A Discrete First-Order Method for Large-Scale MIMO Detection with Provable Guarantees

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***Abstract*—In this paper, we consider a simple and low-complexity discrete first-order method called the Generalized Power Method (GPM) for large-scale MIMO detection. The GPM is essentially a projected gradient method and exploits the fact that the projection onto the discrete MPSK or QAM constellation is efficiently computable. As our main contribution, we first show that under certain conditions on the channel and additive noise, the GPM will converge to the true symbol vector in a finite number of iterations. We then show that the aforementioned conditions will be satisfied with high probability under standard probabilistic models of the channel and noise. Besides enjoying strong theoretical guarantees, the proposed method is shown in our simulations to be competitive with existing methods in terms of both detection performance and numerical efficiency. We believe that our techniques will find further applications in the development of high-performance detection methods for massive MIMO.**

I. Introduction

MIMO detection, a topic that has been extensively studied in the early 2000s, has received renewed interest as recent research activities suggest [1]. The major driving force for revisiting such a seemingly well-established topic is massive MIMO, in which the base station can be equipped with tens or hundreds of antennas. This is in sharp contrast to the MIMO systems available in current wireless standards, which have about 4 to 8 antennas and are regarded as small-scale MIMO. Massive MIMO is one of the most sought-after technologies in future wireless standards, most notably in 5G. It allows a large number of users to be served simultaneously in both the uplink and downlink. MIMO detection is an essential component for the uplink MIMO scenario and plays a crucial role in minimizing the impact of multiuser interference, especially in the presence of a massive number of users. MIMO detection is also important to massive connectivity for machine type communications [2] (with applications such as autonomous vehicles and internet of things) – yet another exciting emerging concept from massive MIMO.

One of the new challenges that arises in recent studies of MIMO detection is the large problem size. While some high-performance MIMO detection methods, such as sphere decoding [3], work extremely well in small-scale MIMO, their complexities tend to increase very quickly with the problem size, thus rendering them impractical in large-scale MIMO. This makes low-complexity detection methods like zero-forcing (ZF), minimum mean square error (MMSE) and their variants—which are previously seen as sub-optimal in small-scale MIMO—attractive again [4], [5]. There has also been some interest in simple heuristics for maximum-likelihood (ML) MIMO detection, such as the likelihood ascent search detectors [6]. It is worthwhile to mention the new developments in [7], [8], in which massive MIMO detection under practical and stringent signal quantization constraints is considered and efficient methods are developed.

In this paper, we propose a discrete first-order method for handling ML MIMO detection. The terminology stems from the fact that our proposed method takes a negative gradient step at the current iterate and projects the resulting point onto the discrete MPSK or QAM constellation to obtain the next iterate. The viability of such a method relies crucially on the fact that the projection onto the MPSK or QAM constellation is efficiently computable, even though neither constellation forms a convex set. Since the idea of applying gradient-type methods to handle ML MIMO detection problems is rather natural and may not be new (see, e.g., [9]), we focus on addressing some fundamental theoretical issues concerning the proposed method. Our contribution is twofold. First, we identify the conditions under which the iterates generated by our discrete first-order method will converge to the *true symbol vector*. Note that such a result does not follow from and is in fact stronger than existing convergence results for projected gradient methods. Indeed, the existing results at best assert only the convergence of the iterates to an *ML estimator* (i.e., an optimal solution to the non-convex ML estimation problem) of the true symbol vector. However, there is no guarantee that the ML estimator is the true symbol vector that we wish to recover. Second, we show that the our method enjoys finite convergence and provide an explicit bound on the number of iterations needed for convergence. As a by-product of our analysis, we show that for the MPSK constellation, our proposed method has the same recovery guarantee as the computationally more expensive semidefinite relaxation (SDR) detector in a certain signal-to-noise ratio (SNR) regime; cf. [10]. This demonstrates that our discrete first-order method is competitive with the SDR detector in terms of both theoretical guarantees and numerical efficiency.

Our notations are standard. We use $\mathbb{C}^n$ to denote the set of

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n-dimensional complex vectors. For a complex number $z \in \mathbb{C}$, we use $\Re(z)$ and $\Im(z)$ to denote its real and imaginary parts, respectively. For a complex vector $\boldsymbol{z} \in \mathbb{C}^n$, we denote its 2-norm and ∞-norm by $\|\boldsymbol{z}\|_2$ and $\|\boldsymbol{z}\|_\infty$, respectively. For a complex matrix $\boldsymbol{Z} \in \mathbb{C}^{m\times n}$, we denote its conjugate transpose by $\boldsymbol{Z}^*$ and its operator norm by $\|\boldsymbol{Z}\|_{\mathrm{op}}$. Given a closed set $\mathcal{C} \subseteq \mathbb{C}^n$ and a point $\boldsymbol{z} \in \mathbb{C}^n$, we use $\Pi_{\mathcal{C}}(z)$ to denote a projection of $\boldsymbol{z}$ onto $\mathcal{C}$; i.e.,

$$\Pi_{\mathcal{C}}(\boldsymbol{z}) \in \arg\min_{\boldsymbol{x}\in\mathcal{C}} \|\boldsymbol{z}-\boldsymbol{x}\|_2^2.$$

II. ML MIMO Detection

Consider a complex-valued linear MIMO model

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}^\star + \boldsymbol{\nu}, \tag{1}$$

where $\boldsymbol{y} \in \mathbb{C}^m$ is the received signal vector, $\boldsymbol{H} \in \mathbb{C}^{m\times n}$ is the channel matrix, $\boldsymbol{x}^\star \in \mathbb{C}^n$ is the transmitted symbol vector, and $\boldsymbol{\nu} \in \mathbb{C}^m$ is the noise vector. We assume that $m \geq n$ throughout, but special emphasis will be put on the setting where both m and n are large and $m > n$. We also assume that the entries of $\boldsymbol{\nu}$ are independent and identically (i.i.d.) Gaussian random variables with mean zero and variance σ_ν^2, and that each symbol x_i is drawn from some discrete constellation $\mathcal{S}$. We focus on the case where $\mathcal{S}$ is either the $(4u^2)$-QAM constellation

$$\mathcal{Q}_u = \{z \in \mathbb{C} : \Re(z), \Im(z) = \pm1, \pm3, \cdots, \pm(2u-1)\}$$

or the MPSK constellation

$$\mathcal{S}_M = \{\exp(2\pi ik/M) : k = 0, 1, \ldots, M-1\},$$

with $i = \sqrt{-1}$ being the imaginary unit. The goal of MIMO detection is to recover $\boldsymbol{x}^\star$, or to recover as many entries of $\boldsymbol{x}^\star$ as possible. Towards that end, we consider the following ML estimation problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & F(\boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|_2^2 \\ \text{s.t.} \quad & x_j \in \mathcal{S},\ j = 1, \ldots, n. \end{aligned} \tag{2}$$

It has long been known that when $|\mathcal{S}| > 1$, Problem (2) is NP-hard in general [11]. In view of this hardness result, there has been much interest in the past in designing *polynomial-time* algorithms, such as the SDR-based detectors (see [12] and the references therein), for finding an approximate solution to the ML estimation problem (2). However, motivated by the developments in massive MIMO, the interest has recently shifted to the design of low-complexity detection methods. In the next section, we present a numerically efficient discrete first-order method for solving Problem (2) and analyze its convergence behavior.

III. Generalized Power Method for MIMO Detection

As alluded to in Section I, our proposed method is iterative in nature. We summarize the method in Algorithm 1. Each iteration consists of a gradient step (line 3) and a projection step (line 4). We remark that Algorithm 1 can be viewed as an instantiation of a general algorithmic scheme called the *Generalized Power Method* (GPM) [13]. Hence, for convenience, we shall refer to Algorithm 1 by the same name. However, one should not confuse our instantiation of the GPM with those in [13], [14], [15]. The former is developed for MIMO detection, while the latter are developed for other problems.

Algorithm 1 Generalized Power Method for MIMO Detection

1: **input**: initial point $\boldsymbol{x}^0 \in \mathcal{S}^n$ and step sizes $\{\alpha_k\}_{k\geq0}$
2: **if** stopping criterion is not met **then**
3: $\quad \nabla F(\boldsymbol{x}^k) \longleftarrow 2\boldsymbol{H}^*(\boldsymbol{H}\boldsymbol{x}^k - \boldsymbol{y})$
4: $\quad \boldsymbol{x}^{k+1} \longleftarrow \Pi_{\mathcal{S}^n}\left(\boldsymbol{x}^k - \frac{\alpha_k}{m}\nabla F(\boldsymbol{x}^k)\right)$
5: $\quad k \longleftarrow k+1$
6: **end if**

The above description leaves some flexibility in the choice of the stopping criterion, the initial point, and the step sizes. In this work, we terminate the GPM if $\boldsymbol{x}^{k+1} = \boldsymbol{x}^{k'}$ for some $k' \leq k+1$. Note that such a stopping criterion will eventually be met, as the number of feasible solutions is finite. One obvious drawback of this stopping criterion is that we need to save all the previous iterates, which means that the memory requirement could be large. However, in our experiments, the GPM always stops in at most 100 or so iterations; see Section IV for more details. For the initial point $\boldsymbol{x}^0 \in \mathcal{S}^n$ and step sizes $\{\alpha_k\}_{k\geq0}$, our theoretical results indicate that an arbitrary initialization and constant step size suffice to ensure finite convergence of the method. In practice, however, more judicious choices will greatly enhance the detection performance of the method. These will be detailed in Section IV.

Since the GPM requires only two matrix-vector multiplications, one vector addition, and one projection onto the very structured set $\mathcal{S}^n$ in each iteration, it is extremely efficient and scalable, making it particularly suitable for large array sizes and high-order digital modulation schemes. Furthermore, the GPM enjoys strong theoretical guarantees. The following theorem, which constitutes the main contribution of this paper, shows that under certain conditions on the channel matrix $\boldsymbol{H}$ and the noise vector $\boldsymbol{\nu}$, the iterates generated by the GPM will converge to the true symbol vector $\boldsymbol{x}^\star$ in a finite number of iterations.

Theorem 1. *Consider the MIMO model* (1). *Let $\{\boldsymbol{x}^k\}_{k\geq0}$ be the sequence of iterates generated by Algorithm 1 with step sizes $\{\alpha_k\}_{k\geq0}$ satisfying*

$$\left\|\frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{\nu}\right\|_\infty < \frac{1}{c} \quad \text{and} \quad \left\|\boldsymbol{I} - \frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{H}\right\|_{\mathrm{op}} \leq \beta < \frac{1}{4}, \tag{3}$$

where $c = \frac{4}{\min_{s\neq s'\in\mathcal{S}}|s-s'|} < \infty$ (hence, we have $c = 2$ for $\mathcal{S} = \mathcal{Q}_u$ and $c = \frac{2}{\sin(\pi/M)}$ for $\mathcal{S} = \mathcal{S}_M$). Then, we have

$$\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^\star\|_2 \leq 4\beta\|\boldsymbol{x}^k - \boldsymbol{x}^\star\|_2$$

for all $k \geq 0$. In particular, after at most $k^\star = \left\lceil \ln\left(\frac{2}{c\|\boldsymbol{x}^0-\boldsymbol{x}^\star\|_2}\right)/\ln(4\beta)\right\rceil$ iterations, we have $\boldsymbol{x}^k = \boldsymbol{x}^\star$ for all $k \geq k^\star$; i.e., the GPM admits finite convergence.

In view of Theorem 1, the natural next step is to study when condition (3) holds. The following theorem provides a setting under which condition (3) will hold with high probability.

Theorem 2. *Suppose that the entries of the channel matrix $\boldsymbol{H}$ are i.i.d. standard complex Gaussian random variables, the noise variances satisfies $\sigma_\nu^2 \leq \frac{m}{4c^2\log n}$, and the aspect ratio satisfies $\gamma := \frac{m}{n} \geq \frac{20}{\beta^2} > 1$. Then, with the constant step size $\alpha_k = \frac{1}{2}$ for all $k \geq 0$, condition (3) will hold with probability at least $1 - \sqrt{\frac{2}{\pi}} \cdot \frac{1}{n} - 4\exp\left(-\frac{m}{8}\right) - 2\exp(-n)$.*

Before we outline the proofs of Theorems 1 and 2, three remarks are in order. First, most existing detection methods focus on finding an optimal solution to the ML estimation problem (2). At first sight, it seems that the GPM, which is essentially a projected gradient method for solving (2), is doing the same. However, Theorem 1 reveals that the GPM is actually achieving more – it can provably recover the true symbol vector $\boldsymbol{x}^\star$ under certain conditions. Second, when $\boldsymbol{H}$ has i.i.d. standard complex Gaussian entries, the GPM has an interesting connection to some well-known detection methods. To explain the connection, observe that since $\mathbb{E}[\boldsymbol{H}^*\boldsymbol{H}] = m\boldsymbol{I}$, one would expect that $\boldsymbol{H}$ is tightly concentrated around its mean; i.e., $\boldsymbol{H}^*\boldsymbol{H} \approx m\boldsymbol{I}$. Thus, if we take $\alpha_k = \frac{1}{2}$ for all $k \geq 0$ in Algorithm 1, then the gradient in line 3 is approximately $\nabla F(\boldsymbol{x}^k) \approx 2(m\boldsymbol{x}^k - \boldsymbol{H}^*\boldsymbol{y})$ and the projected gradient in line 4 is approximately

$$\boldsymbol{x}^{k+1} \approx \Pi_{\mathcal{S}^n}\left(\frac{1}{m}\boldsymbol{H}^*\boldsymbol{y}\right).$$

Upon noting that $(\boldsymbol{H}^*\boldsymbol{H})^{-1} \approx \frac{1}{m}\boldsymbol{I}$, we see that the GPM can be viewed as an iterative version of the zero-forcing (ZF) detector

$$\boldsymbol{x}_{\rm ZF} = \Pi_{\mathcal{S}^n}\left((\boldsymbol{H}^*\boldsymbol{H})^{-1}\boldsymbol{H}^*\boldsymbol{y}\right).$$

On the other hand, if we fix $\delta > 0$ and take $\alpha_k = \frac{1}{2(m+\delta)}$ for all $k \geq 0$ in Algorithm 1, then the projected gradient in line 4 is approximately

$$\boldsymbol{x}^{k+1} \approx \Pi_{\mathcal{S}^n}\left(\frac{1}{m+\delta}\left(\boldsymbol{H}^*\boldsymbol{y} + \delta\boldsymbol{x}^k\right)\right).$$

In particular, the GPM can be viewed as an iterative version of the minimum mean square error (MMSE) detector

$$\boldsymbol{x}_{\rm MMSE} = \Pi_{\mathcal{S}^n}\left((\boldsymbol{H}^*\boldsymbol{H} + \delta\boldsymbol{I})^{-1}\boldsymbol{H}^*\boldsymbol{y}\right).$$

Third, it is interesting to note that under the setting of Theorem 2, the SDR of the ML estimation problem (2) for the MPSK constellation is tight [10]. In other words, we can recover an ML estimator of the true symbol vector by solving the SDR of Problem (2). However, our convergence result in Theorem 1 is stronger, as it guarantees that the GPM will converge to the true symbol vector itself. Moreover, the GPM is numerically more efficient than the SDR detector. This demonstrates the power and potential of our proposed method.

Proof of Theorem 1. Define

$$\boldsymbol{w}^k = \left(\boldsymbol{I} - \frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{H}\right)(\boldsymbol{x}^k - \boldsymbol{x}^\star)$$

and

$$\boldsymbol{z}^k = \boldsymbol{x}^k - \frac{\alpha_k}{m}\nabla F(\boldsymbol{x}^k) = \boldsymbol{x}^\star + \boldsymbol{w}^k + \frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{\nu},$$

where the second equality follows from (1). Then, the update scheme of the GPM can be expressed as

$$\boldsymbol{x}^{k+1} = \Pi_{\mathcal{S}^n}\left(\boldsymbol{x}^\star + \boldsymbol{w}^k + \frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{\nu}\right).$$

Let $J_k = \{j : |w_j^k| \geq \frac{1}{c}\}$. Using condition (3), we have $|z_l^k - x_l^\star| < \frac{2}{c}$ for any $l \notin J_k$. By definition of c, this implies that

$$x_l^{k+1} = \Pi_{\mathcal{S}}(z_l^k) = x_l^\star. \tag{4}$$

To proceed, we need the following lemma. Due to space limitation, we defer its proof to the full version of this paper.

Lemma 1. *Let $\boldsymbol{z} \in \mathbb{C}^n$ and $\boldsymbol{x} \in \mathcal{S}^n$ be given. Then, for all $q \in [1,\infty]$, we have*

$$\|\Pi_{\mathcal{S}^n}(\boldsymbol{z}) - \boldsymbol{x}\|_q \leq 2\|\boldsymbol{z} - \boldsymbol{x}\|_q. \tag{5}$$

Now, let $\boldsymbol{x}_{J_k}$ denote the projection of $\boldsymbol{x}$ onto the coordinate set J_k. We compute

$$\begin{aligned}
\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^\star\|_2 &= \|\boldsymbol{x}_{J_k}^{k+1} - \boldsymbol{x}_{J_k}^\star\|_2 && (6)\\
&\leq 2\|\boldsymbol{z}_{J_k}^k - \boldsymbol{x}_{J_k}^\star\|_2 && (7)\\
&= 2\left\|\boldsymbol{w}_{J_k}^k + \left(\frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{\nu}\right)_{J_k}\right\|_2 \\
&\leq 4\|\boldsymbol{w}_{J_k}^k\|_2 && (8)\\
&\leq 4\|\boldsymbol{w}^k\|_2 \\
&\leq 4\beta\|\boldsymbol{x}^k - \boldsymbol{x}^\star\|_2, && (9)
\end{aligned}$$

where (6) follows from (4), (7) follows from Lemma 1, (8) follows from the definition of J_k and our assumption that $\left\|\frac{2\alpha_k}{m}\boldsymbol{H}^*\boldsymbol{\nu}\right\|_\infty < \frac{1}{c}$, and (9) follows from condition (3). This completes the proof. ☐

Proof of Theorem 2. The first part of condition (3) is a direct consequence of [10, Proposition 3.6], which says that for any $\theta > \frac{1}{2}$,

$$\begin{aligned}
&\Pr_{\boldsymbol{H},\nu}\left(\|\boldsymbol{H}^*\boldsymbol{\nu}\|_\infty > m^\theta\sigma_\nu\right)\\
&\leq\sqrt{\frac{2}{\pi}} \cdot n \cdot \exp\left(-\frac{m^{2\theta-1}}{2}\right) + 4\exp\left(-\frac{m}{8}\right).
\end{aligned}$$

Indeed, by choosing θ such that $m^\theta = \sqrt{4m\log n}$ and assuming $\sigma_\nu \leq \frac{\sqrt{m}}{2c\sqrt{\log n}}$, we have

$$\begin{aligned}
\Pr_{\boldsymbol{H},\nu}\left(\left\|\frac{1}{m}\boldsymbol{H}^*\boldsymbol{\nu}\right\|_\infty > \frac{1}{c}\right) &\leq \Pr_{\boldsymbol{H},\nu}\left(\|\boldsymbol{H}^*\boldsymbol{\nu}\|_\infty > \sigma_\nu\sqrt{4m\log n}\right)\\
&\leq \sqrt{\frac{2}{\pi}} \cdot \frac{1}{n} + 4\exp\left(-\frac{m}{8}\right).
\end{aligned}$$

To prove the second part of condition (3), we use the fact that for any $t > 0$, the inequalities

$$\sigma_{\max}(\boldsymbol{H}) \leq \sqrt{m}\left(1 + \sqrt{\frac{n}{m}} + t\right)$$
$$\sigma_{\min}(\boldsymbol{H}) \geq \sqrt{m}\left(1 - \sqrt{\frac{n}{m}} - t\right)$$

hold simultaneously with probability at least $1 - 2\exp(-mt^2)$; see, e.g., [16, Exercise 9.5]. By setting $t = \sqrt{\frac{n}{m}} = \sqrt{\frac{1}{\gamma}}$ and noting that $\gamma \geq \frac{20}{\beta^2}$ and $\beta < \frac{1}{4}$, we see that

$$\left\| \boldsymbol{I} - \frac{1}{m}\boldsymbol{H}^*\boldsymbol{H} \right\|_{\text{op}} \leq \frac{4}{\sqrt{\gamma}} + \frac{4}{\gamma} \leq \frac{4\beta}{\sqrt{20}} + \frac{\beta^2}{5} < \beta$$

holds with probability at least $1 - 2\exp(-n)$.

Summarizing the above, we conclude that condition (3) will hold with the probability stated in the theorem. □

IV. Simulations

In this section, we present some numerical results to demonstrate the effectiveness of our proposed GPM and to support our theoretical findings. We study the symbol error rate (SER) and complexity performance of the proposed GPM when applied to the MIMO model (1) with the constellations $\mathcal{Q}_u$ and $\mathcal{S}_M$.

In our experiments, the entries of the channel matrix $\boldsymbol{H} \in \mathbb{C}^{m\times n}$ are generated independently according to the standard complex Gaussian distribution, and the entries of the transmitted symbol vector $\boldsymbol{x}^\star$ are drawn independently and uniformly from the constellation $\mathcal{S}$. Under such a setting, we define the SNR as

$$\text{SNR} = \frac{\mathbb{E}[\|\boldsymbol{H}\boldsymbol{x}^\star\|_2^2]}{\mathbb{E}[\|\boldsymbol{\nu}\|_2^2]} = \frac{m\sigma_x^2}{\sigma_\nu^2} = m\delta,$$

where $\sigma_x^2 = \mathbb{E}[\|\boldsymbol{x}^\star\|_2^2]$ and $\delta := \frac{\sigma_x^2}{\sigma_\nu^2}$. We focus on the $\mathcal{Q}_2$ (i.e., 16-QAM) and $\mathcal{S}_8$ (i.e. 8-PSK) constellations with problem sizes $(m, n) = (64, 64)$ and $(m, n) = (128, 64)$. We generate 10^6 problem instances $(\boldsymbol{H}, \boldsymbol{x}^\star, \boldsymbol{\nu})$ for each SNR value. For 16-QAM, we initialize our GPM using the result of the lattice reduction-aided MMSE with decision feedback method (LRA MMSE DF); for 8-PSK, we initialize our GPM by the MMSE estimator.

Note that our theoretical results do not cover the above choices of the problem size. Nevertheless, as we shall see shortly, our proposed GPM still exhibits excellent detection and numerical performance.

A. Symbol Error Rate

We begin by studying the SER versus SNR performance of various methods. For 16-QAM, we compare our proposed GPM with the zero-forcing decision feedback detector (ZF DF), the minimum mean square error detector (MMSE), the lattice reduction-aided MMSE with decision feedback method (LRA MMSE DF), and the lattice reduction-aided ZF with decision feedback method (LRA ZF DF). The results for 16-QAM are summarized in Figure 1. For $(m, n) = (64, 64)$, the performance of GPM is very close to that of LRA MMSE DF. For $(m, n) = (128, 64)$, however, the GPM outperforms all other methods. In particular, at SER $= 10^{-5}$, the SNR gain of GPM over LRA MMSE DF and LRA ZF DF is about 2 dB .

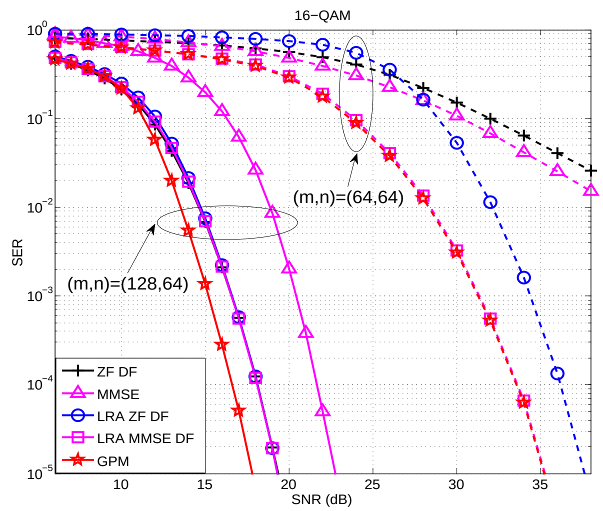

Fig. 1. SER vs SNR for 16-QAM MIMO Detection

For 8-PSK, we compare the GPM with the ZF DF, MMSE, and MMSE DF methods. As shown in Figure 2, the GPM improves the SER of the MMSE method and even outperforms the MMSE DF method in both the $(m, n) = (64, 64)$ and $(m, n) = (128, 64)$ cases. More precisely, Figure 2 shows that at SER $= 10^{-5}$, the SNR gains of the GPM over the MMSE DF method are around 4 dB and 2 dB for $(m, n) = (64, 64)$ and $(m, n) = (128, 64)$, respectively. It is also worth noting that the case where $m = 2n$ requires smaller SNR to achieve the same SER for both 16-QAM and 8-PSK constellations. This is in part due to the fact that a bigger ratio between m and n implies better conditioning of the channel matrix $\boldsymbol{H}$.

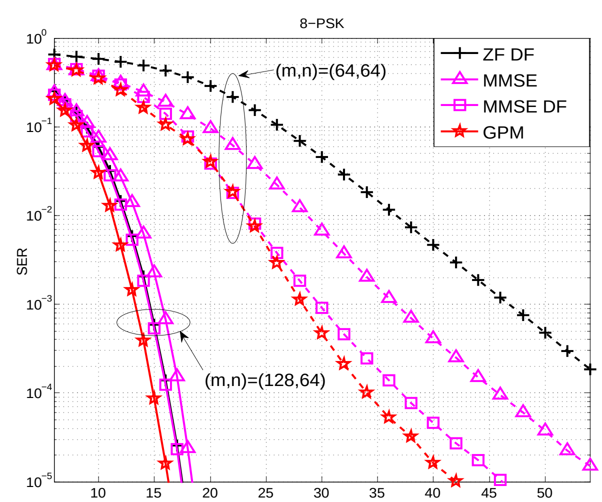

Fig. 2. SER vs SNR for 8-PSK MIMO Detection

B. Step Size Rules

Next, we discuss the choice of step sizes for the GPM. For the case where $m = 2n$, the choice $\alpha_k = \frac{1}{2(1+\delta/m)}$ for all $k \geq 0$ works well for both the 16-QAM and 8-PSK constellations. However, for the case where $m = n$, it is not clear what a good choice would be. We summarize the choices we made in our simulations in Table I. We leave the problem of determining how the choice of step sizes impacts the performance of the GPM to a future study.

TABLE I
CHOICE OF STEP SIZES IN THE GPM

(m,n)	**8-PSK**	**16-QAM**
$(64,64)$	$\frac{1}{2(1+8\delta/m)}$	$\frac{0.87-0.006\cdot\text{SNR}}{2(1+\delta/m)}$
$(128,64)$	$\frac{1}{2(1+\delta/m)}$	$\frac{1}{2(1+\delta/m)}$

C. Iteration Complexity

Lastly, we summarize the iterations needed by the GPM in Table II. By convention, if the initial point $\boldsymbol{x}^0$ already satisfies the stopping criterion, the number of GPM iterations is counted as zero. As can be seen from the table, the maximum number of iterations (over all tested values of SNR and all generated problem instances) is about 100. However, the average number of iterations needed is much less. In particular, for the problem size $(m,n) = (128,64)$, the average number of iterations is less than 3 for both the 16-QAM and 8-PSK constellations.

TABLE II
MAXIMUM AND AVERAGE NUMBER OF GPM ITERATIONS

16-QAM	(64,64)	**SNR**	6	10	14	18	22	26	30	34
		Max	45	74	84	84	88	62	54	28
		Average	1.85	1.92	2.26	1.96	1.09	0.23	0.016	0.0003
	(128,64)	**SNR**	6	8	10	12	14	16	18	
		Max	86	104	106	89	60	28	3	
		Average	5.82	7.25	6.20	2.98	0.79	0.11	0.007	
8-MPSK	(64,64)	**SNR**	6	12	18	24	30	36	42	48
		Max	5	59	81	78	54	49	46	25
		Average	1.05	2.92	5.94	2.18	1.23	1.02	0.99	0.98
	(128,64)	**SNR**	6	8	10	12	14	16	18	
		Max	42	61	36	16	9	5	3	
		Average	4.71	4.30	3.03	1.79	1.13	0.90	0.87	

V. CONCLUSION

We developed a discrete first-order method called the Generalized Power Method (GPM) for MIMO detection with MPSK and QAM constellations. The method is simple and has low per-iteration complexity, thus making it highly attractive for large-scale MIMO. Our convergence analysis and simulation results demonstrate that the GPM is competitive with some existing methods in terms of theoretical guarantees, detection performance, and numerical efficiency. An interesting future direction would be to study whether the algorithmic framework developed in this paper can be extended or further enhanced to tackle other detection problems that arise in massive MIMO.

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