Beyond Convex Relaxation: A Polynomial–Time Non–Convex Optimization Approach to Network Localization

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***Abstract*—The successful deployment and operation of location–aware networks, which have recently found many applications, depends crucially on the accurate localization of the nodes. Currently, a powerful approach to localization is that of convex relaxation. In a typical application of this approach, the localization problem is first formulated as a rank–constrained semidefinite program (SDP), where the rank corresponds to the target dimension in which the nodes should be localized. Then, the non–convex rank constraint is either dropped or replaced by a convex surrogate, thus resulting in a convex optimization problem. In this paper, we explore the use of a *non–convex* surrogate of the rank function, namely the so–called *Schatten quasi–norm*, in network localization. Although the resulting optimization problem is non–convex, we show, for the first time, that a first–order critical point can be approximated to arbitrary accuracy in *polynomial time* by an interior–point algorithm. Moreover, we show that such a first–order point is already sufficient for recovering the node locations in the target dimension if the input instance satisfies certain established uniqueness properties in the literature. Finally, our simulation results show that in many cases, the proposed algorithm can achieve more accurate localization results than standard SDP relaxations of the problem.**

I. Introduction

Determining the locations of nodes is a fundamental task in many wireless network applications. From target tracking [1] to emergency response [2], from logistics support [3] to mobile advertising [4], the information collected or transmitted by a node depends crucially on its location. As it is typically impractical to manually position the nodes or equip them with Global Positioning System (GPS) receivers, a key research question is how signal metrics (such as received signal strength, time of arrival, angle of arrival, etc. [5]) obtained by individual nodes through direct communication with their neighbors can be used to localize the entire network. One of the most common settings under which the above question is studied is when distances between neighboring nodes can be measured or estimated (this can be achieved using various ranging techniques; see, e.g., [3]). Under this setting, the network localization problem becomes that of determining the node positions in $\mathbb{R}^2$ or $\mathbb{R}^3$ so that they are consistent with the given distance measurements. As is well known, such a fixed–dimensional localization problem is intractable in general [6], [7]. Consequently, there has been significant research effort in developing algorithms that can accurately and efficiently localize the nodes in a given dimension; see, e.g., [8] and the references therein. One powerful approach is that of convex relaxation, which was first adopted by Doherty et al. [9] and has since been extensively developed in the literature; see, e.g., [10]–[19]. Such an approach is very natural as far as polynomial–time solvability is concerned, and it often produces accurate localization results in practice. However, due to the intractability of the fixed–dimensional localization problem, existing convex relaxation–based localization algorithms will most likely not be able to localize *all* input instances in the required dimension in polynomial time. Thus, it is reasonable to ask whether there are other polynomial–time implementable approaches that can achieve better localization performance.

To address the above question, it is instructive to start by revisiting the semidefinite programming (SDP) relaxation proposed by Biswas and Ye [20]. The crucial observation underlying Biswas and Ye's derivation is that the fixed–dimensional localization problem can be formulated as a *rank–constrained* SDP feasibility problem, i.e., problem of the form

$$\begin{aligned} \text{find} \quad & Z \\ \text{such that} \quad & \mathcal{E}(Z) = u,\ \text{rank}(Z) = d, \\ & Z \text{ symmetric and positive semidefinite.} \end{aligned} \tag{1}$$

Here, the linear operator $\mathcal{E}$ and vector u are determined by the available distance measurements, and $d \geq 1$ is the target dimension in which the input instance should be localized (see Section II for details). Thus, by dropping the non–convex rank constraint, one immediately obtains an SDP relaxation of the fixed–dimensional localization problem. As it turns out, the Biswas–Ye SDP relaxation has a nice geometric interpretation. Specifically, So and Ye [11] showed that there is a correspondence between the rank–r solutions to the Biswas–Ye SDP relaxation and the r–dimensional localizations of the input instance. Now, in wireless network applications, we are mostly interested in finding a d–dimensional localization of the

input instance, where $d = 2$ or 3. Thus, if the solution obtained after solving the Biswas–Ye SDP relaxation happens to have rank d, then we know by the result of So and Ye that we have found a localization in the required dimension. However, it is known that standard interior–point algorithms for solving SDPs (such as those used in the solver SeDuMi [21]) will always return the solution with the highest rank [11]. Since the Biswas–Ye SDP relaxation does not have any mechanism to eliminate the high–rank solutions, it is unlikely that those algorithms will return a solution with the required rank.

Fortunately, not all is lost, as we could equip the Biswas–Ye SDP relaxation with a suitable penalty function f (also known as a *regularizer*) to filter out some of the high–rank solutions. In other words, instead of solving the original Biswas–Ye SDP relaxation, which is an SDP *feasibility* problem, we consider its *regularized* version:

$$\begin{aligned} \text{minimize} \quad & f(Z) \\ \text{subject to} \quad & \mathcal{E}(Z) = u, \\ & Z \text{ symmetric and positive semidefinite.} \end{aligned} \tag{2}$$

Since our goal is to find a low–rank solution, it is tempting to use $f(\cdot) = \text{rank}(\cdot)$ as the regularizer. However, the resulting optimization problem (2) will be at least as hard as the original localization problem. Thus, we need to take the computational complexity of Problem (2) into account when choosing the regularizer f. Recently, the idea of using regularizers to find low–rank solutions to SDPs has generated significant interest due to its applications in low–rank matrix recovery; see, e.g., the references in [22]. One of the most commonly used and studied regularizers is the so–called *nuclear norm*, which in the context of Problem (2) means taking $f(\cdot) = \text{tr}(\cdot)$. The motivation behind such a choice is twofold. First, since the trace function is linear, Problem (2) is still an SDP and hence can be solved efficiently. Secondly, recall that for a symmetric and positive semidefinite matrix, its rank is equal to the number of non–zero singular values, and its trace is the sum of all its singular values. Thus, by minimizing the trace function in Problem (2), it is hoped that the solution will have a smaller number of non–zero singular values and hence a lower rank. In the context of network localization, trace regularization was first proposed by Biswas et al. in [12]. However, computational experiments show that such a strategy may not be very effective, especially when the number of available distance measurements is small (see Section V for details). This motivates us to search for other regularizers that are more effective in finding low–rank solutions.

In that regard, an attractive proposal is the so–called *Schatten p–regularization*. Specifically, for any given $p \in (0, 1]$, consider the regularizer f_p, where

$$f_p(Z) = \sum_{i \geq 1} (\sigma_i(Z))^p$$

and $\sigma_i(Z)$ is the i–th largest singular value of Z. The value $(f_p(Z))^{1/p}$ is known as the *Schatten p–quasi–norm* of Z, hence the name of the regularization. The Schatten p–quasi–norm has several nice analytical properties that make it a natural candidate for a regularizer. First, when Z is constrained to be symmetric and positive semidefinite, we have $f_1(Z) = \text{tr}(Z)$. Thus, Schatten p–regularization includes trace regularization as a special case. Secondly, as $p \searrow 0$, we have $f_p(Z) \to \text{rank}(Z)$ for all Z. This suggests that Schatten p–regularization can be effective in finding a low–rank solution to Problem (2), especially when p is small. In fact, Schatten p–regularization has already been shown to enjoy some nice theoretical properties in the recovery of general low–rank matrices; see, e.g., [23], [24]. Despite these promising results, there is an obstacle when using the Schatten p–quasi–norm as regularizer, namely, it is $\mathcal{NP}$–hard to minimize the Schatten p–quasi–norm over a system of linear matrix inequalities for any fixed $p \in (0, 1)$; cf. [25]. Although there are algorithms for finding stationary points of a Schatten p–quasi–norm minimization problem (see, e.g, [26]–[28]), none of them has a polynomial–time complexity. Furthermore, when applied to the network localization problem, it is not clear whether the aforementioned algorithms will give better localization results or even preserve the class of instances that can be exactly localized in the required dimension by the original (unregularized) convex relaxation. In fact, to the best of our knowledge, there is no prior work that applies the Schatten p–regularization approach to network localization problems. This necessitates a study on the viability of such approach.

In this paper, we make a first step towards understanding the theoretical and computational issues surrounding the use of Schatten p–regularization in network localization. The starting point of our investigation is the following Schatten p–regularized SDP problem:

$$\begin{aligned} \text{minimize} \quad & \text{tr}(CZ) + \mu \cdot f_p(Z) \\ \text{subject to} \quad & \mathcal{A}(Z) = b, \\ & Z \text{ symmetric and positive semidefinite,} \end{aligned} \tag{3}$$

where the symmetric matrix C, linear operator $\mathcal{A}$ and vector b are arbitrary, and $\mu > 0$ is a given penalty parameter. Such a formulation provides a unified framework for regularizing several existing SDP relaxations of the network localization problem using the Schatten p–quasi–norm. In particular, the regularized Biswas–Ye SDP relaxation (2) is just a special case of Problem (3). Our first contribution is to show that a point satisfying certain first–order optimality conditions of Problem (3) can be approximated to arbitrary accuracy in polynomial time via a *potential reduction algorithm*, which is a type of interior–point algorithm. This extends the result of Ge et al. [25] on ℓ_p–quasi–norm–regularized linear programming and is, to the best of our knowledge, the first polynomial–time complexity result for Problem (3). Although this result only implies that we can find a first–order critical point and not a global minimizer of the Schatten p–regularized Biswas–Ye SDP relaxation (2) in polynomial time, such a point can still be useful in obtaining a low–dimensional localization of the input instance. To see this, observe that the first–order critical point is, by definition, feasible for the original Biswas–Ye SDP relaxation. Moreover, due to the presence of a regularizer, this

feasible point is likely to have a lower rank than the solution returned by standard interior–point algorithms for solving the Biswas–Ye SDP relaxation. Hence, by the aforementioned result of So and Ye [11], the localization corresponding to the first–order critical point will tend to lie in a lower dimension.

Another important issue concerning the proposed regularization approach is whether it preserves the localizability properties of the underlying unregularized problem. Specifically, it is known that the Biswas–Ye SDP relaxation [20] and its trace regularization [12] can localize the input instance in the target dimension if it satisfies certain uniqueness property [11], [12], and a natural question is whether this still holds if we use the potential reduction algorithm to solve the corresponding Schatten p–regularizations. We show that the answer is in fact affirmative. In the language of matrix recovery, this result essentially says that if the original convex relaxation can recover the low–rank matrix that encodes the localization of the input instance in the target dimension, then so can its Schatten p–regularization. This is in sharp contrast with the recovery of *general* low–rank matrices, where it is possible that a target low–rank matrix can be recovered by a convex regularization but not by the Schatten p–regularization; cf. [29].

Finally, we validate our Schatten p–regularization approach via extensive numerical simulations. In particular, we show that our proposed approach is indeed more effective in finding a low–dimensional localization of an input instance than the Biswas–Ye SDP relaxation [20] or its trace regularization [12]. In fact, even when the number of available distance measurements is small, our approach can still succeed in localizing the input instance in the target dimension, whereas the above two approaches would often fail to do so.

The paper is organized as follows. In Section II, we introduce the network localization problem and review some of its convex relaxations. In Section III, we design and analyze an interior–point algorithm for solving the Schatten p–regularized SDP problem (3). In Section IV, we analyze the localizability properties of two Schatten p–regularized SDP relaxations of the network localization problem. We then present our simulation results in Section V and conclude in Section VI.

II. Preliminaries

Let us begin by formally introducing the network localization problem. Let $G = (V, E)$ be a given network, and let $d \geq 1$ be the target dimension in which the network should reside. The nodes of G are partitioned into two sets: the set V_s of *sensors*, and the set $V_a = V \backslash V_s$ of *anchors*. We assume that for all $i \in V_a$, the position $a_i \in \mathbb{R}^d$ of anchor i is known. The partition of V induces two edge sets E_{ss}, E_{as}, where

$$\begin{aligned} E_{ss} &= \{(i,j) \in E : i, j \in V_s\}, \\ E_{as} &= \{(i,j) \in E : i \in V_a,\, j \in V_s\}. \end{aligned}$$

Each edge $(i,j) \in E_{ss}$ (resp. $(i,j) \in E_{as}$) is associated with a positive weight d_{ij} (resp. $\bar{d}_{ij}$), which can be regarded as the measured Euclidean distance between i and j.

When all the distance measurements are noiseless, the goal of the network localization problem is simply to assign a position $x_i \in \mathbb{R}^d$ to each sensor $i \in V_s$, so that the assigned positions are consistent with the distance measurements; i.e., they satisfy

$$\begin{aligned} \|x_i - x_j\|_2^2 &= d_{ij}^2 \quad \text{for } (i,j) \in E_{ss}, \\ \|a_i - x_j\|_2^2 &= \bar{d}_{ij}^2 \quad \text{for } (i,j) \in E_{as}. \end{aligned} \tag{4}$$

On the other hand, when the distance measurements are noisy, there may not be any solution to the constraints in (4). In this case, the assigned positions should be as consistent with the distance measurements as possible. One way to formalize this is to adopt a maximum likelihood estimation approach; see, e.g., [10]. Specifically, we model the noisy distance measurements as

$$\begin{aligned} d_{ij} &= d'_{ij} + \epsilon_{ij} \quad \text{for } (i,j) \in E_{ss}, \\ \bar{d}_{ij} &= \bar{d}'_{ij} + \bar{\epsilon}_{ij} \quad \text{for } (i,j) \in E_{as}, \end{aligned}$$

where d'_{ij} (resp. $\bar{d}'_{ij}$) is the actual Euclidean distance between $i, j \in V_s$ (resp. $i \in V_a$ and $j \in V_s$); ϵ_{ij} and $\bar{\epsilon}_{ij}$ are mean–zero normal random variables with variance σ_{ij}^2 and $\bar{\sigma}_{ij}^2$, respectively, and all random variables involved are statistically independent of each other. Then, the maximum likelihood estimates of the actual sensor positions can be found by solving the following optimization problem [10]:

$$\min_{x_i \in \mathbb{R}^d,\, i \in V_s} \left\{ \sum_{(i,j) \in E_{ss}} \frac{(\|x_i - x_j\|_2 - d_{ij})^2}{\sigma_{ij}^2} + \sum_{(i,j) \in E_{as}} \frac{(\|a_i - x_j\|_2 - \bar{d}_{ij})^2}{\bar{\sigma}_{ij}^2} \right\}. \tag{5}$$

As is well known, both (4) and (5) are non–convex and intractable [6]. In an early work, Biswas et al. [10] have shown that both problems can be reformulated as rank–constrained SDPs. To state the results of Biswas et al., let us first introduce some notation. Let e_i be the i–th standard basis vector whose dimension will be clear from the context; I_d be the $d \times d$ identity matrix; $\mathcal{S}^n$, $\mathcal{S}^n_+$, $\mathcal{S}^n_{++}$ be the sets of $n \times n$ symmetric, symmetric positive semidefinite and symmetric positive definite matrices, respectively; $(u; v) \in \mathbb{R}^{m+n}$ be the column vector obtained by placing the column vector $u \in \mathbb{R}^m$ on top of the column vector $v \in \mathbb{R}^n$.

Proposition 1. *(cf. [10]) The following hold:*

1) *Let*

$$\mathcal{F}_0 = \left\{ Z \;\middle|\; \begin{array}{ll} \mathrm{tr}(K_{ij} Z) = d_{ij}^2 & \text{for } (i,j) \in E_{ss}, \\ \mathrm{tr}(\bar{K}_{ij} Z) = \bar{d}_{ij}^2 & \text{for } (i,j) \in E_{as}, \\ Z_{1:d,1:d} = I_d, & \\ Z \in \mathcal{S}_+^{d+|V_s|} & \end{array} \right\},$$

where $K_{ij} = (\mathbf{0}; e_i - e_j)(\mathbf{0}; e_i - e_j)^T \in \mathcal{S}_+^{d+|V_s|}$ *for* $(i,j) \in E_{ss}$, *and* $\bar{K}_{ij} = (a_i; -e_j)(a_i; -e_j)^T \in \mathcal{S}_+^{d+|V_s|}$ *for* $(i,j) \in E_{as}$. *Then, Problem (4) is equivalent to the following rank–constrained SDP feasibility problem:*

$$\begin{aligned} &\textit{find} && Z \\ &\textit{such that} && Z \in \mathcal{F}_0, \mathrm{rank}(Z) = d. \end{aligned} \tag{6}$$

2) *Let K_{ij} and $\bar{K}_{ij}$ be as above, and*

$$\mathcal{F}_1 = \left\{ W \;\middle|\; \begin{array}{l} W = \left[\begin{array}{ccc} Z & & \\ & D & \\ & & \bar{D} \end{array}\right], \\ D = \mathrm{Diag}(\{D_{ij}\}_{(i,j)\in E_{ss}}), \\ \bar{D} = \mathrm{Diag}(\{\bar{D}_{ij}\}_{(i,j)\in E_{as}}), \\ D_{ij} = \left[\begin{array}{cc} 1 & u_{ij} \\ u_{ij} & v_{ij} \end{array}\right] \textit{ for } (i,j) \in E_{ss}, \\ \bar{D}_{ij} = \left[\begin{array}{cc} 1 & \bar{u}_{ij} \\ \bar{u}_{ij} & \bar{v}_{ij} \end{array}\right] \textit{ for } (i,j) \in E_{as}, \\ \mathrm{tr}(K_{ij}Z) = v_{ij} \quad \textit{for } (i,j) \in E_{ss}, \\ \mathrm{tr}(\bar{K}_{ij}Z) = \bar{v}_{ij} \quad \textit{for } (i,j) \in E_{as}, \\ Z_{1:d,1:d} = I_d, \\ Z \in \mathcal{S}_+^{d+|V_s|},\ D \in \mathcal{S}_+^{2|E_{ss}|},\ \bar{D} \in \mathcal{S}_+^{2|E_{as}|} \end{array} \right\},$$

where $D = \mathrm{Diag}(\{D_{ij}\}_{(i,j)\in E_{ss}})$ denotes the $2|E_{ss}| \times 2|E_{ss}|$ block–diagonal matrix whose (i,j)–th diagonal block is D_{ij}, and similarly for $\bar{D}$. Then, Problem (5) is equivalent to the following rank–constrained SDP problem:

$$\begin{array}{ll} \textit{minimize} & \mathrm{tr}(CW) \\ \textit{subject to} & W \in \mathcal{F}_1,\ \mathrm{rank}(W) = d + |E_{ss}| + |E_{as}|, \end{array} \tag{7}$$

where C takes the form $C = \mathrm{Diag}(\mathbf{0}, C_0, C_1)$, C_0 is the $2|E_{ss}| \times 2|E_{ss}|$ block–diagonal matrix whose (i,j)–th diagonal block is the 2×2 matrix $\sigma_{ij}^{-2}(-d_{ij};1)(-d_{ij};1)^T$ (where $(i,j) \in E_{ss}$), and C_1 is the $2|E_{as}| \times 2|E_{as}|$ block–diagonal matrix whose (i,j)–th diagonal block is the 2×2 matrix $\bar{\sigma}_{ij}^{-2}(-\bar{d}_{ij};1)(-\bar{d}_{ij};1)^T$.

In view of Proposition 1 and the discussions in Section I, we formulate the Schatten p–regularizations of (6) and (7) as

$$\min\left\{f_p(Z) : Z \in \mathcal{F}_0\right\} \tag{8}$$

and

$$\min\left\{\mathrm{tr}(CW) + \mu \cdot f_p(W) : W \in \mathcal{F}_1\right\}, \tag{9}$$

respectively. Note that both (8) and (9) are instances of Problem (3). In the next section, we will focus on Problem (3) and show that a first–order critical point can be approximated to arbitrary accuracy in polynomial time. Before we proceed, however, let us remark that besides (4) and (5), several other variants of the network localization problem (such as those in [12], [30], [31]) can also be reformulated as rank–constrained SDPs. As such, their corresponding Schatten p–regularizations can be written in the form shown in (3). In particular, our algorithm and complexity analysis apply to these regularizations as well.

III. A Potential Reduction Algorithm and Its Complexity Analysis

In this section, we design and analyze a potential reduction algorithm for approximating a first–order critical point of the Schatten p-regularized SDP problem (3), thereby extending both the techniques and results in Ye [32] and Ge et al. [25]. To begin, let us write our problem of interest, namely Problem (3), in a more explicit form:

$$\begin{array}{rll} \Gamma^* = & \text{minimize} & \mathrm{tr}(CZ) + \mu \cdot f_p(Z) \\ & \text{subject to} & \mathrm{tr}(A_i Z) = b_i \quad \text{for } i = 1, \ldots, m, \\ & & Z \in \mathcal{S}_+^n. \end{array} \tag{10}$$

Here, $C, A_1, \ldots, A_m$ are given $n \times n$ symmetric matrices, $b \in \mathbb{R}^m$ is a given vector, and $\mu > 0$ is a given penalty parameter. For notational convenience, let $\pi : \mathcal{S}_+^n \to \mathbb{R}$ and $\mathcal{F} \subset \mathcal{S}_+^n$ be the objective function and feasible region of Problem (10), respectively; i.e.,

$$\begin{array}{rcl} \pi(Z) & = & \mathrm{tr}(CZ) + \mu \cdot f_p(Z), \\ \mathcal{F} & = & \{Z \in \mathcal{S}_+^n : \mathrm{tr}(A_i Z) = b_i \text{ for } i = 1, \ldots, m\}. \end{array}$$

We shall make the following assumptions:

1) The feasible region $\mathcal{F}$ is bounded; i.e., there exists an $R < \infty$ such that $\|Z\|_F^2 = \mathrm{tr}(Z^2) \le R^2$ for all $Z \in \mathcal{F}$.
2) A lower bound $\underline{\theta} > -\infty$ on the optimal value of the (unregularized) SDP problem
$$\min\{\mathrm{tr}(CZ) : Z \in \mathcal{F}\} \tag{11}$$
is known or can be efficiently estimated.
3) A strictly feasible solution Z_0 to Problem (10) (i.e., $Z_0 \in \mathcal{F}$ satisfies $Z_0 \in \mathcal{S}_{++}^n$) is available or can be efficiently constructed, whose smallest eigenvalue $\lambda_{\min}(Z_0)$ satisfies $\lambda_{\min}(Z_0) \ge r$ for some $r > 0$.

It should be noted that these assumptions are quite mild. Indeed, for the network localization formulations we considered, viz. (8) and (9), the feasible regions are bounded whenever the underlying network G is connected. Moreover, the objective matrix C in both formulations is positive semidefinite, which implies that $\underline{\theta} = 0$ is a lower bound on the optimal values of their unregularized counterparts (cf. Problem (11)). In fact, we shall further assume that $\underline{\theta} = 0$ in our analysis. This is just to simplify the exposition and will not compromise the generality of our results.

Regarding the third assumption, one can show that Problem (10) can be transformed into one whose feasible region contains a known strictly feasible point. Due to space limitation, we shall not discuss the details here. Instead, we defer the complete treatment to the full version of this paper.

Since finding an optimal solution to Problem (10) is intractable in general, let us consider the task of computing its first–order critical points. Note that some care must be taken when defining such a point, because the objective function π is non–differentiable on $\mathcal{S}_+^n \backslash \mathcal{S}_{++}^n$. In the sequel, we shall adopt the following definitions:

Definition 1. *Let $Z \in \mathcal{F}$ and $\epsilon \ge 0$ be given. Suppose that rank$(Z) = r$, and let $Z = U\Lambda U^T$ be the spectral decomposition of Z, where $U \in \mathbb{R}^{n\times r}$ has orthonormal columns and $\Lambda = Diag(\lambda_1, \ldots, \lambda_r) \in \mathcal{S}_{++}^r$ is diagonal. For any $s \in \mathbb{R}$, define $\Lambda^s = Diag(\lambda_1^s, \ldots, \lambda_r^s)$ and $Z^s = U\Lambda^s U^T$. We say that*

1) *Z is an ϵ–*optimal solution *to Problem (10) if $\pi(Z) \leq \Gamma^* + \epsilon$;*
2) *Z is an ϵ–*first–order critical point *of Problem (10) if there exists a multiplier $y \in \mathbb{R}^m$ such that*

$$U^T C U + \mu p \Lambda^{p-1} - \sum_{i=1}^m y_i \left(U^T A_i U\right) \in \mathcal{S}_+^r \qquad (12)$$

and

$$0 \leq \frac{\text{tr}\left(CZ + \mu p Z^p - \sum_{i=1}^m y_i A_i Z\right)}{\pi(Z)} \leq \epsilon. \qquad (13)$$

A 0–first–order critical point will simply be called a first–order critical point.

We remark that the above definitions resemble the optimality conditions for problems with differentiable objective functions. Indeed, condition (12) can be regarded as dual feasibility, and condition (13) with $\epsilon = 0$ is complementarity between the primal and dual variables. Furthermore, the term $\pi(Z)$ in (13) is to ensure that an ϵ–first–order critical point has certain invariance properties; see [33] for a more thorough discussion.

Before delving into the details, let us give a high level overview of our algorithm. Essentially, the algorithm has two ingredients: a potential function that measures its progress, and an update rule that computes the next iterate based on the current one. Roughly speaking, in each iteration, the update rule will choose as the next iterate a *strictly feasible* point that achieves the maximum potential reduction. Now, if we can show that the potential value decreases by at least a constant (say $\delta > 0$) in each step, and that the algorithm can terminate when the potential value is below a certain threshold, then we know that the algorithm will terminate in finite number of steps. To establish polynomial–time convergence, it remains to show that δ can be made not too small, and that the potential value corresponding to the initial iterate is not too large.

To implement the above idea, we first define a potential function $\phi : \mathcal{S}_+^n \to \mathbb{R}$ via

$$\phi(Z) = \rho \log(\pi(Z)) - \log\det(Z),$$

where $\rho > 0$ is a parameter to be determined later. Intuitively, the term $-\log\det(Z)$ serves as a barrier and forces the iterates of the algorithm to stay in the interior of $\mathcal{F}$. This is because whenever Z tends to a positive semidefinite but not positive definite matrix, the value $-\log\det(Z)$ will tend to ∞.

Now, consider a generic iteration of the algorithm. Suppose that the current iterate $\bar{Z}$ is strictly feasible for Problem (10). The following proposition gives a condition under which the algorithm can terminate and declare $\bar{Z}$ an approximately *global* optimal solution to Problem (10).

Proposition 2. *Let $\epsilon > 0$ and $\rho > n/p$ be fixed. Suppose that*

$$\phi(\bar{Z}) \leq \underline{\phi} = \left(\rho - \frac{n}{p}\right)\log(\epsilon) + \frac{n}{p}\log(\mu n). \qquad (14)$$

Then, $\bar{Z}$ is an ϵ–optimal solution to Problem (10).

Proof: By the arithmetic–mean geometric–mean inequality, we have

$$\frac{f_p(Z)}{n} = \frac{1}{n}\sum_{i=1}^n (\sigma_i(Z))^p \geq \left(\prod_{i=1}^n \sigma_i(Z)\right)^{\frac{p}{n}} = (\det(Z))^{p/n}$$

for all $Z \in \mathcal{S}_+^n$. In particular, for any $Z \in \mathcal{F}$,

$$\frac{n}{p}\log(\pi(Z)) - \log\det(Z) \geq \frac{n}{p}\log(\mu n),$$

where we use the assumption that $\text{tr}(CZ) \geq \underline{\theta} = 0$ for all $Z \in \mathcal{F}$. Thus, if condition (14) is satisfied, then we have $\pi(\bar{Z}) \leq \epsilon$, which, together with the fact that $\Gamma^* \geq 0$, implies the desired result. ∎

Suppose then condition (14) is not satisfied at the current iterate $\bar{Z}$. Let $D_{\bar{Z}} \in \mathcal{S}^n$ be such that $\text{tr}(A_i D_{\bar{Z}}) = 0$ for $i = 1, \ldots, m$, and consider the next iterate $\bar{Z}^+ = \bar{Z} + D_{\bar{Z}}$. The change in potential value is given by

$$\begin{aligned}\phi(\bar{Z}^+) - \phi(\bar{Z}) \quad &= \quad \rho\left[\log(\pi(\bar{Z}^+)) - \log(\pi(\bar{Z}))\right] \\ &+ \quad \log\det(\bar{Z}) - \log\det(\bar{Z}^+). \qquad (15)\end{aligned}$$

To bound this change, we need the following proposition. Due to space limitation, we omit the proof in this paper.

Proposition 3. *The following hold:*

1) *The function $Z \mapsto \log(\pi(Z))$ is concave on $\mathcal{S}_{++}^n$.*
2) *For each $p \in (0,1)$, the function $Z \mapsto f_p(Z)$ is differentiable on $\mathcal{S}_{++}^n$, and $\nabla f_p(Z) = pZ^{p-1}$.*

Now, let $D = \bar{Z}^{-1/2} D_{\bar{Z}} \bar{Z}^{-1/2}$ and $\bar{C} = \bar{Z}^{1/2} C \bar{Z}^{1/2}$. By the concavity of $Z \mapsto \log(\pi(Z))$, we have

$$\begin{aligned}\log(\pi(\bar{Z}^+)) - \log(\pi(\bar{Z})) \quad &\leq \quad \frac{\text{tr}\left[(C + \mu p \bar{Z}^{p-1}) D_{\bar{Z}}\right]}{\pi(\bar{Z})} \\ &= \quad \frac{\text{tr}\left[(\bar{C} + \mu p \bar{Z}^p) D\right]}{\pi(\bar{Z})}. \qquad (16)\end{aligned}$$

Moreover, if we restrict $\|D\|_F^2 \leq \beta < 1$, then it can be shown that (cf. [34, Section 9.3])

$$\log\det(\bar{Z}) - \log\det(\bar{Z}^+) \leq -\text{tr}(D) + \frac{\beta^2}{2(1-\beta)}, \qquad (17)$$

which, together with the fact that $\text{tr}(A_i \bar{Z}^+) = b_i$ for $i = 1, \ldots, m$, implies that $\bar{Z}^+$ is strictly feasible for Problem (10). Upon substituting (16) and (17) into (15), we obtain

$$\begin{aligned}\phi(\bar{Z}^+) - \phi(\bar{Z}) \quad &\leq \quad \frac{\rho}{\pi(\bar{Z})}\text{tr}\left[(\bar{C} + \mu p \bar{Z}^p) D\right] \\ &- \quad \text{tr}(D) + \frac{\beta^2}{2(1-\beta)}. \qquad (18)\end{aligned}$$

In order to achieve maximum potential reduction, we choose D (and hence $D_{\bar{Z}}$) to minimize the right–hand side of (18). Considering the constraints on D, this is equivalent to solving

$$\begin{aligned}\text{minimize} \quad & \text{tr}\left[\left(\frac{\rho}{\pi(\bar{Z})}(\bar{C} + \mu p \bar{Z}^p) - I\right) D\right] \\ \text{subject to} \quad & \text{tr}(\bar{Z}^{1/2} A_i \bar{Z}^{1/2} D) = 0 \quad \text{for } i = 1, \ldots, m, \\ & \|D\|_F^2 \leq \beta^2. \qquad (19)\end{aligned}$$

Let $Q = (\rho/\pi(\bar{Z}))(\bar{C} + \mu p \bar{Z}^p) - I$. Since Problem (19) is convex and satisfies the Slater condition, its KKT conditions, which are given below, are both necessary and sufficient for optimality.

$$\begin{aligned} 2\lambda D &= \mathcal{A}^T(\nu) - Q, && \text{(20a)} \\ \mathcal{A}(D) &= \mathbf{0}, && \text{(20b)} \\ \|D\|_F^2 &= \beta^2, && \text{(20c)} \\ \lambda &\geq 0. && \text{(20d)} \end{aligned}$$

Here, $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is the linear operator given by

$$\mathcal{A}(D) = \left(\text{tr}\left(\bar{Z}^{1/2} A_1 \bar{Z}^{1/2} D\right), \ldots, \text{tr}\left(\bar{Z}^{1/2} A_m \bar{Z}^{1/2} D\right)\right),$$

and $\mathcal{A}^T : \mathbb{R}^m \to \mathcal{S}^n$ is its adjoint operator; i.e.,

$$\mathcal{A}^T(\nu) = \sum_{i=1}^m \nu_i \bar{Z}^{1/2} A_i \bar{Z}^{1/2}.$$

Upon applying $\mathcal{A}$ to both sides of (20a) and using (20b) and the assumption that $\mathcal{A}$ has full row rank, we have

$$\nu = (\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}(Q). \tag{21}$$

Then, using (20a), (20c) and (20d), we find $\lambda = (2\beta)^{-1} \left\| \left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q \right\|_F$. Together with (20a), this implies that

$$D = -\beta \frac{\left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q}{\left\|\left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q\right\|_F}. \tag{22}$$

Since the linear operator $\mathcal{H} = I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}$ is a projection (and hence $\mathcal{H}^2 = \mathcal{H}$), after substituting (22) into (18), we can bound the change in potential value by

$$\phi(\bar{Z}^+) - \phi(\bar{Z}) \leq -\beta \left\| \left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q \right\|_F + \frac{\beta^2}{2(1-\beta)}. \tag{23}$$

Now, consider two cases.

Case 1: $\left\| \left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q \right\|_F \geq 1$. Then, upon setting $\beta = 1/3$, say, we have $\phi(\bar{Z}^+) - \phi(\bar{Z}) \leq -1/4$ from (23). In other words, by taking $\bar{Z}^+$ as the next iterate, we can achieve a potential reduction of at least $1/4$.

Case 2: $\left\| \left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q \right\|_F < 1$. In this case, we show that $\bar{Z}$ is an ϵ–first–order critical point of Problem (10), from which we conclude that the algorithm can terminate. First, observe that $\left\| \left(I - \mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A}\right) Q \right\|_F < 1$ is, by (21), equivalent to $\|Q - \mathcal{A}^T(\nu)\|_F^2 < 1$, or more explicitly,

$$\left\| \frac{\rho}{\pi(\bar{Z})}(\bar{C} + \mu p \bar{Z}^p) + \mathcal{A}^T(\mu) - I \right\|_F^2 < 1. \tag{24}$$

This, together with the Cauchy–Schwarz inequality and the definition of $\bar{C}$, implies that

$$\frac{n - \sqrt{n}}{\rho} < \frac{\text{tr}\left[\bar{Z}\left(C + \mu p \bar{Z}^{p-1} - \sum_{i=1}^m \bar{y}_i A_i\right)\right]}{\pi(\bar{Z})} < \frac{n + \sqrt{n}}{\rho},$$

where $\bar{y} = -(\pi(\bar{Z})/\rho)\nu \in \mathbb{R}^m$. Upon setting $\Delta = \min\{p, \epsilon\}$, $\rho = (n + \sqrt{n})/\Delta$ and $\bar{S} = C + \mu p \bar{Z}^{p-1} - \sum_{i=1}^m \bar{y}_i A_i$, we obtain

$$\rho > \frac{n}{p} \quad \text{and} \quad 0 < \frac{\text{tr}(\bar{Z}\bar{S})}{\pi(\bar{Z})} < \epsilon.$$

Now, it remains to prove that $\bar{S} \in \mathcal{S}_+^n$, which would imply that (12) holds. Towards that end, it suffices to show that $\bar{Z}^{1/2}\bar{S}\bar{Z}^{1/2} \in \mathcal{S}_+^n$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\bar{Z}^{1/2}\bar{S}\bar{Z}^{1/2}$. By (24), we have $\sum_{i=1}^n \left(\lambda_i - \pi(\bar{Z})/\rho\right)^2 \leq \left(\pi(\bar{Z})/\rho\right)^2$. This implies that $\lambda_n \geq 0$, as desired.

In summary, the above derivation shows that in a generic iteration, we either reduce the potential value by at least $1/4$ (Case 1), in which case we continue to the next iteration; or we reach an ϵ–first–order critical point (Case 2), in which case we terminate.

To complete the description of the algorithm and establish polynomial–time convergence, it remains to show that the algorithm can be initialized with a strictly feasible solution to Problem (10) whose potential value is not too large. Towards that end, recall that by assumption, we have a strictly feasible solution Z_0 to Problem (10) that satisfies $\|Z_0\|_F \leq R$ and $\lambda_{\min}(Z_0) \geq r$. It can then be shown that

$$\phi(Z_0) \leq \overline{\phi} = \rho \log\left(R\|C\|_F + \mu R^p n^{1-p/2}\right) - n \log r. \tag{25}$$

In particular, if we initialize the algorithm with Z_0, then Proposition 2, (25), and the discussion in the preceding paragraph imply that the algorithm terminates in at most

$$\eta = \frac{\overline{\phi} - \underline{\phi}}{1/4} = \mathcal{O}\left(\frac{n}{\Delta} \log\left(\frac{R}{\epsilon r'}\left(\|C\|_F + \mu n^{1-p/2}\right)\right)\right)$$

iterations, where $r' = \min\{r, 1\}$. Note that for every fixed $p \in (0,1)$ and $\epsilon > 0$, η is polynomially bounded by the input parameters. Moreover, each iteration is clearly polynomial–time implementable. Thus, we have proven the following theorem, which is the main result of this paper:

Theorem 1. *Let $p \in (0,1)$ and $\epsilon > 0$ be fixed. The potential reduction algorithm described above returns either an ϵ–optimal solution or an ϵ–first–order critical point of the Schatten p–regularized SDP problem (10) in polynomial time.*

IV. Localizability Analysis

One of the nice features of the Biswas–Ye SDP relaxation [20] and its trace regularization [12] is that for input instances satisfying certain uniqueness properties, positions of the sensors in the required dimension can be exactly recovered by solving those relaxations [11], [12], [16]. Thus, it is natural to ask whether this feature is preserved by the Schatten p–regularizations of those relaxations, especially in view of the fact that we can only compute a first–order critical point in polynomial time. In this section, we show that the answer is affirmative. We begin with some definitions.

Definition 2. *Consider an instance of the network localization problem $\mathcal{L} = (G, (\mathbf{d}, \bar{\mathbf{d}}), \mathbf{a}, d)$, where $G = (V_s \cup V_a, E_{ss} \cup E_{as})$ is the network, $\mathbf{d} = (d_{ij})_{(i,j)\in E_{ss}}$ and $\bar{\mathbf{d}} = (\bar{d}_{ij})_{(i,j)\in E_{as}}$ are the vectors of distance measurements, $\mathbf{a} = (a_i)_{i\in V_a}$ is the vector of anchor positions, and $d \geq 1$ is the target dimension (see Section II). We say that*

1) *(cf. [11], [16]) $\mathcal{L}$ is* uniquely d–localizable *if (i) $V_a \neq \emptyset$, (ii) there is a unique set of sensor positions $\{\bar{x}_i \in$*

$\mathbb{R}^d : i \in V_s\}$ *in* $\mathbb{R}^d$ *that satisfy the given distance measurements, and (iii) for any* $l > d$*, when the position of each anchor* $i \in V_a$ *is regarded as the point* $(a_i; \mathbf{0})$ *in* $\mathbb{R}^l$*, there is no set of sensor positions in* $\mathbb{R}^l$ *that satisfy the given distance measurements;*

2) *(cf. [12])* $\mathcal{L}$ *is* d–localizable *if (i)* $V_a = \emptyset$*, (ii) there is a set of sensor positions* $\{\bar{x}_i \in \mathbb{R}^d : i \in V_s\}$ *in* $\mathbb{R}^d$ *that satisfy the given distance measurements, and (iii) for any* $l \neq d$*, there is no set of sensor positions in* $\mathbb{R}^l$ *that satisfy the given distance measurements (except the set* $\{(\bar{x}_i; \mathbf{0}) \in \mathbb{R}^l : i \in V_s\}$ *when* $l > d$*).*

We remark that these notions of localizability are closely related to those of rigidity, which have been used extensively in the localizability analysis of network localization problems; see, e.g., [16] and the references therein.

Our first result in this section states that when the input instance $\mathcal{L}$ is uniquely d–localizable, the unique set of sensor positions in $\mathbb{R}^d$ can still be recovered if we apply our potential reduction algorithm to the Schatten p–regularized Biswas–Ye SDP relaxation (8). It extends the corresponding result in [11, Theorem 2].

Proposition 4. *Let* $\mathcal{L}$ *be a given instance of the network localization problem and* $p \in (0, 1)$ *be fixed. Consider the corresponding Schatten* p*–regularized Biswas–Ye SDP relaxation (8). Suppose that* $\mathcal{L}$ *is uniquely* d*–localizable, and let* $\bar{Z}$ *be any first–order critical point of Problem (8). Then, we have* $\bar{Z} \in \mathcal{F}_0$ *and rank*$(\bar{Z}) = d$*; cf. (4) and (6).*

Proof: By definition, we have $\bar{Z} \in \mathcal{F}_0$. Moreover, the results in [11, Theorem 2] show that when $\mathcal{L}$ is uniquely d–localizable, $\mathcal{F}_0$ contains only one element, and its rank is d. This completes the proof. ∎

In a similar fashion, we can prove the following result concerning the recoverability of d–localizable instances by certain Schatten p–regularization. It extends the corresponding result in [12, Theorem 1]. We defer the proof to the full version of this paper.

Proposition 5. *Let* $\mathcal{L}$ *be a given instance of the network localization problem with* $V_a = \emptyset$ *and* $p \in (0, 1)$ *be fixed. Consider its Schatten* p*–regularized SDP relaxation:*

$$\min \left\{ f_p(Y) : \mathrm{tr}(K_{ij} Y) = d_{ij}^2 \text{ for } (i,j) \in E_{ss},\ Y \in \mathcal{S}_+^{|V|} \right\}. \tag{26}$$

Suppose that $\mathcal{L}$ *is* d*–localizable. Let* $\bar{Y}$ *be any first–order critical point of Problem (26). Then,* $\bar{Y}$ *is feasible for (26) and rank*$(\bar{Y}) = d$*. In particular, we have* $\bar{Y} = \bar{X}^T \bar{X}$ *for some* $\bar{X} \in \mathbb{R}^{d \times |V|}$*. Moreover, the positions of the sensors* $\bar{x}_1, \ldots, \bar{x}_{|V|} \in \mathbb{R}^d$*, which form the columns of* $\bar{X}$*, are* centered*; i.e., they satisfy* $\sum_{i \in V} \bar{x}_i = \mathbf{0}$*.*

V. Simulation Results

In this section, simulation results are presented to demonstrate the effectiveness of the Schatten p–regularization approach to network localization. For simplicity, we consider the setting where the distance measurements are accurate. All simulations are implemented in MATLAB and run on a 2.66GHz CPU PC with 3 GB memory. We shall compare the following three approaches:

1) SDR: the original Biswas–Ye SDP relaxation [20] (i.e., Problem (6) without the rank constraint)
2) TRACE: trace regularization of the Biswas–Ye SDP relaxation [12]
3) SCHATTEN: Schatten 0.5–regularization of the Biswas–Ye SDP relaxation

Specifically, TRACE (resp. SCHATTEN) corresponds to Problem (8) with $p = 1$ (resp. $p = 0.5$) when there are anchors, or Problem (26) with $p = 1$ (resp. $p = 0.5$) when there is no anchor. In particular, the former is a convex optimization problem, while the latter is not. The value $p = 0.5$ for the Schatten regularization is chosen just for illustration. Due to space limitation, we shall discuss the effects of different choices of $p \in (0, 1)$ in the full version of this paper.

To evaluate the performance of the above approaches, we consider three criteria. The first is *position error* (PE), which measures the discrepancy between the computed locations and true locations of the sensors. It is defined as

$$\mathsf{PE} = \sqrt{\frac{1}{|V|} \sum_{i \in V} \|x_i - \bar{x}_i\|_2^2},$$

where $x_i \in \mathbb{R}^d$ is the position of sensor i as computed by a particular approach, and $\bar{x}_i \in \mathbb{R}^d$ is the true position. The second is *edge error* (EE), which measures the discrepancy between the computed distances and the given distance measurements. It is defined as

$$\mathsf{EE} = \sqrt{(\Delta_{ss} + \Delta_{as})/|E|},$$

where $\Delta_{ss} = \sum_{(i,j) \in E_{ss}} \left| \|x_i - x_j\|_2 - d_{ij} \right|^2$ and $\Delta_{as} = \sum_{(i,j) \in E_{as}} \left| \|a_i - x_j\|_2 - \bar{d}_{ij} \right|^2$. An advantage of using EE over PE as the evaluation criterion is that the true positions of the sensors need not be known. The last criterion is the rank of the solution matrix obtained by a particular approach. Recall that if the rank of the solution equals to d, which is the target dimension in which the nodes should reside, then we have found a localization of the network in $\mathbb{R}^d$ [11]. Thus, this criterion is meaningful in evaluating how faithful the solution is from the dimension requirement.

A. Performance on Unit–Disk Graphs

In our first experiment, we randomly place 50 sensors and 3 anchors over a unit square, and connect two nodes by an edge if their distance is at most ρ, where $\rho = 0.16, 0.17, \ldots, 0.30$. We generate 100 such networks and solve them using the aforementioned three approaches. The results are shown in Figures 1 and 2. As can be seen from the figures, the SCHATTEN approach outperforms the other two in terms of EE and the solution rank when the network is sparse (i.e., when ρ is small). In particular, the SCHATTEN approach tends to produce localizations that are more faithful to the distance measurements and dimension requirement. On the other hand, the SCHATTEN approach incurs a slightly higher

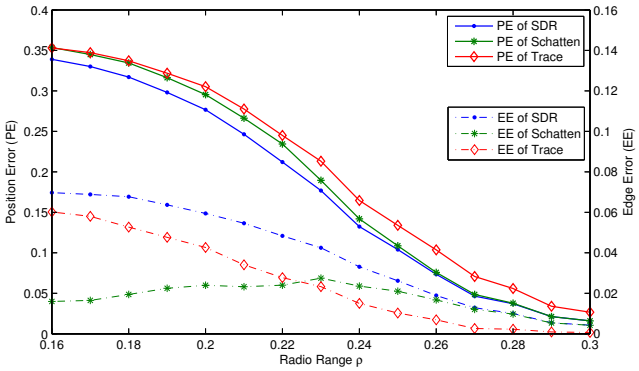

Fig. 1. The average PE and EE with 50 sensors and 3 anchors, unit–disk graphs

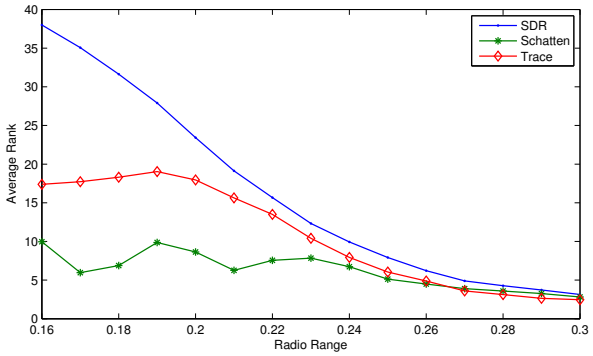

Fig. 2. The average solution rank with 50 sensors and 3 anchors, unit–disk graphs

PE than SDR. However, this does not necessarily imply that the approach cannot localize the nodes accurately, as the input instance could have multiple localizations in $\mathbb{R}^2$.

B. Performance on Globally Rigid Graphs

In our second experiment, we focus on input instances that have a unique localization in $\mathbb{R}^2$. In other words, the underlying graphs of these instances are *globally rigid* in $\mathbb{R}^2$; see, e.g., [16]. It is known (see, e.g., [35]) that every globally rigid graph in $\mathbb{R}^2$ can be constructed from K_4, the complete graph on 4 vertices, by applying a sequence of two types of operations: (i) *edge addition*, where we add an edge to connect some pair of non–adjacent vertices, and (ii) 1*–extension*, where we subdivide an edge (i, j) by a new vertex k and add a new edge (k, l) for some $l \notin \{i, j\}$. Thus, to create an input instance with a unique localization in $\mathbb{R}^2$, we first generate a random globally rigid graph in $\mathbb{R}^2$ with a given number of vertices by repeatedly applying 1–extension to K_4. Then, we perform a number of edge additions. Finally, we randomly place the vertices of the resulting graph over the unit square.

Using the above procedure, we generate two sets of test instances. In the first set, for a given number of extra edges to be added by the edge addition operation, we generate 100 input instances with 50 sensors and no anchor. Then, we solve these instances using the TRACE and SCHATTEN approaches. We count the number of instances for which the TRACE and SCHATTEN approaches return a rank–2 solution. The result is shown in Figure 3. It demonstrates the effectiveness of the Schatten p–regularization in finding a localization in the target dimension, especially when the number of extra edges added is small.

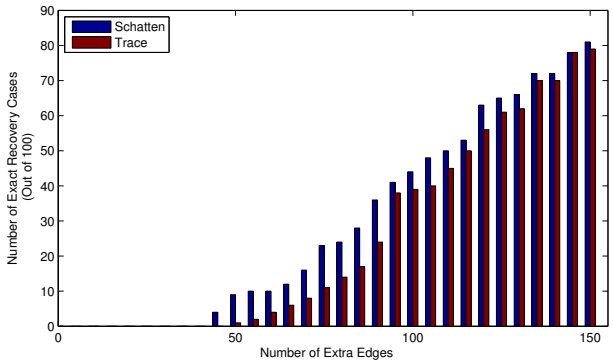

Fig. 3. Number of exact recovery (i.e., rank–2 solution) cases with 50 sensors, globally rigid graphs

In the second set, for a given number of extra edges to be added by the edge addition operation, we generate 100 input instances with 50 sensors and 3 anchors. Then, we solve these instances using all three approaches. The results are shown in Figure 4. We see that the SCHATTEN approach outperforms the other two both in terms of EE and solution rank. Figure 5 shows the localization results for one particular instance. As can be seen from the figure, the localization produced by the SCHATTEN approach is exact, while that produced by SDR contains some error.

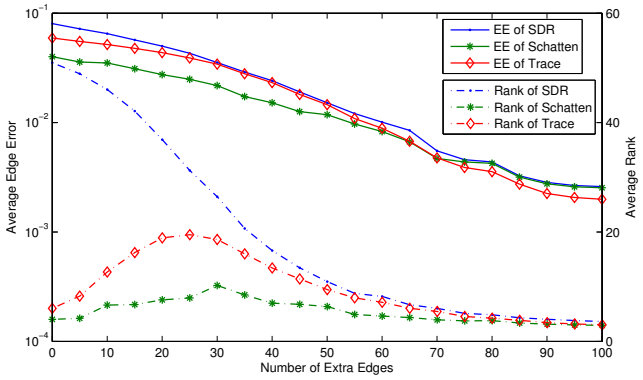

Fig. 4. The average EE and solution rank with 50 sensors and 3 anchors, globally rigid graphs

VI. CONCLUSION

In this paper, we investigated the use of the non–convex Schatten quasi–norm as a regularizer for inducing low–rank solutions to various SDP relaxations of the fixed–dimensional localization problem. We showed that a first–order critical

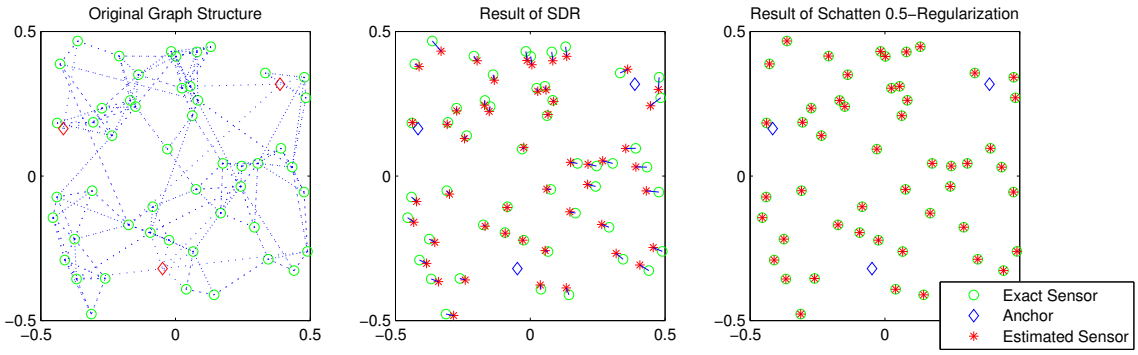

Fig. 5. Localization by SDR and SCHATTEN approaches with 50 sensors and 3 anchors, globally rigid graph

point of the Schatten–regularized SDP problem can be approximated to arbitrary accuracy in polynomial time by an interior–point algorithm. We then demonstrated the viability of our approach via both localizability analysis and simulations.

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