ABSTRACT

Recently, robust transmit beamforming has drawn considerable attention because it can provide guaranteed receiver performance in the presence of channel state information (CSI) errors. Assuming complex Gaussian distributed CSI errors, this paper investigates the robust beamforming design problem that minimizes the transmission power subject to probabilistic signal-to-interference-plus-noise ratio (SINR) constraints. The probabilistic SINR constraints in general have no closed-form expression and are difficult to handle. Based on a Bernstein-type inequality for quadratic forms of complex Gaussian random variables, we propose a conservative formulation to the robust single-cell beamforming design problem. The semidefinite relaxation technique can be applied to efficiently handle the proposed conservative formulation. Simulation results show that, in comparison with existing methods, the proposed method is more power efficient and is able to support higher target SINR values for receivers.

Index Terms— Robust transmit beamforming, semidefinite relaxation, convex optimization.

1. INTRODUCTION

Linear transmit beamforming has been recognized as a powerful technique since it can achieve a large fraction of capacity with low implementation complexity. Conventionally, it is assumed that the transmitter has perfect channel state information (CSI) of the receivers, and the beamforming vectors are optimized such that the signal-to-interference-plus-noise ratio (SINR) requirements of the receivers can be satisfied. In practical situations, however, the CSI at the transmitter is inevitably subject to errors due to finite-energy training and limited feedback. The presence of CSI errors will result in receivers’ performance outage. Therefore, robust transmit beamforming designs that take the CSI errors into consideration [1, 2, 3, 4] are of great importance.

In this paper, we assume that the CSI errors are complex Gaussian distributed and study the stochastic robust beamforming design problem under a single-cell system with multiple single-antenna receivers. Specifically, we study the robust design formulation that minimizes the transmission power subject to probabilistic SINR constraints on the receivers [2]. The probabilistic SINR constraints guarantee the receivers’ SINR requirements to be satisfied with a probability that a sum of random variables deviates from its mean. Roughly speaking, a Bernstein-type inequality is one which bounds the probability that a sum of random variables deviates from its mean.
that $s_i(t)$ has zero mean and $\mathbb{E}[|s_i(t)|^2] = 1$ for all $i$. The SINR of receiver $i$ can be obtained from (1) as
\[
\text{SINR}_i = \frac{|h_i^H w_i|^2}{\sum_{k \neq i} |h_i^H w_k|^2 + \sigma_i^2}.
\]

(2)

The goal of transmit beamforming is to design the beamforming vectors $\{w_i\}_{i=1}^K$ such that each of the receivers can achieve a desired SINR performance. To this end, the following design formulation has been frequently employed:
\[
\min_{w_i \in \mathbb{C}^{N_t}} \sum_{i=1}^K ||w_i||^2
\]
\[
\text{s.t. } \frac{|h_i^H w_i|^2}{\sum_{k \neq i} |h_i^H w_k|^2 + \sigma_i^2} \geq \gamma_i, \quad i = 1, \ldots, K,
\]

(3a)

where $|| \cdot ||$ denotes the vector Euclidean norm, and $\gamma_i > 0$ stands for the preset target SINR value of receiver $i$. The design problem (3) aims to find a most power efficient beamforming solution such that the target SINR requirements $\gamma_i$ are satisfied. There is more than one way to solve problem (3); e.g., by using uplink-downlink duality, by using SDR, or by using a second-order cone program (SOCP) reformulation [7, 8]. Readers are referred to the literature, such as [9], for coverage of this aspect.

2.1. Probabilistic SINR Constrained Robust Beamforming

The conventional design formulation in (3) assumes that the transmitter has perfect knowledge of the channels $\{h_i\}_{i=1}^K$. In practical wireless environments, however, the transmitter may only have inaccurate CSI due to imperfect channel estimation and limited feedback. Let $\bar{h}_i, \ldots, h_K \in \mathbb{C}^{N_t}$ denote the channel estimates at the transmitter. The true channels can be expressed as
\[
h_i = \bar{h}_i + e_i, \quad i = 1, \ldots, K,
\]

(4)

where $e_i \in \mathbb{C}^{N_t}$ represents the CSI error vector. In the presence of CSI errors, the beamforming solution of problem (3), denoted by $\{w_i^*\}_{i=1}^K$, may no longer guarantee the SINR requirements in (3b), that is, for some $e_i$, it is possible to have
\[
\frac{|(\bar{h}_i + e_i)^H w_i|^2}{\sum_{k \neq i} |(\bar{h}_i + e_i)^H w_k|^2 + \sigma_i^2} < \gamma_i.
\]

(5)

It is desirable to design the beamforming vectors $\{w_i\}_{i=1}^K$ such that (5) occurs only with a small probability.

To this end, we assume that the CSI errors are complex Gaussian random vectors with zero mean and covariance matrix $C_i \succeq 0$ (positive semidefinite, PSD), i.e.,
\[
e_i \sim \mathcal{CN}(0, C_i), \quad i = 1, \ldots, K.
\]

(6)

This model is particularly suitable for CSI errors caused by imperfect channel estimation. Let $\rho_i \in (0, 1]$ denotes the maximum tolerable SINR outage probability of receiver $i$. We consider the following robust beamforming design formulation [2]:
\[
\min_{w_i \in \mathbb{C}^{N_t}} \sum_{i=1}^K ||w_i||^2
\]
\[
\text{s.t. } \Pr\left\{ \frac{|(\bar{h}_i + e_i)^H w_i|^2}{\sum_{k \neq i} |(\bar{h}_i + e_i)^H w_k|^2 + \sigma_i^2} \geq \gamma_i \right\} \geq 1 - \rho_i,
\]

(7a)

\[
i = 1, \ldots, K.
\]

(7b)

It can be seen that problem (7) finds a most power efficient beamforming solution such that the $1 - \rho_i$ SINR satisfaction probability is achieved. Solving problem (7) is a challenging task because the probabilistic SINR constraints in (7b) have no closed-form expression and are not convex in general. To obtain approximate solutions satisfying (7b), efficient convex conservative formulations have been proposed; see [2, 4]. In the next section, we present a new conservative formulation for problem (7) that will be shown to outperform the existing methods.

3. PROPOSED CONSERVATIVE FORMULATION

3.1. Bernstein-type Inequality Based Conservative Approach

To present the proposed method, let us express the CSI errors as
\[
e_i = C_i^{1/2} v_i, \quad i = 1, \ldots, K,
\]

(8)

where $C_i^{1/2} \succeq 0$ is the PSD square root of $C_i$, and $v_i \in \mathbb{C}^{N_t}$ is a normalized complex Gaussian random vector with zero mean and covariance matrix $I$ (the $N_t$ by $N_t$ identity matrix, i.e., $v_i \sim \mathcal{CN}(0, I)$. With this expression, the probabilistic constraints in (7b) can be expressed as
\[
\Pr\left\{ v_i^H Q_i (w_i, \ldots, w_K) v_i + 2 v_i^H u_i (w_i, \ldots, w_K) \right\} \geq c_i (w_i, \ldots, w_K) \geq 1 - \rho_i, \quad i = 1, \ldots, K,
\]

(9)

where $\Pr\{\cdot\}$ represents the real part of the associated argument, and
\[
Q_i (w_i, \ldots, w_K) \triangleq C_i^{1/2} \left( \frac{1}{\gamma_i} w_i w_i^H - \sum_{k \neq i} w_k w_k^H \right) C_i^{1/2},
\]

(10a)

\[
u_i (w_i, \ldots, w_K) \triangleq C_i^{1/2} \left( \frac{1}{\gamma_i} w_i w_i^H - \sum_{k \neq i} w_k w_k^H \right),
\]

(10b)

\[
c_i (w_i, \ldots, w_K) \triangleq \sigma_i^2 - \bar{h}_i^H \left( \frac{1}{\gamma_i} w_i w_i^H - \sum_{k \neq i} w_k w_k^H \right) \bar{h}_i.
\]

(10c)

Note that (9) is a probability inequality involving a quadratic form of complex Gaussian random variables. The idea of conservative approaches is to find computationally tractable forms that are sufficient to achieve (9). To implement this idea, we use the following lemma:

Lemma 1 [6]:

\[
\Pr\left\{ G \geq \text{Tr}(Q) - \sqrt{2} \sqrt{\mathbb{E}[Q] \mathbb{E}[||u||^2]} - \delta s^+(Q) \geq 1 - e^{-\delta},
\]

(11)

where $s^+(Q) = \max \{ \lambda_{\max}(-Q), 0 \}$ in which $\lambda_{\max}(-Q)$ denotes the maximum eigenvalue of matrix $-Q$, and $|| \cdot ||_F$ denotes the matrix Frobenius norm.

The inequality in (11) is a Bernstein-type inequality, which bounds the probability that the quadratic form $G$ of complex Gaussian random variables deviates from its mean $\text{Tr}(Q)$. Let $\delta = - \ln(\rho)$ where $\rho \in (0, 1]$. Lemma 1 implies that the inequality
\[
\Pr\left\{ v_i^H Q v_i + 2 v_i^H u \geq c_i \right\} \geq 1 - \rho
\]

(12)

Here, $\mathbb{H}^N$ is the set of all $N$-by-$N$ complex Hermitian matrices.
holds true if the following inequality is satisfied
\[ \text{Tr}(Q) - \sqrt{2x} \sqrt{\|Q\|^2 + 2\|u\|^2} - \delta s^y(Q) \geq c. \] (13)

Equation (13) thus serves as a conservative formulation for (12). Now, a crucial observation is that (13) can be represented by
\[ \text{Tr}(Q) - \sqrt{2x} \delta y \geq c, \] (14a)
\[ \sqrt{\|Q\|^2 + 2\|u\|^2} \leq x, \] (14b)
\[ y + Q \geq 0, \] (14c)
\[ y \geq 0, \] (14d)
where \( x, y \in \mathbb{R} \) are slack variables. By defining
\[ \delta_i \triangleq -\ln(\rho_i), \quad i = 1, \ldots, K, \] (15)
and applying (14) to (9), we obtain the following problem
\[ \min_{w_i \in C_{Nt}^{N}, x_i, y_i \in \mathbb{R}} \sum_{i=1}^{K} \|w_i\|^2 \] (16)
subject to \[ \text{Tr}(Q_i(w_1, \ldots, w_K)) - \sqrt{2x_i \delta y_i} \geq c_i(w_1, \ldots, w_K), \quad i = 1, \ldots, K, \] (17a)
\[ \left\| \sqrt{2u_i(w_1, \ldots, w_K)} \right\| \leq x_i, \quad i = 1, \ldots, K, \] (17b)
\[ y_i + Q_i(w_1, \ldots, w_K) \geq 0, \quad i = 1, \ldots, K, \] (17c)
\[ y_i \geq 0, \quad i = 1, \ldots, K, \] (17d)
as a conservative formulation for problem (7), where \( \text{vec}(\cdot) \) denotes the column-by-column matrix vectorization.

Our derived conservative formulation (16) is desirable since it has closed-form expressions with the constraints. However, problem (16) is nonconvex, owing to the fact that \( Q_i(w_1, \ldots, w_K), u_i(w_1, \ldots, w_K) \) and \( c_i(w_1, \ldots, w_K) \) are indefinite quadratic in \( \{w_1, \ldots, w_K\} \) [see (10)]. Next, we will handle this problem by using the SDR technique [10].

3.2. Semidefinite Relaxation

To apply SDR to the conservative formulation (16), we let \( W_i = w_i w_i^H \), for \( i = 1, \ldots, K \). By replacing \( w_i w_i^H \) with \( W_i \) in (10) and then constraining \( W_i \) to be positive semidefinite only, we obtain the following problem
\[ \min_{w_i \in C_{Nt}^{N}, x_i, y_i \in \mathbb{R}} \sum_{i=1}^{K} \text{Tr}(W_i) \] (17a)
subject to \[ \text{Tr}(Q_i(W_1, \ldots, W_K)) - \sqrt{2x_i \delta y_i} \geq c_i(W_1, \ldots, W_K), \] (17b)
\[ \left\| \sqrt{2u_i(W_1, \ldots, W_K)} \right\| \leq x_i, \] (17c)
\[ y_i + Q_i(W_1, \ldots, W_K) \geq 0, \] (17d)
\[ y_i \geq 0, \quad W_i \geq 0, \quad i = 1, \ldots, K, \] (17e)
where, with a slight abuse of notations, we define
\[ Q_i(W_1, \ldots, W_K) \triangleq C_i^{1/2} \left( \frac{1}{\gamma_i} W_i - \sum_{k \neq i} W_k \right) C_i^{1/2}, \]
\[ u_i(W_1, \ldots, W_K) \triangleq C_i^{1/2} \left( \frac{1}{\gamma_i} W_i - \sum_{k \neq i} W_k \right) \bar{h}_i, \]
\[ c_i(W_1, \ldots, W_K) \triangleq \sigma_i^2 - \bar{h}_i^H \left( \frac{1}{\gamma_i} W_i - \sum_{k \neq i} W_k \right) \bar{h}_i, \]
for \( i = 1, \ldots, K \). Note that the constraints in (17b), (17c) and (17d) are respectively a linear constraint, a convex second-order cone (SOC) constraint and a convex PSD constraint. Hence problem (17) is a convex conic problem and can be efficiently solved by standard solvers such as CVX [11].

The SDR problem (17) is in general a relaxation of problem (16) because the associated optimal \( \{W_i\}_{i=1}^{K} \) may not be of rank one. If the optimal \( \{W_i\}_{i=1}^{K} \) of (17) is of rank one, i.e., \( W_i = w_i w_i^H \) for all \( i \), then \( \{W_i\}_{i=1}^{K} \) is an optimal solution to problem (16); otherwise additional solution approximation procedures to turn the optimum \( \{W_i\}_{i=1}^{K} \) into a rank-one approximate solution of problem (16) is needed [10]. Fortunately and rather surprisingly, it is found by simulations that problem (17) always yields rank-one optimal \( \{W_i\}_{i=1}^{K} \). This implies that the globally optimal solution of the conservation formulation (16) can be attained by SDR, at least for all the problem instances we tested in simulations. Also, under the same argument, the feasibility of (16) may be equivalent to that of (17).

3.3. Reducing the Level of Conservatism by Bisection

Analogous to the methods presented in [2, 4], it is found that the proposed conservative formulation (16) with \( \delta_i \) set as in (15) may yield beamforming solutions that correspond to an SINR satisfaction probability [in (7b)] much higher than \( 1 - \rho_i \). According to (11), the SINR satisfaction probability achieved by formulation (16) can be reduced by decreasing the parameter \( \delta_i \). Hence the bisection method presented in [2, 4] can be used to reduce the level of conservatism of problem (16).

To illustrate how this method works, let us assume that all the receivers have the same SINR outage probability, i.e., \( \rho_1 = \rho_2 = \ldots = \rho_K \). Thus we can let \( \delta = \delta_1 = \ldots = \delta_K \) [see (15)]. For a given \( \delta \), one can obtain a beamforming solution by solving (16) and use the validation procedure in [5] to test if the associated probabilistic SINR constraints in (7b) are empirically satisfied or not. If yes, the parameter \( \delta \) can be reduced; otherwise it should be increased. The procedure is repeated until a predefined stopping criterion is met. Readers are referred to [2, 4] for further details.

4. SIMULATION RESULTS AND DISCUSSIONS

In this section, we present some simulation results to demonstrate the performance of the proposed method. We consider the wireless system as described in Section 2 with three antennas at the transmitter and with three receivers (\( N_t = K = 3 \)). For simplicity, we consider independent and identically distributed (i.i.d.) complex Gaussian CSI errors, i.e., \( C_i = \mathcal{C}_i \) for all \( i \), and set \( \epsilon = 0.002 \). The noise variances of all receivers are set to 0.01 (\( \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0.01 \)), and the outage probabilities are set to 0.1 (\( \rho_1 = \rho_2 = \rho_3 = 0.1 \)), i.e., 90% satisfaction probability. The target SINR values of all receivers are also set to be the same, i.e., \( \gamma_1 = \gamma_2 = \gamma_3 \). We compare the proposed conservative formulation in (16) with the method presented in [4] and the Formulation I in [2]. The bisection technique mentioned in Section 3.3 was also implemented for the three methods under test. The parameter setting of this technique follows that in [4]. All three methods were implemented using CVX [11].

Since a less conservative method is more likely to be feasible, in the first example, we examine the feasibility rates of the three methods under test and their bisection counterparts. To this end, 500 sets of channel estimates \( \{\bar{h}_i\}_{i=1}^{K} \) were generated according to complex Gaussian with zero mean and covariance matrix \( \mathbf{I} \). Figure 1 shows the simulation results of feasibility rate (%) versus target SINR \( \gamma \).
the proposed method has a larger computation time than the method
feasible channel realizations are shown in Fig. 3. We can see that in [4]; this shows that there is a tradeoff between complexity and performance for these two methods. On the other hand, we can observe from this figure that both the proposed method and that in [4] always consumes the least power for every feasible channel realization. The results in Fig. 1 and Fig. 2 imply that the proposed method is amenable to support a wider range of target SINRs.

In the last example, we examine the average computation time (in seconds) of CVX for solving the proposed method and the methods in [2] and [4]. The simulation results for $\gamma = 7$ dB over 50 feasible channel realizations are shown in Fig. 3. We can see that the proposed method has a larger computation time than the method in [4]; this shows that there is a tradeoff between complexity and performance for these two methods. On the other hand, we can observe from this figure that both the proposed method and that in [4] are computationally cheaper than the method in [2], especially when $N_t = K \geq 3$.

5. REFERENCES


