A Supplementary Note for “Unraveling the Rank-One Solution Mystery of Robust MISO Downlink Transmit Optimization: A Verifiable Sufficient Condition via a New Duality Result”

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Abstract

In the manuscript “Unraveling the Rank-One Solution Mystery of Robust MISO Downlink Transmit Optimization: A Verifiable Sufficient Condition via a New Duality Result,” the rank-one solution analysis of a robust transmit optimization problem in the multiuser MISO downlink scenario is considered. As a companion note of the aforementioned manuscript, this technical report provides some auxiliary results for the rank-one solution analysis problem. In particular, we prove the limitation of an alternative approach for tackling the rank-one solution analysis problem, namely, that by semidefinite program rank reduction.

1 Purpose

This technical report is a companion note of the main manuscript [1]. In [1], we consider a robust rate-constrained problem in the multiuser MISO downlink scenario

$$\min_{\boldsymbol{\mathcal{W}}} \ \sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \tag{1a}$$

$$\text{s.t.} \ \max_{\boldsymbol{h}_i \in \mathcal{U}_i} \varphi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{h}_i) \leq 0, \ i = 1, \ldots, K, \tag{1b}$$

$$\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K \succeq \mathbf{0}, \tag{1c}$$

where

$$\varphi_i(\boldsymbol{\mathcal{W}}, \boldsymbol{h}_i) = \sigma_i^2 + \boldsymbol{h}_i^H \left(\sum_{j \neq i} \boldsymbol{W}_j - \frac{1}{\gamma_i} \boldsymbol{W}_i \right) \boldsymbol{h}_i,$$

$$\mathcal{U}_i = \{ \boldsymbol{h}_i \in \mathbb{C}^N \mid \| \boldsymbol{h}_i - \bar{\boldsymbol{h}}_i \|_2 \leq \varepsilon_i \},$$

and study the corresponding rank-one solution analysis problem; see the main manuscript for details. The purpose of this report is to provide a proof on the limitation of an alternative approach for tackling the rank-one solution analysis, namely, the SDP rank reduction approach. While we do not adopt this approach in the main manuscript, the results may be of interest to some readers.

2 SDP Rank Reduction

In Section II-D of the main manuscript, we discussed SDP rank reduction and its application to the rank-one solution analysis problem. In particular, recall from the main manuscript that Problem (1) can be recast as an SDP

$$\min_{\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{t}} \ \sum_{i=1}^{K} \mathrm{Tr}(\boldsymbol{W}_i) \tag{2a}$$

$$\text{s.t.} \ \boldsymbol{Z}_i = \begin{bmatrix} \boldsymbol{Q}_i + t_i \boldsymbol{I} & \boldsymbol{r}_i \\ \boldsymbol{r}_i^H & s_i - t_i \varepsilon_i^2 \end{bmatrix}, \ i = 1, \ldots, K, \tag{2b}$$

$$\boldsymbol{W}_i \succeq \mathbf{0}, \boldsymbol{Z}_i \succeq \mathbf{0}, t_i \geq 0, \ i = 1, \ldots, K, \tag{2c}$$

where $\boldsymbol{Q}_i = \frac{1}{\gamma_i} \boldsymbol{W}_i - \sum_{j \neq i} \boldsymbol{W}_j$, $\boldsymbol{r}_i = \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i$, $s_i = \bar{\boldsymbol{h}}_i^H \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i - \sigma_i^2$, $\boldsymbol{W}_i \in \mathbb{H}^N$, $\boldsymbol{Z}_i \in \mathbb{H}^{N+1}$, $t_i \in \mathbb{R}$ for all i. Also, recall the following SDP rank reduction result.

Fact 1 (SDP rank reduction [2]) *Consider a complex-valued separable SDP*

$$\min_{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \in \mathbb{H}^n} \ \sum_{i=1}^{k} \mathrm{Tr}(\boldsymbol{C}_i \boldsymbol{X}_i) \tag{3a}$$

$$\text{s.t.} \ \sum_{l=1}^{k} \mathrm{Tr}(\boldsymbol{A}_{i,l} \boldsymbol{X}_l) \trianglerighteq_i b_i, \ i = 1, \ldots, m, \tag{3b}$$

$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \succeq \mathbf{0}, \tag{3c}$$

where $\boldsymbol{A}_{i,l}, \boldsymbol{C}_i \in \mathbb{H}^n$, $b_i \in \mathbb{R}$ for all i, l, and the notation $\trianglerighteq_i$ can be either '$\geq$' or '$=$' for each i. Suppose that Problem (3) has an optimal solution. Then, there exists a solution $(\boldsymbol{X}_1^\star, \ldots, \boldsymbol{X}_k^\star)$ to Problem (3) such that

$$\sum_{i=1}^{k} \mathrm{rank}(\boldsymbol{X}_i^\star)^2 \leq m. \tag{4}$$

In particular, if $\boldsymbol{X}_i^\star \neq \boldsymbol{0}$ *for all* i *and* $m \leq k+2$*, then every* $\boldsymbol{X}_i^\star$ *has* $\mathrm{rank}(\boldsymbol{X}_i^\star) = 1$.

The problem of interest is to apply Fact 1 to obtain a sufficient condition on the ranks of the optimal solution $\boldsymbol{\mathcal{W}}^\star$ to Problem (2). The result is as follows: If Problem (2) has an optimal solution, then there exists an optimal solution $(\boldsymbol{W}_i^\star, \boldsymbol{Z}_i^\star, t_i^\star)_{i=1}^K$ to Problem (2) such that

$$\sum_{i=1}^K \mathrm{rank}(\boldsymbol{W}_i^\star)^2 \leq K(N^2+2N) - \sum_{i=1}^K \mathrm{rank}(\boldsymbol{Z}_i^\star)^2. \tag{5}$$

Also, $\boldsymbol{Z}_i^\star$ must satisfy

$$\mathrm{rank}(\boldsymbol{Z}_i^\star) \leq N, \quad \text{for all } i. \tag{6}$$

The implication of (5)–(6) is that even the "best-case" bound in (5), i.e., $\sum_{i=1}^K \mathrm{rank}(\boldsymbol{W}_i^\star)^2 \leq K(N^2+2N) - KN^2 = 2NK$, does not guarantee the rank-one result $\mathrm{rank}(\boldsymbol{W}_i^\star) = 1$ for all i.

Herein, we give the proof of the above result.

Step 1: We reformulate Problem (2) as Problem (3), and thereby use (4) to deduce a rank result. Observe that each constraint in (2b) can be represented by

$$\mathrm{Re}([\boldsymbol{Z}_i]_{k,l}) = \sum_{j,p,q} \left(a_{j,p,q}^{(i,k,l)} \mathrm{Re}([\boldsymbol{W}_j]_{p,q}) + b_{j,p,q}^{(i,k,l)} \mathrm{Im}([\boldsymbol{W}_j]_{p,q}) \right) + c^{(i,k,l)} t_i + d^{(i,k,l)}, \quad \text{for all } k \leq l, \tag{7a}$$

$$\mathrm{Im}([\boldsymbol{Z}_i]_{k,l}) = \sum_{j,p,q} \left(\bar{a}_{j,p,q}^{(i,k,l)} \mathrm{Re}([\boldsymbol{W}_j]_{p,q}) + \bar{b}_{j,p,q}^{(i,k,l)} \mathrm{Im}([\boldsymbol{W}_j]_{p,q}) \right) + \bar{c}^{(i,k,l)} t_i + \bar{d}^{(i,k,l)}, \quad \text{for all } k < l, \tag{7b}$$

and for some coefficients $a_{j,p,q}^{(i,k,l)}$, $b_{j,p,q}^{(i,k,l)}$, $c^{(i,k,l)}$, $d^{(i,k,l)}$, $\bar{a}_{j,p,q}^{(i,k,l)}$, $\bar{b}_{j,p,q}^{(i,k,l)}$, $\bar{c}^{(i,k,l)}$, $\bar{d}^{(i,k,l)}$. Note that there are totally $(N+1)^2$ equations in (7). Let us denote

$$\boldsymbol{F}_{k,l} = \begin{array}{c} \begin{matrix} & k & & l & \\ & \downarrow & & \downarrow & \end{matrix} \\ \left[\begin{matrix} & & & \\ & & \frac{1}{2} & \\ & & & \\ & \frac{1}{2} & & \\ & & & \end{matrix} \right] \begin{matrix} \\ \leftarrow k \\ \\ \leftarrow l \\ \\ \end{matrix} \end{array} \qquad \boldsymbol{G}_{k,l} = \begin{array}{c} \begin{matrix} & k & & l & \\ & \downarrow & & \downarrow & \end{matrix} \\ \left[\begin{matrix} & & & \\ & & -\frac{\jmath}{2} & \\ & & & \\ & \frac{\jmath}{2} & & \\ & & & \end{matrix} \right] \begin{matrix} \\ \leftarrow k \\ \\ \leftarrow l \\ \\ \end{matrix} \end{array}$$

where the empty entries are all zeros. It can be verified that given a Hermitian matrix $\boldsymbol{X}$, we have the identities

$$\mathrm{Re}(X_{k,l}) = \mathrm{Tr}(\boldsymbol{F}_{k,l}\boldsymbol{X}), \qquad \mathrm{Im}(X_{k,l}) = \mathrm{Tr}(\boldsymbol{G}_{k,l}\boldsymbol{X}).$$

Using the above identities, the equations in (7) can be re-expressed as

$$\mathrm{Tr}(\boldsymbol{F}_{k,l}\boldsymbol{Z}_i) = \sum_j \mathrm{Tr}\left(\left(\sum_{p,q} a_{j,p,q}^{(i,k,l)} \boldsymbol{F}_{k,l} + b_{j,p,q}^{(i,k,l)} \boldsymbol{G}_{k,l} \right) \boldsymbol{W}_j \right) + c^{(i,k,l)} t_i + d^{(i,k,l)}, \quad \text{for all } k \leq l, \tag{8a}$$

$$\mathrm{Tr}(\boldsymbol{G}_{k,l}\boldsymbol{Z}_i) = \sum_j \mathrm{Tr}\left(\left(\sum_{p,q} \bar{a}_{j,p,q}^{(i,k,l)} \boldsymbol{F}_{k,l} + \bar{b}_{j,p,q}^{(i,k,l)} \boldsymbol{G}_{k,l} \right) \boldsymbol{W}_j \right) + \bar{c}^{(i,k,l)} t_i + \bar{d}^{(i,k,l)}, \quad \text{for all } k < l. \tag{8b}$$

We see that every equation in (8) takes the form in (3b).

Moreover, we should note that Fact 1 can be extended to handle situations where the sizes of $\boldsymbol{X}_i$'s are unequal; i.e., $\boldsymbol{X}_i \in \mathbb{H}^{n_i}$, where $n_i > 0$ can be unequal w.r.t. i. As alluded to in the SDP rank reduction proof, e.g., that of [2,3], such an extension is almost immediate. Now, consider connecting Problem (2) and Problem (3) via setting $m = K(N+1)^2$, $\boldsymbol{X}_i = \boldsymbol{W}_i, \boldsymbol{X}_{i+K} = \boldsymbol{Z}_i, \boldsymbol{X}_{i+2K} = t_i$ for $i = 1, \ldots, K$, and $k = 3K$. We see that Problem (2) is equivalent to Problem (3). Hence, by Fact 1, we have the following rank result.

$$\sum_{i=1}^{K} \text{rank}(\boldsymbol{W}_i^\star)^2 + \sum_{i=1}^{K} \text{rank}(\boldsymbol{Z}_i^\star)^2 + \sum_{i=1}^{K} \text{rank}(t_i^\star)^2 \leq K(N+1)^2. \tag{9}$$

Step 2: We show that (9) can be reduced to (5). The idea is to prove $t_i^\star > 0$ for all i, which, when applied to (5), results in (9). The proof for $t_i^\star > 0$ is as follows. Suppose that $t_i^\star = 0$. Let us simply denote $\boldsymbol{Q}_i = \frac{1}{\gamma_i}\boldsymbol{W}_i^\star - \sum_{j \neq i} \boldsymbol{W}_j^\star$, $\boldsymbol{r}_i = \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i$, $s_i = \bar{\boldsymbol{h}}_i^H \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i - \sigma_i^2$. By applying the Schur complement to (2b), with $t_i^\star = 0$, we get $s_i - \boldsymbol{r}_i^H \boldsymbol{Q}_i^\dagger \boldsymbol{r}_i \geq 0$. On the other hand, we have

$$s_i - \boldsymbol{r}_i^H \boldsymbol{Q}_i^\dagger \boldsymbol{r}_i = \bar{\boldsymbol{h}}_i^H \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i - \sigma_i^2 - \bar{\boldsymbol{h}}_i^H \boldsymbol{Q}_i \bar{\boldsymbol{h}}_i = -\sigma_i^2 < 0.$$

Thus, by contradiction, we must not have $t_i^\star = 0$.

Step 3: We complete the proof by showing $\text{rank}(\boldsymbol{Z}_i^\star) \leq N$. Recall that the size of $\boldsymbol{Z}_i$ is $(N+1) \times (N+1)$. Suppose that $\text{rank}(\boldsymbol{Z}_j^\star) = N+1$ for some j. Then, we can show that there exists a feasible solution that yields a lower objective value than that of $\boldsymbol{\mathcal{W}}^\star$, a contradiction. To prove this, assume $j = 1$ for convenience. Let $\boldsymbol{W}_1' = (1-\alpha)\boldsymbol{W}_1^\star$ for some $0 < \alpha < 1$, $\boldsymbol{W}_i' = \boldsymbol{W}_i^\star$, $i = 2, \ldots, K$, and $\boldsymbol{t}' = \boldsymbol{t}^\star$. The corresponding $\boldsymbol{Z}_i$'s in (2b) are given by

$$\boldsymbol{Z}_i' = \begin{cases} \boldsymbol{Z}_1^\star - \frac{\alpha}{\gamma_1}\boldsymbol{B}_1, & i = 1, \\ \boldsymbol{Z}_i^\star + \alpha \boldsymbol{B}_i, & \text{otherwise}, \end{cases}$$

where

$$\boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{W}_1^\star & \boldsymbol{W}_1^\star \bar{\boldsymbol{h}}_i \\ \bar{\boldsymbol{h}}_i^H \boldsymbol{W}_1^\star & \bar{\boldsymbol{h}}_i^H \boldsymbol{W}_1^\star \bar{\boldsymbol{h}}_i \end{bmatrix} \succeq \boldsymbol{0}.$$

Since $\boldsymbol{Z}_1^\star$ has full rank, or is positive definite, there exists a sufficiently small α such that $\boldsymbol{Z}_1^\star - \frac{\alpha}{\gamma_1}\boldsymbol{B}_1 \succeq \boldsymbol{0}$ is satisfied. Also, $\boldsymbol{Z}_i^\star + \alpha \boldsymbol{B}_i \succeq \boldsymbol{0}$ holds by nature. Therefore, for a sufficiently small α, $(\boldsymbol{W}_i', \boldsymbol{Z}_i', t_i')_{i=1}^K$ is a feasible solution to Problem (2). Since $\sum_i \text{Tr}(\boldsymbol{W}_i') < \sum_i \text{Tr}(\boldsymbol{W}_i^\star)$ (for $\boldsymbol{W}_1^\star \neq \boldsymbol{0}$, which can be easily verified), we obtain the desired result.

References

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