Theorem (Set-Set Separation)

Let \( S_1, S_2 \) be non-empty, closed convex sets with \( S_1 \cap S_2 = \emptyset \). Suppose further that \( S_2 \) is bounded. Then, \( \exists y \) such that

\[
\max_{z \in S_1} y^Tz < \min_{u \in S_2} y^Tu
\]

Remark: Since \( S_2 \) is closed and bounded, it is compact.

Proof: Define \( S_1 - S_2 \equiv \{ z - u : z \in S_1, u \in S_2 \} \). Note that this is non-empty and convex (exercise). Since \( S_1 \cap S_2 = \emptyset \), \( 0 \notin S_1 - S_2 \). Before we can apply point-set separation, we need to check \( S_1 - S_2 \) is closed.

Suppose for now that \( S_1 - S_2 \) is closed. Then, \( \exists y \) s.t.

\[
\max_{v \in S_1 - S_2} y^Tv < y^T0 = 0
\]

Observe

\[
v \in S_1 - S_2 \iff \exists z \in S_1, u \in S_2 : v = z - u
\]

\[
\max_{v \in S_1 - S_2} y^Tv = \max_{z \in S_1, u \in S_2} y^T(z - u) = y^Tz + y^Tu
\]

Closedness of \( S_1 - S_2 \):

Show: Let \( x_1, x_2, ... \) be a sequence in \( S_1 - S_2 \) such that \( x_k \to x \). Need: \( x \notin S_1 - S_2 \)

Since \( x_k \in S_1 - S_2 \), \( \exists z_k \in S_1, u_k \in S_2 \) s.t \( x_k = z_k - u_k \).
Since $x_k \in S_1 - S_2$, there exists $z_k \in S_1$, $u_k \in S_2$ such that $x_k = z_k - u_k$.

**Fact:** Since $S_2$ is compact, every sequence in $S_2$ has a convergent subsequence.

Hence, there exists a subsequence $\{u_{k_i}\}$ of $u$ (i.e., $u_{k_1}, u_{k_2}, u_{k_3}, \ldots$) such that $u_{k_i} \to u$ as $i \to \infty$.

Note: Since $u_k \in S_2$ and $S_2$ is closed, $u \in S_2$. Also, since $x_k \to x$, all subsequences of $\{x_k\}$ converge to $x$ implies $x_k \to x$.

Now, $x_{k_i} = z_{k_i} - u_{k_i}$

$\Rightarrow z_{k_i} = x_{k_i} + u_{k_i}$

$\Rightarrow \lim_{i \to \infty} z_{k_i} = \lim_{i \to \infty} (x_{k_i} + u_{k_i}) = x + u$

Since $z_{k_i} \in S_1$, $z_{k_i} \to x + u$, and $S_1$ is closed, we have $x + u \in S_1$.

Hence, $x = (x + u) - u \in S_1 - S_2$, as desired.

**Convex Functions**

**Definition:** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function that is not identically $+\infty$.

1. We say $f$ is **convex** if for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1],

   \[
   f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)
   \]

We say $f$ is **concave** if $-f$ is convex.
2. The epigraph of $f$ is the set
\[ \text{epi}(f) = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t \} \subseteq \mathbb{R}^{n+1} \]

3. The effective domain of $f$ is the set
\[ \text{dom}(f) = \{ x \in \mathbb{R}^n : f(x) < +\infty \} \subseteq \mathbb{R}^n \]

4. Let $S \subseteq \mathbb{R}^n$ be a set. The indicator of $S$ is
\[ I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S \end{cases} \]

Remark: With the indicator,
\[ \inf_{x \in S} f(x) \iff \inf_{x \in \mathbb{R}^n} \{ f(x) + I_S(x) \} \]

Constrained Unconstrained