ESTR 2004: Discrete Mathematics for Engineers **2021–22 First Term**

Homework Set 6

Instructor: Anthony Man–Cho So   Due: 5pm, December 6, 2021

Required Reading: LLM Chapters 11.1 — 11.3, 11.8, 11.9.1, 11.10.1 — 11.10.2, 16.2, 16.5.1 — 16.5.3; notes from November 8, 10, 11, 15, 18, 22, 24, and 25.

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (10pts). Let r be a given real number and $k \geq 0$ be a given integer. Show that

$$\binom{r}{k}\binom{r-1/2}{k} = \frac{1}{4^k}\binom{2r}{2k}\binom{2k}{k}.$$

Problem 2 (15pts). Suppose that you repeatedly flip a fair coin until three consecutive flips match the pattern `HHT` or the pattern `TTH`. What is the probability you will see `HHT` first? Show all your calculations. In particular, specify the sample space and the event you are interested in.

Problem 3 (15pts). A hand of 5 cards from a standard 52-card deck is called a *sequence* if it consists of 5 consecutive cards of any suit; e.g., 3♠–4♠–5♢–6♡–7♣. Note that an ace can either be high (as in 10–J–Q–K–A) or low (as in A–2–3–4–5), but cannot go around (such as Q–K–A–2–3). What is the probability of getting a sequence? Show your calculations. In particular, specify the sample space and the event you are interested in.

Problem 4 (25pts). Suppose that we have n identical balls and n bins numbered $\{1, \ldots, n\}$, where $n \geq 2$. For each of the n balls, we pick one of the n bins uniformly at random and place the ball in it.

(a) **(10pts).** Show that for $k = 0, 1, \ldots, n$, the probability of having at least k balls in bin 1 is at most

$$\binom{n}{k}\left(\frac{1}{n}\right)^k. \tag{1}$$

(b) **(15pts).** Using the result in (a), or otherwise, show that with probability at least $1 - O(n^{-1})$, the bin with the most balls among the n bins has $O\left(\frac{\log n}{\log\log n}\right)$ balls. (*Hint: Determine the level of k at which the quantity in* (1) *is at most* $1/n^2$. *Then, observe that the bin with the most balls can be any one of the n bins.*)

Problem 5 (20pts). Prove or disprove the following statements.

(a) **(10pts).** Let u, v, w be three distinct vertices of a simple graph G. Suppose that there is an even-length path from u to v and an even-length path from v to w. Then, there is an even-length path from u to w.

(b) **(10pts).** Suppose that u, v are the only vertices of odd degree in a simple graph G. Then, G contains a path between u and v.

Problem 6 (10pts). Let G be a simple graph such that every vertex of G has degree k for some $k \geq 1$. What is the smallest possible number of vertices in G? Justify your answer and give an explicit description of the graph that attains the smallest number of vertices.

Problem 7 (20pts). Given an integer $n \geq 1$, we define a graph called the n-dimensional hypercube H_n as follows: The vertex set is $\{0, 1\}^n$; i.e., H_n has exactly 2^n vertices, each of which is labeled by a distinct binary string of length n. There is an edge between two vertices u and v if and only if u and v, when viewed as binary strings of length n, differ in exactly one position.

(a) **(10pts).** Show that H_n has $n2^{n-1}$ edges.

(b) **(10pts).** Show that H_n is bipartite.

Problem 8 (20pts). Let G be a connected bipartite graph with bipartition L and R. Suppose that every vertex of G has degree k for some $k \geq 2$.

(a) **(5pts).** Show that $|L| = |R|$.

(b) **(15pts).** Show that G has no *cut–edge*; i.e., an edge whose deletion will result in a disconnected graph.