Rank-Two Beamforming and Stochastic Beamforming for MISO Physical-Layer Multicasting with Finite-Alphabet Inputs

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***Abstract*—This paper considers multi-input single-output (MISO) downlink multicasting with finite-alphabet inputs when perfect channel state information is known at the transmitter. Two advanced transmit schemes, namely the beamformed (BF) Alamouti scheme and the stochastic beamforming (SBF) scheme, for maximizing the finite-alphabet-constrained multicast rate are studied. We show that the transmit optimization for these two schemes can be formulated as an SNR-based max-min-fair (MMF) problem with Gaussian inputs, which can be handled via the semidefinite relaxation (SDR) technique. Apart from transmit optimization, we analyzed the rate performance of the two schemes. Our analytical results show that for BF Alamouti, the multicast rate degrades with the number of users M at a rate of $\sqrt{M}$, which is better than the traditional transmit beamforming scheme. For SBF, the multicast rate degradation is less sensitive to the increase in the number of users and outperforms BF Alamouti for large M. All the results were verified by numerical simulations.**

***Index Terms*—multicast, transmit beamforming, finite-alphabet input, semidefinite relaxation (SDR).**

I. Introduction

As an efficient way of delivering common information to multiple users simultaneously, transmit beamforming for physical-layer multicasting has received considerable attention in the last decade [1]. The signal-to-noise ratio (SNR)-based max-min-fair (MMF) formulation, together with the semidefinite relaxation (SDR) technique [1], has been demonstrated to offer a reasonably good multicast rate; see [2] and the references therein. In our recent work [3], extensions to rank-two beamforming and stochastic beamforming were proposed to further improve the multicast rate, especially for those large-scale multicast systems [4]. We should mention that in existing beamforming studies, Gaussian signaling is usually assumed. While such an assumption is crucial to quantifying the limit of the system, it may not be adequate for practical systems as they usually involve non-Gaussian finite-alphabet inputs, such as M-PSK and M-QAM. Recently, there has been a growing interest in transmit optimizations with non-Gaussian, finite-alphabet inputs; notable works include [5]–[9] on various single/multi-user MIMO transceiver designs.

In this letter, driven by practical considerations, we focus on the scenario of multiuser multi-input single-output (MISO) downlink multicasting with *finite-alphabet inputs*. We assume that perfect channel state information is known at the transmitter and our goal is to maximize the finite-alphabet-constrained multicast rate by judiciously designing the transmit schemes. As the main contribution of this paper, two transmit schemes are investigated—beamformed (BF) Alamouti and stochastic beamforming (SBF). For the proposed transmit schemes, by converting the MISO channel into an equivalent single-input single-output (SISO) one, we show that the finite-alphabet-constrained multicast rate optimization problems can both be turned into the classic SNR-based MMF problem with Gaussian inputs. Hence, the semidefinite relaxation (SDR) technique [10] can be employed to deliver an (approximate) solution for our considered problem. To quantify the performance of the BF Alamouti and SBF schemes, we study their approximation accuracy—i.e., the rate gap between the SDR-based approximate solution and the optimal solution to the SDR problem. Building upon existing SDR approximation results [1], [3], we derive analytic rate gaps of the two schemes under finite-alphabet inputs. In particular, our results reveal that the BF Alamouti and SBF schemes are generally better than transmit beamforming. Moreover, BF Alamouti is suitable for small number of users while SBF can achieve better rate performance for large number of users. All these results are new for MISO multicasting with finite-alphabet inputs and they are consistent with and complement those in [3], where Gaussian inputs are assumed.

II. The Beamformed Alamouti Scheme

A. System Model and Problem Statement

Consider a multi-user MISO downlink, where a multi-antenna base station sends common information to M single-antenna users. Assuming quasi-static and flat-fading channels, the receive signal at user i ($i = 1, \ldots, M$) is given by

$$y_i(n) = \mathbf{h}_i^H \mathbf{x}(n) + v_i(n), \quad n = 1, \ldots, 2T, \qquad (1)$$

where $\mathbf{h}_i \in \mathbb{C}^N$ is the channel vector from an N-antenna base station to the ith user, $2T$ represents the data frame

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length, $v_i(n) \sim \mathcal{CN}(0,1)$ is the additive white Gaussian noise, and $\mathbf{x}(n)$ is the transmit signal carrying the common information. For the classic multicast beamforming scheme, $\mathbf{x}(n)$ is generated as

$$\mathbf{x}(n) = \sqrt{P}\mathbf{w}s(n), \quad n = 1, \ldots, 2T, \tag{2}$$

where P is the transmit power, $\mathbf{w} \in \mathbb{C}^N$ is the beamforming vector, and $s(n) \in \mathbb{C}$ is the information symbol, which is usually assumed to be Gaussian distributed [1]. Herein, deviating from (2), we introduce a new rank-two beamforming scheme, called beamformed (BF) Alamouti. Particularly, we consider the multicast system with *finite-alphabet inputs*.

The BF Alamouti scheme was first proposed in [3], [11], which can be seen as a rank-two generalization of the transmit beamforming scheme [1]. It consists of the following three main steps: 1) Group the consecutive information symbols into multiple 2×1 vectors $\boldsymbol{s}(m) = [\ s(2m-1)\ s(2m)\ ]^T$ for $m = 1, \ldots, T$; 2) map $\boldsymbol{s}(m)$ into a 2×2 Alamouti code $\mathbf{C}(\boldsymbol{s}(m))$, i.e.,

$$\mathbf{C}(\boldsymbol{s}(m)) = \begin{bmatrix} s(2m-1) & s(2m) \\ -s^*(2m) & s^*(2m-1) \end{bmatrix};$$

3) multiply $\mathbf{C}(\boldsymbol{s}(m))$ by an $N \times 2$ rank-two beamforming matrix $\mathbf{B}$. As a result, the transmit signal $\boldsymbol{X}(m) \in \mathbb{C}^{N \times 2}$ for the mth transmission block is given by

$$\boldsymbol{X}(m) = [\,\mathbf{x}(2m-1)\,\mathbf{x}(2m)\,] = \sqrt{P}\mathbf{B}\mathbf{C}(\boldsymbol{s}(m)),\ m = 1, \ldots, T.$$

Herein we consider the case where $\boldsymbol{s}(m)$ is drawn uniformly from a given finite-alphabet set $\mathcal{S}$, e.g., $\mathcal{S} = \{\pm 1 \pm j\}$ for QPSK. Then, at the receiver side, by treating $\mathbf{B}$ as part of the channel and applying the standard Alamouti detection, the MISO received signal model

$$\boldsymbol{y}_i(m) = \sqrt{P}\mathbf{h}_i^H\mathbf{B}\mathbf{C}(\boldsymbol{s}(m)) + \boldsymbol{v}_i(m),\ m = 1, \ldots, T$$

can be turned into the following equivalent SISO one

$$y_i(n) = \sqrt{\mathsf{snr}_i(\mathbf{B}\mathbf{B}^{\mathbf{H}})}s(n) + v_i(n),\ n = 1, \ldots, 2T,$$

where we define $\mathsf{snr}_i(\mathbf{A}) \triangleq P\mathbf{h}_i^H\mathbf{A}\mathbf{h}_i$ for $\mathbf{A} \succeq \mathbf{0}$, $\boldsymbol{y}_i(m) = [y_i(2m-1), y_i(2m)]^T$, and $\boldsymbol{v}_i(m) = [v_i(2m-1), v_i(2m)]^T$. This completes the description of the BF Alamouti scheme. Now, our problem of interest is to design the beamforming matrix $\mathbf{B}$ such that the finite-alphabet-constrained multicast rate is maximized; i.e.,

$$\max_{\mathbf{B} \in \mathbb{C}^{N \times 2}} \min_{i=1,\ldots,M} R_i(\mathbf{B}\mathbf{B}^H) \qquad \text{s.t. } \mathrm{Tr}(\mathbf{B}\mathbf{B}^H) \le 1, \tag{3}$$

where $R_i(\mathbf{B}\mathbf{B}^H)$ is user-i's achievable rate under finite-alphabet inputs [5]:

$$R_i(\mathbf{B}\mathbf{B}^H) = \log|\mathcal{S}| - |\mathcal{S}|^{-1}\sum_{s_m \in \mathcal{S}} \mathbb{E}_{v_i}\Big[\log \sum_{s_k \in \mathcal{S}} e^{d_{m,k}(\mathbf{B}\mathbf{B}^H)}\Big].$$

Here, $d_{m,k}(\mathbf{A}) \triangleq |v_i|^2 - |\sqrt{P\mathsf{snr}_i(\mathbf{A})}s_{m,k} + v_i|^2$, $s_{m,k} \triangleq s_m - s_k, \forall s_m, s_k \in \mathcal{S}$, and $v_i \sim \mathcal{CN}(0,1)$.

In contrast to the case of Gaussian inputs, problem (3) appears to be more challenging due to the complex form of the rate function R_i. Nevertheless, the following lemma reveals some nice properties of R_i:

Lemma 1 ([5], [12]) *The rate function $R_i(\mathbf{B}\mathbf{B}^H)$ is nondecreasing and concave w.r.t. $\mathsf{snr}_i(\mathbf{B}\mathbf{B}^H)$.*

The nondecreasing property follows directly from Theorem 1 in [12] and the concavity is due to Theorem 1 in [5]. In light of Lemma 1, we can express problem (3) as the following much simpler SNR-based max-min-fair (MMF) problem:

$$(\mathsf{MMF}) \quad \max_{\mathbf{B} \in \mathbb{C}^{N \times 2}} \min_{i=1,\ldots,M} \mathbf{h}_i^H\mathbf{B}\mathbf{B}^H\mathbf{h}_i \quad \text{s.t. } \mathrm{Tr}(\mathbf{B}\mathbf{B}^H) \le 1.$$

Problem (MMF) is identical to the multicasting problem under Gaussian inputs [3], which is known to be NP-hard in general [3], [11]. To generate an approximate solution to (MMF) in an efficient manner, a widely used technique is SDR. Specifically, by letting $\mathbf{W} = \mathbf{B}\mathbf{B}^H \succeq \mathbf{0}$ and dropping the rank-two constraint on $\mathbf{W}$, we get an SDR of (MMF):

$$(\mathsf{SDR}) \quad \max_{\mathbf{W} \succeq \mathbf{0}} \min_{i=1,\ldots,M} \mathbf{h}_i^H\mathbf{W}\mathbf{h}_i \qquad \text{s.t. } \mathrm{Tr}(\mathbf{W}) \le 1.$$

Problem (SDR) is a convex problem and can be efficiently solved. Let $\mathbf{W}^\star$ be an optimal solution to (SDR). If $\mathrm{rank}(\mathbf{W}^\star) \le 2$, then an optimal solution to problem (3) can be obtained through eigendecomposition; otherwise a rank-two Gaussian randomization in Algorithm 1 can be employed to generate an approximate solution $\hat{\mathbf{B}}$ to (3). In the sequel, we shall develop a sufficient condition under which (SDR) has an optimal rank-two solution and analyze the SDR approximation accuracy of the Gaussian randomization procedure.

B. Approximation Accuracy Analysis for BF Alamouti

To facilitate the analysis of the SDR approximation accuracy, let us denote by

$$R^\star(P) \triangleq \min_{i=1,\ldots,M} R_i(\mathbf{W}^\star) \tag{4}$$

the "multicast rate" associated with the optimal solution to (SDR). Clearly, $R^\star(P)$ serves as an upper bound on the maximum multicast rate of BF Alamouti; i.e.,

$$R^\star(P) \ge R^\star_{\mathsf{ALAM}}(P) \ge \min_{i=1,\ldots,M} R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H)$$

holds for any feasible $\hat{\mathbf{B}}$ of (3), where $R^\star_{\mathsf{ALAM}}(P)$ denotes the optimal value of (3). Let $\rho_{\min} = \min_{i=1,\ldots,M} \mathbf{h}_i^H\mathbf{W}^\star\mathbf{h}_i$. The following proposition reveals a further relationship among $R^\star(P)$, $R^\star_{\mathsf{ALAM}}(P)$, and $\min_{i=1,\ldots,M} R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H)$:

Proposition 1 *Consider problems* (MMF) *and* (SDR)*:*

a) *When $M \le 8$, one can always find in polynomial time an optimal $\mathbf{W}^\star$ of* (SDR) *such that* $\mathrm{rank}(\mathbf{W}^\star) \le 2$. *Thus, $R^\star(P) = R^\star_{\mathsf{ALAM}}(P)$ holds and problem* (MMF)*, or equivalently problem* (3)*, can be optimally solved.*
b) *When $M > 8$, by using the rank-two Gaussian randomization in Algorithm 1, one can generate from $\mathbf{W}^\star$ a feasible solution $\hat{\mathbf{B}}$ to problem* (3) *that satisfies*

$$R^\star_{\mathsf{ALAM}}(P) - \min_{i=1,\ldots,M} R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H) \le \min\{R^\star_{\mathsf{ALAM}}(P), \gamma(P)\}$$

with probability at least $1 - (5/6)^L$, where L denotes the number of Gaussian randomizations and

$$\gamma(P) \triangleq 1 - \log 2 + \frac{1}{|\mathcal{S}|}\sum_{s_m \in \mathcal{S}} \log\Big(\frac{\sum_{s_k \in \mathcal{S}} e^{-\frac{|s_{m,k}|^2 P\rho_{\min}}{12.22\sqrt{M}}}}{\sum_{s_k \in \mathcal{S}} e^{-|s_{m,k}|^2 P\rho_{\min}}}\Big).$$

The proof of Proposition 1 is relegated to Appendix A. The first part of Proposition 1 gives a sufficient condition under which the SDR is tight, while the second part identifies a worst-case multicast rate gap of the SDR-based approximation. Roughly speaking, the rate gap scales up at a rate of $\sqrt{M}$ as the number of users M increases.[1] While this result is better than that for transmit beamforming [1], [3], BF Alamouti may still suffer from rate degradation when M is large. In the next, we will introduce another beamforming scheme—stochastic beamforming, which is able to achieve better multicast rate performance than BF Alamouti, especially for large M.

Algorithm 1 Gaussian Randomization Procedure for (3)

1: For $j = 1$ to L, generate two independent random vectors $\boldsymbol{\xi}_1^j, \boldsymbol{\xi}_2^j \sim \mathcal{CN}(\mathbf{0}, \mathbf{W}^\star)$, define $\tilde{\mathbf{B}}_j = \frac{1}{\sqrt{2}}[\ \boldsymbol{\xi}_1^j \quad \boldsymbol{\xi}_2^j\ ]$ and $\hat{\mathbf{B}}_j = \tilde{\mathbf{B}}_j \Big/ \sqrt{\mathrm{Tr}(\tilde{\mathbf{B}}_j \tilde{\mathbf{B}}_j^H)}$;

2: Let $j^\star := \arg \max_{j=1,\ldots,L} \min_i \ \mathsf{snr}_i(\hat{\mathbf{B}}_j \hat{\mathbf{B}}_j^H)$ and $\hat{\mathbf{B}} = \hat{\mathbf{B}}_{j^\star}$.

III. The Stochastic Beamforming Scheme

Stochastic beamforming (SBF) was originally introduced in [3] for multicast beamforming under Gaussian inputs. The key idea is to adopt a randomize-in-time beamforming strategy [3], rather than keeping $\mathbf{w}$ invariant over the whole transmission [cf. (2)]. Specifically, the transmit signal of SBF takes the form

$$\mathbf{x}(n) = \sqrt{P}\mathbf{w}(n)s(n), \quad n = 1, \ldots, 2T, \tag{5}$$

where the beamformer $\mathbf{w}(n)$ varies randomly in time according to some prespecified distribution (to be specific shortly). At the receiver side, by treating $\mathbf{w}(n)$ as part of the channel, SBF renders a virtual fast-fading SISO received signal model:

$$y_i(n) = \sqrt{P}\mathbf{h}_i^H \mathbf{w}(n)s(n) + v_i(n), \quad n = 1, \ldots, 2T. \tag{6}$$

By letting the transmitter send the random seed for generating $\mathbf{w}(n)$ and its covariance $\mathbf{W}$ to the users as part of the preamble of the transmitted data frame, SBF receivers can presume simple coherent symbol reception and channel decoding. Hence, SBF is just as efficient as those of fixed beamforming with channel coding in terms of implementation. Consequently, the multicast rate of SBF under finite-alphabet inputs can be deduced as follows:

$$R_{\mathsf{SBF}}(\mathbf{W}) = \min_{i=1,\ldots,M} \mathbb{E}_{\boldsymbol{w}}\big[R_i(\boldsymbol{w}\boldsymbol{w}^H)\big], \tag{7}$$

where $\boldsymbol{w}$ denotes a generic random variable of $\mathbf{w}(n)$ with mean zero and covariance matrix $\mathbb{E}[\boldsymbol{w}\boldsymbol{w}^H] = \mathbf{W}$. Notice that $\mathrm{Tr}(\mathbf{W}) \leq \mathbf{1}$ is implicitly assumed in order to satisfy the transmit power constraint.

To maximize the SBF multicast rate $R_{\mathsf{SBF}}(\mathbf{W})$, we need to optimize the distribution of $\boldsymbol{w}$, which is a challenging task. In this letter, as a compromise, we consider an easy-to-generate distribution, namely the Gaussian distribution [3], which is able to achieve a provably good multicast rate.[2]

[1] To obtain this result, we have used the fact that a log-sum-exp function can be well approximated by a pointwise maximum function [13].

[2] There are also other choices of SBF distributions, e.g., elliptic and Bingham SBFs; readers may refer to [3] for details.

For the Gaussian SBF, $\boldsymbol{w}$ follows a complex Gaussian distribution $\mathcal{CN}(\mathbf{0}, \mathbf{W})$. In this case, it remains to optimize the covariance matrix $\mathbf{W}$, viz.,

$$\max_{\mathbf{W}\succeq\mathbf{0}} \ \min_{i=1,\ldots,M} \ \mathbb{E}_{\boldsymbol{w}}\big[R_i(\boldsymbol{w}\boldsymbol{w}^H)\big] \quad \text{s.t. } \mathrm{Tr}(\mathbf{W}) \leq 1. \tag{8}$$

Problem (8) is a stochastic optimization problem. By evaluating the expectation, it can be shown that

Claim 1 Problem (8) is equivalent to problem (SDR).

The proof of Claim 1 is provided in Appendix B. Claim 1 implies that the optimal covariance for generating Gaussian SBF beamformers is identical to the optimal $\mathbf{W}^\star$ to (SDR). Moreover, since $R_i(\boldsymbol{w}\boldsymbol{w}^H)$ is concave w.r.t. $\mathsf{snr}_i(\boldsymbol{w}\boldsymbol{w}^H)$ (cf. Lemma 1), by Jensen's inequality, it can be shown that the objective of (8) is upper bounded by $R^\star(P)$ [cf. (4)], i.e., $R_{\mathsf{SBF}}(\mathbf{W}^\star) \leq R^\star(P)$. Conversely, we have

Proposition 2 *For Gaussian SBF, it holds that*

$$0 \leq R^\star(P) - R_{\mathsf{SBF}}(\mathbf{W}^\star) \leq \min\{R^\star(P), \gamma_{\mathsf{Gauss}}(P)\}, \tag{9}$$

where

$$\gamma_{\mathsf{Gauss}}(P) = 1 - \log 2 + \frac{1}{|\mathcal{S}|} \sum_{s_m \in \mathcal{S}} \log\left(\frac{\sum_{s_k \in \mathcal{S}} \frac{2}{2 + |s_{m,k}|^2 P \rho_{\min}}}{\sum_{s_k \in \mathcal{S}} e^{-|s_{m,k}|^2 P \rho_{\min}}}\right).$$

The proof of Proposition 2 is relegated to Appendix C. Compared to Proposition 1(b), the rate gap of Gaussian SBF in (9) is insensitive to M, which implies that Gaussian SBF may achieve better rate performance than BF Alamouti, especially for large number of users. This will be further confirmed by our numerical results in Sec. IV.

IV. Simulation Results and Conclusions

In this section, we provide numerical results to compare the performance of the three transmit schemes, namely the transmit beamforming scheme, BF Alamouti scheme, and Gaussian SBF scheme. The simulation settings are as follows: The base station has N antennas and serves M users; QPSK and 16-ary QAM modulation schemes are adopted; the number of Gaussian randomization is $L = 30MN$. All the channels are randomly generated with each entry being i.i.d. $\mathcal{CN}(0, 1)$.

Fig. 1 investigates the rate behaviors of various methods when we set the number of users $N = 8$ and increase the number of users M by fixing $P = 0\,$dB. For ease of presentation, we normalize each user's achievable rate by $\log|\mathcal{S}|$, and thus the maximum rate is 1 bps/Hz. This figure shows that as expected, Gaussian SBF is less sensitive to M, as compared with transmit beamforming and BF Alamouti. Particularly, it outperforms other schemes when $M > 32$. This suggests that for large-scale system, it is more desirable to adopt the SBF scheme. In addition, for $M = 8$ we see that BF Alamouti coincides with the SDR upper bound, which is consistent with Proposition 1 (a). In Fig. 2, we show the bit error rate (BER) results for different transmit strategies. Specifically, we show a relatively long code–a rate-$1/3$ turbo code with an information length of 960 bits–for $N = 8$,

$M = 24$, 16-ary QAM modulated case and a relatively short code–rate-$1/2$ turbo code with an information length of 286 bits–for $N = 8$, $M = 32$, 64-ary QAM modulated case. Note that we adopt the channel coding scheme in [14] with 10 decoding iterations. As a performance lower bound, we plot the result of "SDR bound", which runs a virtual single-user SISO channel with SNR $P\rho_{\min}$. From the plots, we see that even with a short code (which means that T is not relatively large), both BF Alamouti and Gaussian SBF own better BER performance than transmit beamforming, and they are within 1dB away from the SISO lower bound.

To conclude, we have considered the multicast rate optimization for MISO downlink with finite-alphabet inputs. Two new beamforming schemes, namely beamformed (BF) Alamouti and Gaussian stochastic beamforming (SBF), were investigated. We have shown that for both schemes, the multicast rate optimization problem under finite-alphabet inputs can be recast as that under Gaussian inputs. From there, we have also analyzed the rate performances of the two schemes. Simulation results demonstrate that BF Alamouti and Gaussian SBF can achieve better rate performance than traditional transmit beamforming, and Gaussian SBF is superior to other schemes when there are many users in the system.

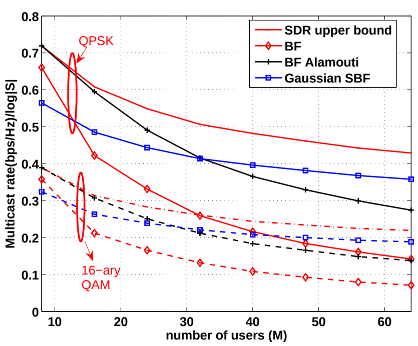

Fig. 1. The normalized multicast rate scaling with the number of users.

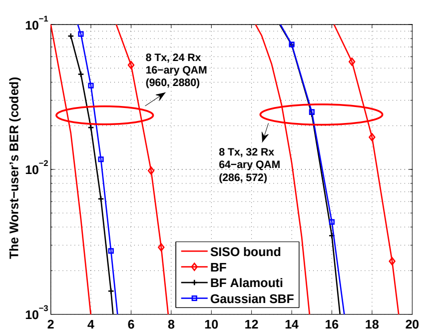

Fig. 2. The worst-user's BER performance for various schemes.

Appendix

A. Proof of Proposition 1

The proof relies on the existing SDR approximation results for problem (MMF) [1], [3]. Specifically, the first part of Proposition 1 follows directly from Proposition 5 in [3]. To prove the second result, notice that for any $\mathbf{B}$, we have

$$R_i^{\mathsf{LB}}(\mathbf{B}\mathbf{B}^H) \le R_i(\mathbf{B}\mathbf{B}^H) \le R_i^{\mathsf{UB}}(\mathbf{B}\mathbf{B}^H) \tag{10}$$

where $R_i^{\mathsf{LB}}(\mathbf{B}\mathbf{B}^H) \triangleq \log|\mathcal{S}| - 1 + \log 2 - |\mathcal{S}|^{-1}\sum_{s_m\in\mathcal{S}} \log \sum_{s_k\in\mathcal{S}} \exp(-|s_{m,k}|^2 P \mathbf{h}_i^H \mathbf{B}\mathbf{B}^H \mathbf{h}_i/2)$ and $R_i^{\mathsf{UB}}(\mathbf{B}\mathbf{B}^H) \triangleq \log|\mathcal{S}| - \frac{1}{|\mathcal{S}|}\sum_{s_m\in\mathcal{S}} \log \sum_{s_k\in\mathcal{S}} \exp(-|s_{m,k}|^2 P \mathbf{h}_i^H \mathbf{B}\mathbf{B}^H \mathbf{h}_i)$. The first inequality in (10) follows from Theorem 1 in [6] and the second inequality is due to Jensen's inequality $\mathbb{E}_{v_i}[\log\sum_{s_k\in\mathcal{S}} \exp(d_{m,k}(\mathbf{B}\mathbf{B}^H))] \ge \log\sum_{s_k\in\mathcal{S}} \exp(\mathbb{E}_{v_i}[d_{m,k}(\mathbf{B}\mathbf{B}^H)])$. It also follows from Theorem 4 in [3] that the Gaussian randomization procedure in Algorithm 1 yields a beamforming matrix $\hat{\mathbf{B}}$ such that

$$\min_{i=1,\ldots,M} \mathbf{h}_i^H \hat{\mathbf{B}}\hat{\mathbf{B}}^H \mathbf{h}_i \ge (12.22M)^{-1} \min_{i=1,\ldots,M} \mathbf{h}_i^H \mathbf{W}^\star \mathbf{h}_i \tag{11}$$

holds with probability at least $1 - (5/6)^L$. Combining (10) and (11) yields

$$\begin{aligned} &R^\star_{\mathsf{ALAM}}(P) - \min_i R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H) \le R^\star(P) - \min_i R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H) \\ &\le \min_i R_i^{\mathsf{UB}}(\mathbf{W}^\star) - \min_i R_i^{\mathsf{LB}}(\hat{\mathbf{B}}\hat{\mathbf{B}}^H), \\ &\le \min_i R_i^{\mathsf{UB}}(\mathbf{W}^\star) - \min_i R_i^{\mathsf{LB}}\big((12.22M)^{-1}\mathbf{W}^\star\big) \triangleq \gamma(P), \end{aligned}$$

where the last inequality is due to (11). This, together with the nonnegativity of $\min_i R_i(\hat{\mathbf{B}}\hat{\mathbf{B}}^H)$, produces the desired result.

B. Proof of Claim 1

By using the property of Gaussian SBF [3, Theorem 1], we have

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{w}\sim\mathcal{CN}(\mathbf{0},\mathbf{W})}\left[R_i(\boldsymbol{w}\boldsymbol{w}^H)\right] \\ =&\mathbb{E}_\xi\underbrace{\left\{ c - |\mathcal{S}|^{-1} \sum_{s_m\in\mathcal{S}} \mathbb{E}_{v_i}\left[\log \sum_{s_k\in\mathcal{S}} e^{|v_i|^2 - |\sqrt{P\rho_i\xi}s_{m,k}+v_i|^2}\right]\right\}}_{\triangleq f(\rho_i,\xi)}, \end{aligned}$$

where $c = \log|\mathcal{S}|$, $\rho_i = \mathbf{h}_i^H \mathbf{W} \mathbf{h}_i$, $v_i \sim \mathcal{CN}(0,1)$, and ξ follows an exponential distribution with unit mean. Note that for a given ξ, $f(\rho_i,\xi)$ is nondecreasing w.r.t. ρ_i (recall Lemma 1). Then, it is easy to see that $\mathbb{E}_\xi[f(\rho_i,\xi)]$ is nondecreasing w.r.t. ρ_i, and thus $R_{\mathsf{SBF}}(\mathbf{W}) = \min_{i=1,\ldots,M} \mathbb{E}_\xi[f(\rho_i,\xi)]$ is nondecreasing w.r.t. $\min_{i=1,\ldots,M}\rho_i$. In other words, maximizing $R_{\mathsf{SBF}}(\mathbf{W})$ amounts to maximizing $\min_{i=1,\ldots,M}\rho_i$, which completes the proof.

C. Proof of Proposition 2

Define $\rho_i = \mathbf{h}_i^H \mathbf{W}^\star \mathbf{h}_i$ and $\xi_i = |\mathbf{h}_i^H \boldsymbol{w}|^2/\rho_i$. Since $\boldsymbol{w} \sim \mathcal{CN}(\mathbf{0},\mathbf{W}^\star)$, it can be shown that ξ_i follows an exponential distribution with unit mean. Hence, we have

$$\begin{aligned} &R_{\mathsf{SBF}}(\mathbf{W}^\star) = \min_{i=1,\ldots,M} \mathbb{E}_{\boldsymbol{w}\sim\mathcal{CN}(\mathbf{0},\mathbf{W}^\star)}\left[R_i(\boldsymbol{w}\boldsymbol{w}^H)\right] \\ &\ge \min_{i=1,\ldots,M} \mathbb{E}_{\boldsymbol{w}\sim\mathcal{CN}(\mathbf{0},\mathbf{W}^\star)}\left[R_i^{\mathsf{LB}}(\boldsymbol{w}\boldsymbol{w}^H)\right] \\ &\ge \min_{i=1,\ldots,M}\left\{ c - \frac{1}{|\mathcal{S}|}\sum_{s_m\in\mathcal{S}} \log\Big(\sum_{s_k\in\mathcal{S}} \mathbb{E}_{\xi_i}\left[e^{-|s_{m,k}|^2 P\xi_i\rho_i/2}\right]\Big)\right\} \\ &= c - |\mathcal{S}|^{-1}\sum_{s_m\in\mathcal{S}} \log\Big(\sum_{s_k\in\mathcal{S}} \frac{2}{2+|s_{m,k}|^2 P\rho_{\min}}\Big), \end{aligned} \tag{12}$$

where $c = \log|\mathcal{S}| + \log 2 - 1$, R_i^{LB} is defined in (10), and the second inequality is due to Jensen's inequality. Also, we have

$$R^\star(P) = \min_{i=1,\ldots,M} R_i(\mathbf{W}^\star) \le \min_{i=1,\ldots,M} R_i^{\mathsf{UB}}(\mathbf{W}^\star),$$

which together with (12) and the nonnegativity of $R_{\mathrm{SBF}}(\mathbf{W})$ yields the desired result in (9).

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