L).

(b) **(15pts).** Suppose that the dual of (L) is feasible. Let $(\bar{y}, \bar{\theta})$ be any dual feasible solution, where $\bar{y} = (\bar{y}_{ij})_{(i,j)\in\mathcal{E}}$. Show that the matrix $\sum_{(i,j)\in\mathcal{E}} \bar{y}_{ij} E_{ij}$ is semidefinite (i.e., either positive semidefinite or negative semidefinite). (*Hint: Evaluate* $E_{ij} \bullet ee^T$.)

Problem 2 (40pts). Let $n \ge 1$ be given. Consider the set

$$\mathcal{Q}_r^{n+2} = \left\{ (u,v,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : 2uv \ge \|x\|_2^2,\ u,v \ge 0 \right\}.$$

(a) **(15pts).** Show, by first principles, that $\mathcal{Q}_r^{n+2}$ is a pointed cone.

(b) **(10pts).** Show that $(u,v,x) \in \mathcal{Q}_r^{n+2}$ if and only if $(\bar{u},\bar{v},x) \in \mathcal{Q}^{n+2}$, where $\mathcal{Q}^{n+2} = \left\{ (t,z) \in \mathbb{R} \times \mathbb{R}^{n+1} : t \ge \|z\|_2 \right\}$ and

$$\bar{u} = \frac{1}{\sqrt{2}}(u+v), \quad \bar{v} = \frac{1}{\sqrt{2}}(u-v).$$

(c) **(15pts).** Using the result in (b) and the fact that $(\mathcal{Q}^{n+2})^* = \mathcal{Q}^{n+2}$, or otherwise, show that $(\mathcal{Q}_r^{n+2})^* = \mathcal{Q}_r^{n+2}$.

Problem 3 (20pts). Let $a_1, \ldots, a_n > 0$ be given. Consider the problem

$$\begin{array}{ll} \text{maximize} & x_1 x_2 \cdots x_n \\ \text{subject to} & \displaystyle\sum_{i=1}^n \frac{x_i}{a_i} = 1. \end{array} \tag{P}$$

(a) **(10pts).** Write down the first–order optimality conditions of (P) and explain why they are necessary for optimality.

(b) **(10pts).** Using the result in (a), or otherwise, determine the optimal solution to (P).

Problem 4 (15pts). Consider the standard form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned} \tag{LP}$$

Suppose that (LP) has a *unique* optimal solution. Does this necessarily imply that the dual of (LP) also have a *unique* optimal solution? If so, give a proof. If not, construct an example (any dimension will do) to explain why.