Modeling Financial Systemic Risk — the Network Effect and the Market Liquidity Effect

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Abstract
Financial institutions are interconnected directly by holding debt claims against each other (the network channel), and they are also bound by the market liquidity in selling assets to meet debt liabilities when facing distress (the liquidity channel). The goal of our study is to investigate how these two channels of risk interact to propagate individual defaults to a system-wide catastrophe. We formulate the model as an optimization problem with equilibrium constraints and derive a partition algorithm to solve for the market-clearing equilibrium. The solutions so obtained enables us to identify two factors, the network multiplier and the liquidity amplifier, to characterize the contributions of these two channels to financial systemic risk, whereby we can acquire better understanding of the effectiveness of several policy interventions. The analysis behind the algorithm yields estimates for the contagion probability on the basis of the market value of the institutions’ net worths, underscoring the importance of equity capital as a cushion against systemic shocks in the presence of the liquidity channel. The optimization formulation also provides more structural insights to allow us to extend the study of systemic risk to a system with debts of different seniorities, and meanwhile presents a close connection to the literature of stochastic networks. Finally, we illustrate the impacts of the network and the liquidity channels—in particular, the significance of the latter—in the formation of systemic risk with data on the European banking system.

Keywords: Systemic risk, contagion, liquidity risk, financial network

1 Introduction
Financial institutions knit a complex system, through either active borrowing-and-lending activities among themselves or holding significant amount of marketable securities against each other. This system, in turn, binds the institutions tightly together to an unprecedented degree, such that failure at one or several institutions due to excessive idiosyncratic risk taking can quickly propagate through it to set off cascading disasters. While the 2007-2009 credit crisis in the US and the European sovereignty debt crisis have triggered much debate as to the causes, culprits and lessons learned, they have also brought out the need to understand
how risks evolve and propagate in a tightly coupled financial system, which constitutes a crucial step toward designing regulatory tools to measure, monitor, mitigate and manage the systemic risk before it grows up to an imminent threat to the stability of our financial system.

In this paper we develop an optimization-based approach to characterize analytically how two important channels of financial contagion interact with each other to amplify the systemic risk. The first channel is the direct debt exposures among financial institutions: one firm holds heavy liabilities against another and therefore the loss of one failure will be easily transmitted to the others. We refer this to as the network channel later in this paper. The second channel for contagion, referred to as the liquidity channel below, is that institutions are also interconnected indirectly by the market. A fire sale initiated by one distressed institution for the purpose of fund raising, in particular under difficult aggregate economic conditions, will drive down the asset price sharply. Since the institutions across the system accumulate large positions in the assets of similar nature during normal periods, such price decline creates a serious negative externality for the rest of the financial system. As noted in Brunnermeier [11], these two channels play a prominent role in the amplification mechanism contributing to the severe 2007-2009 credit crisis in the US.

1.1 Contribution of the Current Paper

To capture the interaction of these two channels, the institutions in our model are assumed to possess three classes of assets, including external projects, interbank lending, and liquid and illiquid marketable securities, financed by funds from debt holders inside and outside the banking system. We investigate the repayment equilibrium for the system when it receives income shocks on the external investments. One salient feature of the model is that the security price is determined endogenously in the equilibrium through a market-clearing mechanism, as the institutions with a shortfall are allowed to liquidate those securities to meet their liabilities. In this way, the model reflects better the price effect in developing the systemic risk than the other existing models without such channel.

We formulate the model to an optimization problem with equilibrium constrains and construct a partition algorithm to identify the default institutions in the repayment equilibrium. The key idea underpinning the algorithm is that we first identify some “obvious” defaults, then partition them from the remaining part of the system so that we can form a subsystem with a smaller number of institutions yet to be partitioned. This procedure is iterated until no new defaults are found. In each round of partition, the algorithm attempts to adjust the repayment and asset sales of each institution dynamically so as to satisfy the limited liability principle and clear the security market. On the top of the network channel that the existing literature discusses intensively, the analysis behind the algorithm underscores the role of market liquidity in triggering and accelerating a massive financial contagion. As defaults cascade over the system, large-scaled asset liquidation significantly depresses the market price of the asset, weakening the net worths of every institution to a great degree and making them more susceptible to further contagion. The method lends us computational advantage to track how the failure of one institution propagates, through both the asset price...
and direct bilateral liability exposures, down this highly interdependent system.

To concretize the impact of the price, we further use our algorithm to establish some estimates about the expected number of defaults that one particular failure will cause. This result quantifies the importance of the market value, as opposed to the book value, of the net worth of the banking system as one indispensable component of the systemic resilience against external shocks. If the assets that the institutions hold are mainly illiquid, we show that the market value of the institutions’ net worths may change very drastically when the financial contagion deteriorates. The algorithm closely tracks such value changes after each round of partition and thereby can help to provide more accurate estimates for the expected scale of defaults in the next round.

Based on the identification results yielded by the algorithm, we undertake sensitivity analysis on the obtained equilibrium to assess its dependence on the market liquidity as well as the topology of bilateral liability network. Our optimization formulation grants us methodological convenience in this regard. Applying some standard sensitivity analysis tools developed in the area of optimization, such as the duality formulation of linear programming, we manage to identify analytically network multiplier and liquidity amplifier to characterize the respective impacts of the two channels for risk transmission. They both demonstrate self-reinforcing structures that can cause highly nonlinear amplification effects. Using these two multipliers as major building blocks, we also explore the effectiveness of several intervention policies, including asset purchase and capital injection, aiming to gain more understanding about how to better utilize the roles of the two channels to stem the spillover of individual failures.

Regulators can use our method to develop counterfactual simulation schemes to test the resilience of financial systems against systemic shocks. Inclusion of the liquidity channel in the method will contribute a new aspect to such schemes, as we note that a majority of the existing simulation methodologies, such as those ones reviewed in Upper [40], accounts for only the network effect. The study of the maximal equilibrium also sheds new insights to the problem structure, allowing us to deal with a broader class of financial networks. One important extension in this regard is that our optimization-based approach can solve the repayment equilibrium in the presence of senior debts, contrast to the standard proportional repayment assumption widely used in the existing literature. Moreover, the approach taken in the paper links the study of financial systemic risk to stochastic queueing networks, which opens a possibility of applying the methodologies from the latter to facilitate the research on the former.

The numerical experiments of this paper further illustrate the importance of the liquidity channel in the systemic risk formation. We base the experiments on the data released by the European Banking Authority after its 2011 EU-wide stress test. The participant banks had strengthened their capital positions significantly in the months prior to the test under the coordinated pressure exerted by the EU regulators. As a result, the dataset shows that the interbank exposures across the major European banks have already been brought down to a very small portion of the banks’ total assets. Therefore, it is not surprising to observe from our experiment outcomes that absent the liquidity channel, the network itself cannot trigger a massive financial
contagion among the banks. However, we find that the danger of a large contagion may become imminent, if we introduce the illiquid asset and its market impact hypothetically into the dataset. In spite of the crude estimates we use to recover the European financial system from the limited information disclosed by the test dataset, the message conveyed from the experiments is unequivocal: the market liquidity plays a dominating role in the development of the systemic risk. Our outcomes concur with the recommendation of the EBA Chair Andrea Enria in his opening statement of the publication of the stress test results. In it, he warned that a bank with sizeable exposures to sovereign debts, which became highly illiquid quickly as the sovereign crisis deteriorated, still needed to take further protective measures to enhance its capital position, even though it might fall above the capital benchmark in the stress test. The method presented in the current paper helps to quantify the benefit of such enhancement.

1.2 Related Literature

The contagion effect in a financial system has been well investigated in the literature. Some early papers, such as Rochet and Tirole [37], Kiyotaki and Moore [32], Allen and Gale [3], Freixas, Parigi, and Rochet [23], and Leitner [33], study the economic origin of interbank networks. They demonstrate via some highly stylized models how the network channel, a risk sharing mechanism of a financial system under normal conditions, will help to transmit the systemic crisis when there is a global shortage for liquidity. Another line of research, represented by Shleifer and Vishny [39], Gromb and Vayanos [30], Allen and Gale [4], Glîneanu and Pedersen [28], Brummermeier and Pedersen [12], primarily focus on the contagious effect of asset price. As the price falls, financial institutions have to liquidate a large proportion of their long-term assets, due to either the needs to meet their commitments or the requirement of risk management practice. Their research indicates that such defensive liquidation triggers fire sale, generating adverse welfare consequences for the entire system such as high price volatility, more bank defaults, and market inefficiency. Both threads of research concern mainly the microeconomic foundations of financial contagion, as opposed to characterizing the impacts of interconnectedness for a given financial network via an optimization-based approach, which is the focus of the current paper.

Our paper is largely inspired by the work of Eisenberg and Noe [18]. They consider only the network effect in the formation of the systemic risk. On the basis of some fixed-point arguments, they show that in theory the original shortfall in the payment of a single institution can cascade through the interbank liability network, causing more and more to default in a domino fashion. Their research sets off intensive exploration about the relationship between the systemic risk and the network topology; see Gai and Kapadia [26], Gai, Haldane, and Kapadia [27], Haldane and May Acemoglu, Ozdaglar, and Tahbuz-Salehi [31], Acemoglu, Ozdaglar, and Tahbuz-Salehi [1], Amini, Cont, and Minca [5], and Elliott, Golub, and Jackson [19] for more details. Liu and Staum [34] perform some sensitivity analysis on the Eisenberg-Noe model to examine the systemic impact caused by small changes in the model parameters. Here we introduce an additional liquidity channel on top of the pure network effect to analyze the interaction between these two channels as the
contagion spills over the system.

Our work is consistent with some recent empirical and simulation literatures on the network stress testing, which find that it is difficult to generate significant contagion just via the interbank liabilities. Furfine [25] reports that the probability of multiple rounds of contagion due to explicit financial linkage in the US interbank market was economically insignificant during February and March 1998, a time period foreshadowed by the looming failure of Long Term Capital Management. The research of Elsinger, Lehar, and Summer [20] on the Austria banking data suggests that correlation in banks asset portfolios dominates the network effect as the main source of systemic risk. Degryse and Nguyen [16] simulate the contagion risk on a dataset about the Belgian banking system over the period 1993–2002. They do not find a high spillover effect through the interbank network, even under an unrealistic assumption that one bank will lose 100% of its interbank assets if the counterpart defaults. The estimates of Glasserman and Young [29] based on data of the European banking system also point to that the network effect has only a very limited impact. In addition, Rogers and Veraart [38] investigate the incentive issue in rescuing distressed financial institutions in the presence of the negative externality of bankrupt costs. All the above works and the related literature call for the necessity of a full investigation on other transmission channels of the systemic risk, especially the channel involving the price effect. The current paper aims to develop some analytical tools in this direction.

Several other papers in the literature share similar models as ours to study the financial contagion. Cifuentes, Ferrucci, and Shin [14] and Nier et al. [35] execute simulation exercises on some symmetric networks to examine the impacts of liquidity risk and banks interlinkage to the system stability. The major discovery from their numerical experiments is that systemic resilience demonstrates a highly nonlinear relationship to the interconnectedness of a banking system. Our work eyes more on the methodological aspect of the problem. The analytical approach developed in this paper enables us to quantify the above nonlinear effects observed in the two papers. The research of Amini, Filipovic and Minca [6] explores the risk mitigation effect of a central clearing counterparty in an interbank market. All these studies complement each other from different angles without much overlapping.

1.3 Paper Organization

The rest of the paper is organized as follows. In Section 2 we review the Eisenberg-Noe model but from a new perspective, whereby we develop our optimization-based formulation and a partition algorithm. This approach casts new insights to the problem, allowing us to extend the E-N model to incorporate more structures. Meanwhile, it also builds up the link between systemic risk and stochastic queueing networks. Section 3 is devoted to the most important extension of this paper, incorporating the liquidity effect to the study of the systemic risk. We then examine in Section 4 how to derive estimates of contagion probability based on the net worth value from the partition algorithm. Section 5 contains some numerical experiments on the data of the 2011 EU-wide stress test. All the technical proofs will be deferred to the appendix part.
2 Basic Model: A New Perspective on the Eisenberg-Noe Model

2.1 Notations

The subsequent analysis utilizes a number of standard notations from linear algebra. For ease of reference, we collect them together in this subsection. We assume in the paper that all vectors are row vectors unless otherwise specified. More specifically, any vector that pre-multiplies a matrix is, naturally, a row vector; and this is the majority case below. Only occasionally will we have cases in which a column vector post-multiplies a matrix. The inner product of two vectors, \( v \) and \( u \), will be simply written as \( vu \), with the understanding that \( v \) is a row vector and \( u \) a column vector. We use \( I \) to denote the identity matrix, \( e_i \) the \( i \)th standard basis vector for a Euclidean space, \( 1 \) and \( 0 \) vectors of all 1’s and 0’s, respectively. For a matrix \( M \) and two index sets \( I \) and \( J \), \( M_{I,J} \) represents the submatrix of \( M \) consisting of the rows and columns indexed by \( I \) and \( J \), respectively. If \( I = J \), the notation of the submatrix will be simplified further to \( M_I \) in the paper. We call \( u > v \) (\( u \geq v \)) for two vectors \( u \) and \( v \) if \( u \) is strictly larger (not less than) \( v \) component wise. Denote \( u \wedge v \) be a new vector such that \( u \wedge v = (u_1 \wedge v_1, \ldots, u_n \wedge v_n) \). In addition, we use \( P(\cdot) \) and \( E(\cdot) \) to indicate a probability measure and its related expectation.

Here’s the plan for the remaining part of this section. We start with an overview of the model of Eisenberg and Noe [18] in Section 2.2, but with a focus on using optimization formulation— as opposed to the fixed-point algorithm in their original paper — to obtain the clearing vector. In so doing, we offer a new perspective to the classical problem. Furthermore, it has the advantage over the fixed-point algorithm in dealing with multiple equilibria. We illustrate this by obtaining new results and insights when the relative liability matrix is stochastic, as well as providing new applications in dealing with debts of different seniority. Section 2.5 demonstrates that the optimization-based approach connects directly to the so-called “bottleneck analysis” in stochastic networks.

2.2 Solution via a Partition Algorithm

Consider a financial system of \( n \) banks with interconnected balance sheets: a bank holds liabilities against some banks else in the system. The interconnectedness is represented via a (nominal) liability matrix \( L := (L_{ij}) \), where \( L_{ij} \) denotes the liability of bank \( i \) to bank \( j \). Naturally, assume \( L_{ij} \geq 0 \) for \( i \neq j \) and \( L_{ii} = 0 \). In addition, every bank may also owe some amount of money to creditors outside the system. Denote \( b_i \geq 0 \) to be bank \( i \)’s external liability. From now on, we will use \( \ell := (\ell_i) \) and \( P := (p_{ij}) \) to indicate the liability vector and the relative liability matrix of this banking system, respectively, where

\[
\ell_i := b_i + \sum_{j \neq i} L_{ij} \quad \text{and} \quad p_{ij} := L_{ij}/\ell_i, \quad i, j = 1, \ldots, n. \tag{1}
\]

On the asset side of each bank, there is a vector \( \alpha := (\alpha_i) \), with \( \alpha_i \geq 0 \) representing the value of exogenous assets invested by bank \( i \).

Given a realization of \( \alpha \), our objective is to find a clearing repayment vector \( x := (x_i) \) such that it complies with the principle of limited liability. Namely, bank \( i \) needs to pay all of its liability \( \ell_i \) if it can; or,
if short of that, it should declare default and pay all of what it will receive from both external and internal sources, \( \alpha_i + \sum_{j \neq i} x_j p_{ji} \). Therefore, the repayment vector \( x \) is the solution to the following equation system:

\[
x_i = \ell_i \wedge (\alpha_i + \sum_{j \neq i} x_j p_{ji}), \quad i = 1, \ldots, n.
\] (2)

Putting it into a matrix form, we have

\[
x = \ell \wedge (\alpha + xP).
\] (3)

We assume implicitly in (2) and (3) that all the creditors of the banks are of the same seniority. When a particular bank defaults, its repayment will be proportionally distributed to its creditors, commensurate to the relative size of liabilities specified by \( P \). If we impose an additional assumption, like what Glasserman and Young [29] do, that every bank has nonzero external liability, i.e., \( b_i > 0 \) for all \( i \), the matrix \( P \) will be strictly substochastic with all the row sums strictly less than 1. The above fixed-point formulation (3) then is a contraction, implying that there exists a unique solution \( x \).

Eisenberg and Noe [18] notice that the repayment vector defined in (3) can also be obtained by the following linear program, with \(|x|\) denoting the sum of all the components of \( x \) (\( L_1 \) norm):

\[
\max_x |x| := x1, \quad \text{s.t.} \quad x(I - P) \leq \alpha, \quad 0 \leq x \leq \ell.
\] (4)

No solution to the above LP is attempted in their paper. Instead, they still use fixed-point arguments to establish the solution existence to (3) and propose a “fictitious default” algorithm based on that. Here, we intend to explore more about this optimization approach. The optimal solution to (4) points to a repayment equilibrium in which the minimum number of banks will default. Moreover, we can show that the default banks in this equilibrium will default in any other equilibria (fixed-point solutions) that satisfy (3). Therefore, the approach allows us to isolate the sources of necessary contagion in a banking system. This property is especially valuable when we discuss multiple equilibria caused by network and market liquidity in the subsequent sections.

We point out that the above optimization formulation in fact unifies both the solution existence and the “fictitious default” algorithm in Eisenberg and Noe [18], and beyond that it can accomplish more. For any optimal solution to the LP, obviously either the constraints \( x(I - P) \leq \alpha \) or \( x \leq \ell \) is binding. Let \( D = \{i : x_i < \ell_i\} \) and \( N = \{i : x_i = \ell_i\} \). Partition the \( P \) matrix accordingly such that

\[
P = \begin{pmatrix} P_D & P_{D,N} \\ P_{N,D} & P_N \end{pmatrix}.
\]

Then, the problem (4) is equivalent to finding an optimal partition of the index set \( \{1, \cdots, n\} \), \( D^* \) and \( N^* \), to solve the following optimization:

\[
\max_{x_D, D, N} |x|, \quad \text{s.t.} \quad x_D = \alpha_D + x_N P_{N,D} + x_D P_D, \quad x_N \leq \alpha_N + x_N P_N + x_D P_{D,N}, \quad x_N = \ell_N, \quad x_D < \ell_D.
\] (5)

(6)
The first constraint in (5) is an accounting identity the default banks should comply with: the total repayments they make on its left hand side equal the total incomes they receive on the right hand side. The second constraint in (5), referred to as surplus constraint below, states that the non-default banks should have sufficient funds to meet their liabilities. The second constraint in (6), referred to as shortfall constraint below, states that the amounts of repayments from the default banks should be strictly less than their nominal liabilities.

We develop below an algorithm to search for the optimal partition. Start with setting \( D = \emptyset \) and \( N = \{1, \cdots, n\} \). This is the case when every bank will have enough assets to cover its liability; hence, no bank will default. Such setting is tantamount to letting \( x = \ell \) according to the first constraint in (6). Note that this tentative repayment vector satisfies all constraints in (5-6), except for the surplus constraint in (5).

Therefore, once

\[ \ell(I - P) \leq \alpha, \]  

(7)

which is corresponding to that constraint under this initial setting, is valid, the partition of \( D = \emptyset \) and \( N = \{1, \cdots, n\} \) is feasible for the problem. In addition, we know that \( x = \ell \) is greater than any solution to (4). It hence provides us an optimal solution to (4).

If (7) is violated for a subset of banks, we should augment \( D \) by including them into it and updating \( N = \{1, 2, \cdots, n\} \backslash D \) accordingly. With this updated partition, we proceed to solve for the corresponding \( x_D \) and \( x_N \) from the accounting identity in (5) and the first constraint in (6); that is, let

\[ x_D = (\alpha_D + \ell_N P_{N,D})(I_D - P_D)^{-1} \quad \text{and} \quad x_N = \ell_N. \]  

(8)

It is straightforward to see that such defined \( x_D \) should satisfy \( x_D < \ell_D \), the shortfall constraint. Indeed, since the new \( D \) contains all the banks violating (7), meaning that \( \ell_D - \ell_D P_D - \ell_N P_{N,D} > \alpha_D \), we have

\[ \ell_D(I_D - P_D) > \ell_N P_{N,D} + \alpha_D. \]  

(9)

On the other hand, the assumption that \( P \) is a strict substochastic matrix ensures that \( I - P \) is an M-matrix. We therefore know that each of its principal submatrices, such as \( (I_D - P_D) \), should also be an M-matrix. Recall that an M-matrix is known to have a non-negative inverse; refer to Berman and Plemmons [8] for a comprehensive discussion on the properties of such matrices. Multiplying the non-negative matrix \( (I_D - P_D)^{-1} \) on both sides of (9), we can see that

\[ \ell_D > (\ell_N P_{N,D} + \alpha_D)(I_D - P_D)^{-1} = x_D. \]

In this way, we obtain from (8) an updated clearing vector \( x \), which again satisfies all constraints in (5-6) except for the surplus constraint in (5). More new defaults may be identified by checking the validity of the surplus constraint. We summarize the previous discussion into an algorithm as follows:
### Partition Algorithm

- **Step 0.** Set $\mathcal{D} = \emptyset$ and $\mathcal{N} = \{1, \ldots, n\}$.
- **Step 1.** Set $x_{\mathcal{D}}$ and $x_{\mathcal{N}}$ as in (8).
- **Step 2.** Check the feasibility of the surplus constraint in (5). If it is satisfied, the algorithm terminates. Otherwise, identify the subset of banks for which it is violated, and include them into $\mathcal{D}$. Let $\mathcal{N} = \{1, \ldots, n\} \setminus \mathcal{D}$. Repeat Step 1.

Obviously, the above algorithm terminates in at most $n$ steps. Denote $(\mathcal{D}^*, \mathcal{N}^*)$ to be the partition it ultimately yields. In the proof of the following proposition, we show that the corresponding vectors $x_{\mathcal{D}}$ and $x_{\mathcal{N}}$ obtained from (8) in each intermediate step are indeed infeasible for the problem (4). They remain greater than the optimal value of problem (4). The algorithm keeps reducing the $L_1$-norm of these infeasible solutions by identifying more and more default banks. When it terminates, the partition $(\mathcal{D}^*, \mathcal{N}^*)$ satisfies all the constraints in the problem (5-6), yielding a primal feasible solution. Combining these two facts, we establish the optimality of $x_{\mathcal{D}^*}$ and $x_{\mathcal{N}^*}$.

**Proposition 1.** The optimal clearing vector $x^*$, the solution to the LP in (4), follows (8), with $(\mathcal{D}^*, \mathcal{N}^*)$ identified by the above Partition Algorithm, which takes at most $n$ iterations.

As mentioned earlier, our approach unifies the arguments for both the solution existence and the “fictitious default” algorithm. In particular, Proposition 1 shows that the existence and structure of the repayment vector, as represented by $(x_{\mathcal{D}}, x_{\mathcal{N}})$ in (8), are consequences of the optimization formulation. The partition algorithm is essentially the same as the fictitious default algorithm presented in Eisenberg and Noe [18]. Nevertheless, we can see more clearly through the optimization-based arguments that the gist of the algorithm is to enforce primal feasibility iteratively to generate the optimal partition.

In addition, the linear-programming nature of (4) offers more structural insights to the problem. Note that the constraint $x \geq 0$ is superfluous for a positive $\alpha$, as the objective function of maximizing $|x|$ will never allow any negative $x_i$. Therefore, we omit this constraint in formulating its duality. Let $\delta := (\delta_i)$ and $\eta := (\eta_i)$ be the dual variables corresponding to the two constraints in (4), $x(I - P) \leq \alpha$ and $x \leq \ell$, respectively. We have

\[
\min_{\delta, \eta} \alpha \delta + \ell \eta \quad \text{s.t.} \quad (I - P)\delta + \eta \geq 1, \quad \delta \geq 0, \quad \eta \geq 0.
\]

The dual solution should be

\[
\delta_{\mathcal{D}^*} = (I_{\mathcal{D}} - P_{\mathcal{D}^*})^{-1}1_{\mathcal{D}^*}, \quad \delta_{\mathcal{N}^*} = 0_{\mathcal{N}^*}; \quad \eta_{\mathcal{D}^*} = 0_{\mathcal{D}^*}, \quad \eta_{\mathcal{N}^*} = 1_{\mathcal{N}^*} + P_{\mathcal{N}^*} \cdot (I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}1_{\mathcal{D}^*}.
\]
In light of the “shadow-price” interpretation of dual variables, we can immediately conclude that $e_i(I_D - P_D)^{-1}$ and $e_j P_{N,D}(I_D - P_D)^{-1}$ are the sensitivity of the repayment vector $x$ with respect to $\alpha_i$ and $\ell_j$ for $i \in D^*$ and $j \in N^*$. The matrix

$$(I_{D^*} - P_{D^*})^{-1} = I_{D^*} + P_{D^*} + P_{D^*}^2 + \cdots$$  \hspace{1cm} (11)$$

is referred to as the network multiplier in this paper for obvious reasons: it captures the network effect on the payment vector $x_{D^*}$. To see this, consider a scenario that the exogenous asset value of bank $i \in D^*$ is worth one dollar less. This bank will reduce its repayment to the other banks by $1$. In sequence, its immediate creditors in the subgroup $D^*$ will receive less a fraction of this dollar, captured by $e_i P_{D^*}$, and reduce their repayments accordingly. Such reduction then feeds back again into the repayments of yet more banks: the creditors of the 1-round creditors of bank $i$ will receive a fraction of this $e_i P_{D^*}$ less, namely $e_i P_{D^*}^2$, and so forth. The aggregate effect on the repayments of the banks in $D^*$ generated by the initial reduction of a single dollar on bank $i$ then should be given by the RHS of (11). While similar sensitivity results have been derived in Liu and Staum [34], we will demonstrate in Section 3.2 that the partition-based approach extends to a more general model that accounts for liquidity, and thereby yields sensitivity results in that general setting as well.

To obtain $x_D$ in (8) in each step of the algorithm, we have to invert $I_D - P_D$. When the cardinality of $D$ is large and $P_D$ is sparse, two typical attributes of a banking system data, a more computationally efficient way to solve the accounting identity (cf. the first constraint in (5)) is through an iteration method. Let

$$F(y) := \alpha_D + \ell_N P_{N,D} + y P_D.$$ 

We generate a sequence of vectors $y_i$ via $F$: let $y^0 = \ell_D$ and $y^i = F(y^{i-1})$ for all $i \geq 1$. The inequality (9) implies that $y^1 < \ell_D = y^0$; and hence the sequence of $\{y^i : i \geq 1\}$ must be decreasing and converge to the solution $x_D$. The above iterative method involves only the operation of matrix multiplication in which a sparse matrix $P_D$ will grant us significant computational advantage. The chapter 7 of Berman and Plemmons [8] contains a more comprehensive investigation on the iterative methods in solving linear systems.

### 2.3 Case of a General $P$

In Section 2.2, we have assumed that $P$ is a strictly substochastic matrix. It allows us to claim that $I - P$ and all of its principal submatrices are invertible. Here we relax this assumption, considering the cases in which some row sums of $P$ equal 1; and, hence, $I - P$ may not be invertible. Such $P$ corresponds to a system in which not every bank has external liabilities, i.e., $b_i = 0$ for some $i$. Eisenberg and Noe [18] discuss a special case that $b \equiv 0$. Another example is a banking system with the presence of senior debts, which we will discuss in the next subsection. We illustrate how our approach reveals more about the problem structure than the fixed-point based arguments in the case of $P$ being not strictly substochastic.

When we drop the strictly substochastic assumption about $P$, multiple equilibria may arise under the fixed-point formulation (3), as shown by the following simple example.
Example 2. Suppose that a banking system consists of three banks \{1, 2, 3\}. All of them have neither exogenous assets nor external liabilities, i.e., \((a_1, a_2, a_3) = (b_1, b_2, b_3) = 0\). In addition, assume that their total nominal liabilities are all 1 and the relative liability matrix of the system is given by

\[
P = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Then, a continuum of equilibria satisfy the fixed-point formulation (3). In particular, \(x = (1, 1, 1)\) is a solution, so is \(\theta x\) for any \(\theta \in (0, 1)\). The former one is corresponding to a “good” equilibrium since no bank will default under it. We refer the other ones to as “bad” equilibria because everyone defaults in them. □

The network interdependence in this example causes a self-fulfilling phenomenon. If every bank believes that the equilibrium the system finally reaches will be the good one, paying the full nominal liability correspondingly, then the good clearing vector \(x = (1, 1, 1)\) is fulfilled. On the contrary, the expectation that a bad equilibrium will be reached will cause every bank to pay partially, and consequently everyone defaults in the realized equilibrium.

Example 2, in essence, is the counter-example constructed in Appendix 2 of Eisenberg and Noe [18]. When \(P\) is stochastic, for instance, when \(P\) is the transition matrix of an irreducible Markov chain, there exists a positive (pre-normalized) stationary distribution \(x\) for the matrix \(P\) such that \(xP = x\). Multiplying \(x\) by a positive constant \(\theta\) will give another solution to (3). Therefore, the non-uniqueness is a problem the fixed-point algorithm must confront when we consider a banking system with a stochastic \(P\). In contrast, our optimization-based approach focuses on the maximal clearing vector \(x\). The partition algorithm outputs \(D^* = \emptyset\) and \(x = (1, 1, 1)\) in the above example, which indicates that the failure of the entire banking system should be avoidable, as long as we can change the market expectation in a way to break the self-fulfilling cycle. Moreover, the maximal equilibrium admits more interesting structures than the others, as summarized in the following proposition.

Proposition 3. Suppose that \(P\) is a substochastic matrix such that \(\sum_j p_{ij} \leq 1\) for all \(i\). Let \(D^*\) be the set of the default banks for a solution to LP (4). We have:

(i) If \(P\) is stochastic, i.e., \(\sum_j p_{ij} = 1\) for all \(i\), it is impossible for all banks to default in the largest equilibrium.

(ii) The spectral radius of \(P_{D^*}\) is strictly less than 1. Hence, it cannot be a stochastic matrix, i.e., some of row sums of \(P_{D^*}\) must be strictly less than 1.

Remark 1. Part (i) of Proposition 3 does not hold if the relative liability matrix \(P\) is strictly substochastic as assumed in Section 2.2. Consider a case that all components of \(\ell\) are sufficiently large such that \(\alpha(I-P)^{-1} < \ell\). Then,

\[
x(I - P) \leq \alpha \quad \Rightarrow \quad x \leq \alpha(I - P)^{-1} < \ell.
\]

(Recall, \(I - P\) is an M-matrix; hence, its inverse is a non-negative matrix.) Hence, \(x = \alpha(I - P)^{-1}\) is the solution to the LP in (4), and \(x < \ell\). That implies, all banks will default, i.e., \(D = \{1, \ldots, n\}\), in this case. □
Observe that all of the banks do not default in the maximal equilibrium in Example 2. Part (i) of the above proposition points out that this observation is by no means an coincidence, extending it to any cases that $P$ is stochastic. One important implication of Part (ii) of Proposition 3 is that we still can use the Partition Algorithm in the last section to search for optimal partition even though $I - P$ is not invertible. From the discussion leading to the algorithm and the proof of Proposition 1, we know that the default set $D$ identified in each intermediate step must be a subset of $D^*$. Part (ii) of Proposition 3 ensures the invertibility of $I_{D^*} - P_{D^*}$, which implies that $I_{D} - P_{D}$, as a principal submatrix of an invertible matrix, must be invertible too. Therefore, we still can use (8) to compute $x_D$ although $P$ is not strictly substochastic.

In Eisenberg and Noe [18], to overcome the non-uniqueness issue, its Lemma 1 gives $|\alpha_D| > 0$ as a (sufficient) condition to rule out the possibility for $P_D$ to be stochastic ($D$ is called a “surplus set” in that paper). Part (ii) of the proposition enhances the lemma by removing its “surplus set” condition. The result further illustrates the advantage of our approach in working with maximizing an objective function.

### 2.4 Senior Debts

Recall in Section 2.2 and 2.3 a proportional repayment mechanism applies for the debt holders both inside and outside the banking system in the case of a default. However, outside debt claims, typically held by depositors and other senior-debt holders, will have priority over the interbank claims when a bank files for bankruptcy. Therefore, we discuss in this subsection how to extend the optimization-based approach and its related partition algorithm to derive the clearing repayment vector in the presence of senior debts.

We assume that all interbank claims are with the same seniority for simplicity. In other words, the proportional payment mechanism should only apply to the banks within the network after the external liabilities are paid off. Accordingly the determination of the clearing repayment vector in the presence of senior debts should be modified to

$$x_i = \ell_i \land (\alpha_i + \sum_{j \neq i} (x_j - b_j)^+ p_{ji}), \quad i = 1, \ldots, n, \quad (12)$$

where $x^+ := \max\{x, 0\}$. That is, any bank $j$ will first deduct $b_j$ from its (total) payment $x_j$ to repay the senior debt holders, and any leftover will be paid proportionally to the other banks in the network. Meanwhile, we should change the the definition of $p_{ji}$ to $p_{ji} = L_{ij}/\sum_j L_{ij}$ for all $i$ and $j$. In this sense, the relative liability matrix $P$ is stochastic because its row sums now become equal to 1.

With a transformation of variables, we turn the above fixed-point formulation to an optimization problem, similar as the LP in (4). Let $\tilde{x}_j := (x_j - b_j)^+$, subtract $b_i$ and then take $(\cdot)^+$ on both sides of (12). We have

$$\tilde{x}_i = \left[ (\ell_i - b_i) \land (\alpha_i - b_i + \sum_{j \neq i} \tilde{x}_j p_{ji}) \right]^+ = (\ell_i - b_i) \land (\alpha_i - b_i + \sum_{j \neq i} \tilde{x}_j p_{ji})^+, \quad \ell_i = \ell - b$$

where the second equality takes into account $\ell_i - b_i = \sum_j L_{ij} \geq 0$. Denote $\tilde{\alpha}_i = \alpha_i - b_i$ and $\tilde{\ell}_i = \ell_i - b$ for all $i$. Then, to find the maximal clearing vector satisfying (12), we need to solve

$$\max_{\tilde{x}} |\tilde{x}|, \quad \text{s.t.} \quad \tilde{x} \leq (\tilde{\alpha} + \tilde{x} P)^+, \quad 0 \leq \tilde{x} \leq \tilde{\ell}. \quad (13)$$
Senior debts are often secured by collaterals. Therefore, the vector $\alpha$ in this setting should also include the value of the assets that are used as collaterals to secure the senior debts. The above variable transformation reflects the repayment priority among the creditors of the banking system. The senior debt holders get paid ahead of the interbank claims. The remaining value after such payments, $\tilde{\alpha}$, will be distributed to repay the other banking creditors. It is possible that $\tilde{\alpha}_i < 0$ for some $i$, meaning that the banks do not possess sufficient funds to honor full repayments even to their senior debt holders. In that case, the repayment $\tilde{x}_i$ from bank $i$ to its junior debt holders should be zero. Hence the constraint $\tilde{x} \geq 0$ will become an essential constraint in the presence of senior debts. In contrast, recall that we omit the non-negativity constraint $x \geq 0$ in the original LP (4) because it never binds for the optimal solution when $\alpha$ is positive.

The Partition Algorithm, with some minor modifications, remains valid to solve (13). Still let $\mathcal{N} = \{i : \tilde{x}_i = \tilde{\ell}_i\}$ and $\mathcal{D} = \{i : 0 \leq \tilde{x}_i < \tilde{\ell}_i\}$. It is apparent to see that solving (13) amounts to solving

$$\max_{\tilde{\alpha}, \mathcal{N}} |\tilde{x}|, \quad \text{s.t.} \quad \tilde{x}_D = (\tilde{\alpha}_D + \tilde{\ell}_N P_{N,D} + \tilde{x}_D P_D)^+, \quad \tilde{x}_N = \tilde{\alpha}_N + \tilde{x}_N P_N + \tilde{x}_D P_{D,N}$$

(14)

$$\tilde{x}_N = \tilde{\ell}_N, \quad \tilde{x}_D < \tilde{\ell}_D.$$  

(15)

The problem (14-15) clearly inherits a similar structure as the problem (5-6). A minor change is on the accounting identity for the bank group $\mathcal{D}$ as a result of the repayment priority. Following what we did in the previous algorithm, we try a partition of $\mathcal{N} = \{1, 2, \ldots, n\}$ and $\mathcal{D} = \emptyset$ initially. From the first constraint in (15), the corresponding tentative solution is $\tilde{x} = \tilde{\ell}$, greater than any clearing repayment vector that solves (13). Therefore, it will be an optimal solution to the problem (14-15) if it also satisfies the surplus constraint (cf. the second one in (14)), or specifically under this tentative partition,

$$\tilde{\ell}(I - P) \leq \tilde{\alpha}.$$  

(16)

Note that (16) plays an identical role as (7) in the previous algorithm. If some banks violate (16), we enforce the primal feasibility in a similar way by augmenting the default set $\mathcal{D}$ to include all the violations. We thereby reach a new partition if we update $\mathcal{N} \leftarrow \{1, 2, \ldots, n\} \setminus \mathcal{D}$.

To determine $\tilde{x}_D$ corresponding to this new partition from the accounting identity, we rely on the iterative method established in Section 2.2. Let

$$G(y) := (\tilde{\alpha}_D + \tilde{\ell}_N P_{N,D} + y P_D)^+.$$  

Start with $y^0 = \tilde{\ell}_D$ and generate a sequence of vectors iteratively by setting $y^i = G(y^{i-1})$ for all $i \geq 1$. The sequence $\{y^i : i \geq 1\}$ has several properties akin to the sequence we generated from the function $F(\cdot)$. First, according to the updating mechanism of $\mathcal{D}$, it contains all the banks violating (16), i.e.,

$$\tilde{\ell}_D > \tilde{\alpha}_D + \tilde{\ell}_N P_{N,D} + \tilde{\ell}_D P_D.$$  

In other words, $y^0 > G(y^0) = y^1$. In conjunction with the monotonicity of $G(\cdot)$ in $y$, we can easily see that the sequence of $\{y^i : i \geq 0\}$ must be decreasing. The limit $\lim_{i \to +\infty} y^i$ exists and converges to a
solution to the accounting identity, as all $y^i$ are bounded below by 0. Second, such obtained limit should be strictly less than $\tilde{\ell}_D$ component-wise. This is because $\lim_{i \to +\infty} y^i \leq y^i < y^0 = \tilde{\ell}_D$. Consequently, the shortfall constraint $\tilde{x}_D < \tilde{\ell}_D$ is satisfied automatically for such obtained $\tilde{x}_D$.

Once an updated repayment vector $\tilde{x} = (\tilde{x}_D, \tilde{x}_N)$ is solved, what remains is to check the feasibility of the surplus constraint in (14) again so as to identify more new defaults. The following algorithm summarizes the entire procedure. It is identical as the previous one except only for a minor difference in Step 1, how to determine $\tilde{x}_D$ by incorporating the presence of senior debts.

<table>
<thead>
<tr>
<th>A Partition Algorithm in the Presence of Senior Debts</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Step 0. Set $D = \emptyset$ and $N = {1, \ldots, n}$.</td>
</tr>
<tr>
<td>• Step 1. Set $\tilde{x}_N = \tilde{\ell}_N$ and solve $\tilde{x}_D$ from the accounting identity, the first constraint in (14).</td>
</tr>
<tr>
<td>• Step 2. Check the feasibility of the second constraint in (14). If it is satisfied, the algorithm terminates. Otherwise, identify the subset of banks for which it is violated and include them into $D$. Let $N = {1, \ldots, n} \setminus D$ and repeat Step 1.</td>
</tr>
</tbody>
</table>

We establish the optimality of the above algorithm in Proposition 4.

**Proposition 4.** The algorithm will terminate after at most $n$ augmentations of $D$. The partition $(D^*, N^*)$ identified by it and the corresponding repayment vector $\tilde{x}^*$ constitute an optimal solution to (13).

### 2.5 Connections to Stochastic Network

In this section we map the financial system detailed above to a stochastic network of $n$ service or processing nodes, pointing out the connection of the systemic risk modelling to stochastic network. In the literature of queueing networks, each node $i$ has a nominal service rate (or, capacity) $\ell_i$ (usually denoted $\mu_i$), and $\alpha_i$ is the outside arrival rate to node $i$. The matrix $P$ specifies the routing mechanism: after being served at node $i$, a job will transit to node $j$ with probability $p_{ij}$. The first-order business in stochastic network is to solve for the effective arrival rate to node $i$ (usually denoted $\lambda_i$), by taking into account both external arrivals and internal transitions. Mathematically, that task reduces down to solving the “traffic equation” $x = \alpha + xP$ for $x$, and then taking $x \land \ell$ as the solution. If for any node $i$ the effective arrival rate exceeds the service rate ($\lambda_i \geq \ell_i$), then node $i$ is called a bottleneck, with throughput $x_i = \ell_i$; otherwise, it is a non-bottleneck node, with $x_i = \lambda_i < \ell_i$. Intuitively, a bottleneck node is one that does not have enough capacity to process all the work that is pumped into it, i.e., its effective arrival rate exceeds its service rate; hence, its throughput is limited to its service rate. A non-bottleneck node has an effective arrival rate less than its service rate; hence, its throughput is just the effective arrival rate.
Observe that the above traffic equation is analogous to (2), if we view the capacity and the outside arrival rate as analogies to the nominal liabilities and external asset values, respectively. This observation establishes a correspondence between the financial system in the last subsections and the aforementioned service stochastic network. The clearing repayment of bank \( i \) in the former corresponds to the throughput at node \( i \), the rate of jobs going through (i.e., being processed by) node \( i \), in the latter. The total asset values of the non-default banks exceed their total liabilities, hence their clearing repayments are just the liabilities. In this sense, they correspond to the bottleneck nodes. On the other hand, a bottleneck node maps to a default bank, whose asset value falls short of its total liability. Consequently, its clearing repayment is limited to the total asset value it possesses.

A potentially more profound implication of this connection is that we find it is possible to apply the methodologies developed in the literature of stochastic network to facilitate the research on the systemic risk. For instance, in Chapter 7.3 of Chen and Yao [13], there are analyses and results concerning the existence and uniqueness of the solution to the above traffic equations in (2). They present an algorithm that partitions the \( n \) nodes in the network into bottlenecks and non-bottlenecks. That algorithm is essentially in the same spirit as the Partition Algorithm detailed in the current paper that solves the LP in (4) and separates the banks into the default and non-default sets \( D \) and \( N \).

Needless to stress, however, stochastic networks and financial systems are two subjects of distinct nature. Some important aspects of the financial systemic risk, such as the impact of the market liquidity that we will discuss in the next section, are new to the known literature of queueing systems. They pose intriguing challenges and call for a thorough investigation.

3 General Model: Interconnectedness via the Market Liquidity

Our next step is to build a even more important effect into our network model, the market effect. To do so, we will use a class of “illiquid” assets as a proxy to capture the interconnectedness of the banks through the market. For ease of exposition, we assume that the proportional rule applies for the debt repayment, i.e., the relative liability matrix \( P \) is a strictly substochastic matrix below. One can easily extend the discussion to the cases of \( P \) being stochastic, using the ideas in Sections 2.3 and 2.4.

3.1 Deriving the Market-Clearing Vector

In the previous section, we implicitly assume that the external asset values of the banks will not be affected during the course of the systemic risk development. A substantial body of literature, some representatives of which are reviewed in the introduction, point out that asset prices act in concert with credit interconnection to propagate financial distress across the market. The fire-sale like liquidation of a large distressed institution will significantly depress the market price of the asset it holds, generating a knock-on effect via the price channel to other institutions with similar asset holdings.

To incorporate the market effect, we suppose that each bank \( i \) invests in three categories of assets:
external projects such as loans to households and non-financial sectors, marketable securities, and interbank debts. A detailed breakdown is illustrated in Table 1. The most prominent change we make, compared with the previous E-N model, is that we isolate out the part of the bank’s assets whose value will be considerably influenced by the market condition and collect them under a category of illiquid securities. As shown in Table 1, bank $i$ initially owns amounts $\bar{y}_i$ and $\bar{s}_i$ of liquid and illiquid securities, respectively, for all $1 \leq i \leq n$. When a bank has a shortfall in repayment, it is allowed under this extended model to liquidate part of its security holdings to meet the liabilities. We assume that the liquid securities can be converted into cash at its face value, whereas the corresponding proceeds of illiquid asset liquidation will be $s_i q$, should bank $i$ decide to sell a face value $s_i \in [0, \bar{s}_i]$ of its illiquid holding. Here $q$ is the asset market price determined by the following inverse demand function:

$$q = Q \left( \sum_{j=1}^{n} s_j \right), \quad (17)$$

where $\sum_j s_j$ is the aggregate liquidation amount in the market from the entire banking system.

Our intention is to use the function $Q$ to capture the illiquidity effect. Assume that

**Assumption 5.** (i) $Q(0) = 1$; (ii) $Q(s) \geq 0$ and $Q(s)$ is decreasing in $s$ for all $s \geq 0$.

These two conditions clearly indicate that the asset price suffers from the “discount for immediacy”: the asset market price will drop significantly below its face value, which is normalized at $\$1$ in the model, if too much supply is present in the market concurrently. For differentiable $Q$, we further denote

$$\gamma = -\frac{Q'(s)}{Q(s)},$$

measuring the relative price change at the supply level of $s$ in response to the increment of liquidation amount. A large value of $\gamma$ means a high degree of illiquidity in the sense that even a moderate selling wave will have a severe effect in driving down the asset price. One example used in the numerical experiments for $Q(\cdot)$ is to take $\gamma$ to be a positive constant, i.e., $Q(s) = \exp(-\gamma s)$.

**Remark 2.** Assumption 5 is very general, providing convenience when people attempt to calibrate the real market data to a variety of demand models. For instance, Elliot, Golub, and Jackson [19] use an indicator demand function, $Q(s) = 1_{\{s<a\}}$ for some constant $a$, to capture the price impact. The setting indicates that the market can only provide a limited liquidity to absorb the sales of illiquid asset at its face value. The price will drop to 0 if the total supply exceeds a level. This function obviously satisfies Assumption 5.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities and owner’s equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>External investments: $\beta_i$</td>
<td>External debt claim: $b_i$</td>
</tr>
<tr>
<td>Interbank Loans: $L_{ik}$ for $k \neq i$</td>
<td>Interbank liabilities: $L_{ji}$ for $j \neq i$</td>
</tr>
<tr>
<td>Liquid securities: $\bar{y}_i$</td>
<td>Equity: $e_i$</td>
</tr>
<tr>
<td>Illiquid securities: $\bar{s}_i$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The Balance Sheet of a Representative Bank in the Financial System
Amini, Filipovic, and Minca [6] assume a stronger condition that $sQ(s)$ is increasing. Working with this stronger assumption, they show the uniqueness of the market equilibrium. As it becomes evident later, relaxing it will lead to multiple equilibria, which reveals that limited liquidity is another important source of self-fulfilling equilibrium. With the help of the proposed optimization-based approach, we can work on identifying necessary financial contagion even with the presence of equilibrium multiplicity. This further attests to the advantage of our approach.

Given the values of the external projects of the banking system $\beta$, we allow the banks to liquidate assets to raise funds for the purpose of meeting their debt obligation. If bank $i$ sells amounts $y_i$ and $s_i$ of liquid and illiquid securities, then the total available cash received by the bank is

$$\beta_i + y_i + \sum_{j \neq i} x_{ij}p_{ji} + s_iq.$$ 

The limited-liability principle requires that the bank pays either its nominal liabilities $\ell_i$ if it is solvent, or its total available funds if it defaults. That is,

$$x_i = \ell_i \wedge (\beta_i + y_i + \sum_{j \neq i} x_{ij}p_{ji} + s_iq).$$ 

(18)

We assume that the liquidation of liquid assets precedes illiquid ones. In particular, let

$$y_i = \bar{y}_i \wedge [\ell_i - (\beta_i + \sum_{j \neq i} x_{ij}p_{ji})]^+$$ 

(19)

and

$$s_i = \bar{s}_i \wedge \left\{ \frac{[\ell_i - (\beta_i + \sum_{j \neq i} x_{ij}p_{ji}) - y_i]^+}{q} \right\}.$$ 

(20)

In words, the banks attempt to sell liquid assets first to make up the shortfall between the bank’s due liability $\ell_i$ and its incomes received from the external investments $\beta_i$ and the other banks’ repayments $\sum_{j \neq i} x_{ij}p_{ji}$. Suppose that no short sales are allowed in the model. Under this assumption, the liquidation of liquid assets is capped by its initial holding, as shown in (19), and then liquidation of illiquid assets follows until the bank exhausts all the security holdings. A noticeable feature of (20) is that the sales amount of illiquid security relies on its market price, which is determined by the aggregate supply of the security through the market clearing condition (17). To summarize the previous discussion, we define

**Definition 6.** A quadruple $(x,y,s,q)$ is called a market-clearing repayment equilibrium if it satisfies the market clearing condition (17), the limited liability condition (18), and the asset sale equations (19) and (20), for $i = 1, \ldots, n$.

The following example illustrates that the illiquidity can cause multiple equilibria as the network effect.

**Example 7.** Consider two banks 1 and 2 whose balance sheets are given in Table 7. Both of them are holding 1 and 2 dollars (face value) of illiquid securities, i.e., $\bar{s}_1 = 1$ and $\bar{s}_2 = 2$. Their business are partially financed
by some borrowings from the creditors outside of the system with the nominal values being $b_1 = b_2 = 1$. Note that these two banks are interlinked only through the market channel. We assume that the inverse demand function in this two-bank market is given by $Q(s) = \exp(-s)$.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Bank 1} & \text{External investments } \beta_1 & \text{External debt } b_1 & \text{Equity } e_1 \\
\hline
\text{Iliquid Asset } s_1 & & & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|}
\hline
\text{Bank 2} & \text{External investments } \beta_2 & \text{External debt } b_2 & \text{Equity: } e_2 \\
\hline
\text{Iliquid Asset } s_2 & & & \\
\hline
\end{array}
\]

Table 2: Balance Sheets of the Two-Bank System in Example 7.

Suppose that $\beta_1 = 0.1$ and $\beta_2 = 0.9$. Apparently, both banks have to liquidate part (or all) of the assets to raise cash to pay off the debts in the equilibrium, i.e., $s_1, s_2 > 0$. According to Definition 6, we need to solve the following equations to seek for the repayment equilibria:

\[
s_1 = 1 \wedge \frac{0.9}{q}, \quad s_2 = 2 \wedge \frac{0.1}{q}, \quad \text{and} \quad q = \exp(-(s_1 + s_2)),
\]

where $0.9$ and $0.1$ are the respective shortfalls for both banks. No interior solution exists to (21). In fact, suppose that it is not the case and we have $s_1 < 1$ and $s_2 < 2$ satisfying the above equations. That implies

\[
s_1 q = 0.9, \quad s_2 q = 0.1 \quad \Rightarrow \quad (s_1 + s_2)q = (s_1 + s_2) \exp(-(s_1 + s_2)) = 1,
\]

leading to a contradiction because

\[
\max_{0 \leq s_1 \leq 1, 0 \leq s_2 \leq 2} (s_1 + s_2) \exp(-(s_1 + s_2)) = \exp(-1) < 1.
\]

Therefore, either $s_1 = 1$ or $s_2 = 2$ must hold in any equilibrium. At least two possibilities will then arise: $(s_1, s_2) = (1, 0.4092)$ and $(s_1, s_2) = (1, 2)$. In the first equilibrium, $q = 0.2443, x_1 = 0.3443 < 1$ and $x_2 = 1$, meaning that bank 1 defaults but bank 2 does not. However, neither banks will survive in the second equilibrium because $q = 0.0498, x_1 = 0.1498 < 1$ and $x_2 = 0.9996 < 1$ under it. □

Example 7 illustrates that limited liquidity constitutes another important source of equilibrium multiplicity. The total shortfall of the two banks in the example amounts to $1$, exceeding the maximal liquidity the market can provide, as shown by (22). In this situation, different liquidation order will lead to different equilibria. Suppose that we force bank 1 to be liquidated first, i.e., letting $s_1 = 1$ in (21). This transforms the equations to solving $s_2 \exp(-(1 + s_2)) = 0.1$. Clearly, $s_2 = 0.4092$ is the solution in $[0, 2]$ to the above equation, which leads to the first equilibrium. On the other hand, if we specify that the default and liquidation of bank 2 occur first, we cannot find any $s_1 \in [0, 1]$ satisfying $s_1 \exp(-(s_1 + 2)) = 0.9$. The banking system will end up at the second equilibrium.

Observe that the failure of bank 1, the one with the largest shortfall in the example, is necessary, because it will default no matter which equilibrium is ultimately realized. In contrast, the default of bank 2, together with the worst social welfare outcome such as an ultra-low market-clearing price for the illiquid securities in the second equilibrium, can be avoided, if we handle the repayment order properly as indicated by the
discussion in the last paragraph. That motivates us to use the optimization formulation again to study the maximal equilibrium in the presence of the price effect. To this end, consider the following problem to find an equilibrium with the greatest repayment vector:

$$\max_{x,y,s,q} |x| \quad \text{s.t.} \quad x = \ell \land (\beta + y + sq + xP), \quad y = \bar{y} \land d^1, \quad s = \bar{s} \land (d^2/q), \quad q = Q(|s|),$$

(23)

where we rewrite the four constraints (17-20) in their own matrix forms in order to simplify the notational burden. Here $d^i = (d^i_1, \ldots, d^n_{i})$, $j = 1, 2$, record the shortfalls of individual banks before and after they sell their liquid holdings to meet the liabilities, i.e.,

$$d^i_1 = [\ell_i - (\beta_i + \sum_{j \neq i} x_jp_{ji})]^+ \quad \text{and} \quad d^i_2 = [\ell_i - (\beta_i + \sum_{j \neq i} x_jp_{ji}) - y_i]^+, \quad \forall i = 1, 2, \ldots, n.$$"
tentative partition. That system consists of the accounting identity regarding \( x_D \) in (24) and the market clearing constraint (27). Namely,

\[
\begin{align*}
x_D &= \beta_D + \bar{y}_D + \bar{s}_D q + x_D P_D + \ell_N P_{N,D}; \quad (28) \\
y_N &= \bar{y}_N \land d_N = \bar{y}_N \land [\ell_N - (\beta_N + \ell_N P_N + x_D P_{D,N})^+]; \quad (29) \\
s_N &= \bar{s}_N \land (d_N^2/q) = \bar{s}_N \land \left(\frac{[\ell_N - (\beta_N + \ell_N P_N + x_D P_{D,N}) - y_N]^+}{q}\right); \quad (30) \\
q &= Q(|\bar{s}_D| + |s_N|). \quad (31)
\end{align*}
\]

The equation system (28-31) plays a similar role, but in an extended form, as the accounting identities in the previous two partition algorithm (cf. the first constraints in (5) and (14)). To see this, note that Eq. (28) implies that

\[
x_D = (\beta_D + \bar{y}_D + \bar{s}_D q + \ell_N P_{N,D})(I_D - P_D)^{-1},
\]

which is identical as the expression of \( x_D \) in Section 2.2, except that \( \bar{y}_D + \bar{s}_D q \) counts for an additional income for the banks generated by asset liquidation.

We still use an iteration-based method to solve for its solution. Following the idea behind introducing the functions \( F(\cdot) \) and \( G(\cdot) \) in the last section, define \( H(\cdot) \) to be a mapping from the space \( \mathcal{R} := \prod_{i \in D}[0, \ell_i] \otimes \prod_{i \in N}[0, \bar{y}_i] \otimes \prod_{i \in N}[0, \bar{s}_i] \otimes [0, 1] \) to itself, such that for any \( u \in \prod_{i \in D}[0, \ell_i], v \in \prod_{i \in N}[0, \bar{y}_i], r \in \prod_{i \in N}[0, \bar{s}_i], \) and \( t \in [0, 1], \) we have \((x, y, s, q) := H(u, v, r, t), \) in which

\[
x = (\beta_D + \bar{y}_D + \bar{s}_D t + u P_D + \ell_N P_{N,D}) \land \ell_D, \quad y = \bar{y}_N \land [\ell_N - (\beta_N + \ell_N P_N + u P_{D,N})]^+
\]

and

\[
s = \bar{s}_N \land \left(\frac{\ell_N - (\beta_N + \ell_N P_N + u P_{D,N}) - v}{t}\right)^+, \quad q = Q(|\bar{s}_D| + |r|).
\]

Start with \((u^0, v^0, r^0, t^0) = (\ell_D, 0_N, 0_N, 1)\) and repeatedly apply \( H. \) In this way, we can generate a sequence of vectors \( \{(u^i, v^i, r^i, t^i) : i \geq 1\}, \) which will converge to the maximal solution to (28-31); see the supplementary document for more details about the properties of \( H \) and how to establish the convergence result.

Once \((x_D, y_N, s_N, q)\) is found via the above iteration, we reach a tentative equilibrium \((x, y, s, q)\) corresponding to the attempted partition. This quadruple can be shown to satisfy all the constraints of the problem (23-26) except the surplus constraint about \( x_N \) in (24). As what we have discussed in the last two algorithms, we force the feasibility by augmenting \( D \) to collect all the violating banks.

The following algorithm is indeed a modification of the ones in Section 2 to encapsulate the above procedure to identify the optimal partition:
A Partition Algorithm in the Presence of Asset Liquidation

- Step 0. Set $D = \emptyset$ and $N = \{1, \ldots, n\}$.

- Step 1. Set $x_N = \ell_N$, $y_D = \bar{y}_D$, and $s_D = \bar{s}_D$. Solve $(x_D, y_N, s_N, q)$ from the equation system (28-31). Let $x = (x_N, x_D)$, $y = (y_N, y_D)$, and $s = (s_N, s_D)$.

- Step 2. Check the feasibility of the second constraint in (24) under $(x, y, s, q)$. If it is satisfied, the algorithm terminates. Otherwise, identify the subset of banks for which it is violated and include them into $D$. Let $N = \{1, \ldots, n\} \setminus D$ and repeat Step 1.

**Theorem 8.** The above algorithm terminates in at most $n$ iterations of Step 1. When it stops, it yields an optimal partition $(D^*, N^*)$ for the problem (24-27).

### 3.2 Liquidity Amplifier

With the optimal partition $(D^*, N^*)$ and its related market-clearing equilibrium $(x^*, y^*, s^*, q^*)$ obtained from Theorem 8, we proceed in this subsection to characterize the sensitivity of $x^*$ and $q^*$ with respect to small changes in the model parameters. (Here “small” means the partition set will not be affected.)

One important consequence of such analysis is that we identify a liquidity amplifier, the counterpart of the network multiplier in the liquidity channel, to capture the effect of asset price to the systemic risk. Moreover, the analysis demonstrates analytically the interplay of two amplification mechanisms of liability network and market liquidity, whereby we are able to examine the effectiveness of several intervention policies in stalling spillover of the systemic risk.

Denote $L^* = \{i \in N^* : s^*_i > 0\}$. By it, we split the non-default banks further into two subgroups, $N^* = L^* \cup (N^* \setminus L^*)$. The former subgroup of banks relies partly on illiquid asset sales to meet their liabilities, whereas the latter one sells no illiquid assets at all in the equilibrium. As we can expect, the banks in $L^*$ play a crucial role in the liquidity amplifier, because their competition for the limited market liquidity will drive down the price of the illiquid security. As their financial health deteriorates, they will turn to more liquidation in order to raise cash. That presses the asset price down further and in turn impairs the asset value of every bank in the system against the external shocks. In contrast, the banks in $N^* \setminus L^*$ should have no effect on the equilibrium because they fully repay their liabilities and do not participate in asset sales. Therefore they will not appear in the expressions of the network and liquidity multipliers. More precisely, we have

**Theorem 9.** Assume that $Q(\cdot)$ is differentiable. Then,

\[
\frac{\partial q^*}{\partial \beta_i} = \frac{\gamma e_i (I_{D^*} - P_{D^*})^{-1} P_{D^* \setminus L^*} 1}{1 - \gamma (|s_{L^*}^*| + \bar{s}_D (I_{D^*} - P_{D^*})^{-1} P_{D^* \setminus L^*} 1)}, \quad \text{for } i \in D^*
\]
and

\[ \frac{\partial q^*}{\partial \beta_i} = \frac{1 - \gamma (|s^*_{L\cdot}| + \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot)}{\gamma}, \quad \text{for } i \in L^*, \]

where \( \gamma = -Q'(s^*)/Q(s^*) \) and \( e_i \) denotes the \( i \)-th unit vector of dimension \( |D^*| \).

Theorem 9 presents a complex knock-on effect due to the liquidity. To see that, consider the impact of a reduction of $1 in \( \beta_i \), the value of external assets of bank \( i \), to the market price of the illiquid security. If this reduction does not change the equilibrium partition, namely, bank \( i \) is still in \( L^* \), paying its liability fully, the bank will have to sell an additional amount of \( 1/q \) of illiquid security, at unit price \( q = Q(\cdot) \), to compensate for this reduction. This extra sale subsequently will lower the price of the illiquid asset by a factor of

\[ Q(\cdot) - Q(\cdot + \frac{1}{q}) \approx -\frac{Q'(\cdot)}{Q(\cdot)} = \gamma. \]

This is the first-order liquidity effect.

Such price effect will feed back into the incomes of the banks in \( L^* \) through two channels. First, it immediately causes a reduction in the sale proceeds of asset liquidation of those banks. Note that the amounts of the illiquid security sold from the banks in \( L^* \) are given by \( s^*_{L\cdot} \). The impact via this channel is therefore that these banks will receive \( \gamma s^*_{L\cdot} \) dollars less, as the price declines by \( \gamma \).

The second channel is from the network effect, more specifically, the repayments of the banks in \( D^* \). All the banks in this subgroup, defaulting in the equilibrium, are forced to liquidate all the asset holdings; hence, \( s^*_i = \bar{s}_i \) for \( i \in D^* \). When the market price declines by \( \gamma \), the income for these banks will shrink by \( \gamma \bar{s}_{D\cdot} \). Observing the amplification effect captured by the network multiplier \( (I_{D\cdot} - P_{D\cdot})^{-1} \), this income reduction to \( D^* \) will result in a loss to the repayments to \( L^* \) by \( \gamma \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot \cdot \). As a whole, the second-order effect from the above two channels to the bank subgroup \( L^* \) is that these banks lose

\[ (\gamma s^*_{L\cdot} + \gamma \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot) 1 = \gamma (|s^*_{L\cdot}| + \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot) \]

aggregately in terms of the incomes.

Higher-order effects will be triggered because the banks in \( L^* \) have to sell more to offset the impact of the above loss. That further lowers the asset price. Continuing this argument leads to that the ultimate price decline, taking all orders of effect into account, should be

\[ \gamma + \gamma \left[ \gamma (|s^*_{L\cdot}| + \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot) \right] + \gamma \left[ \gamma^2 (|s^*_{L\cdot}| + \bar{s}_{D\cdot} (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot)^2 \right] + \cdots. \]

The sum of this geometric series is exactly the expression of \( \partial q/\partial \beta_i \) for \( i \in L^* \).

We can interpret \( \partial q/\partial \beta_i \), \( i \in D^* \), in an almost same way. Observe that, if we reduce $1 in the value of \( \beta_i \) for \( i \in D^* \), that reduction will cause a value decrease in the repayments to \( L^* \), amounting to

\[ \delta_i := e_i (I_{D\cdot} - P_{D\cdot})^{-1} P_{D\cdot, \cdot \cdot \cdot \cdot \cdot \cdot} \cdot \cdot \cdot. \]
Therefore,

\[
\delta \gamma + \delta \gamma \left[ \gamma \left( |s^*_L| + \bar{s}_D \left( I_{D^*} - P_{D^*} \right)^{-1} P_{D^*,L^*} \right) \right] + \delta \gamma \left[ \gamma^2 \left( |s^*_L| + \bar{s}_D \left( I_{D^*} - P_{D^*} \right)^{-1} P_{D^*,L^*} \right)^2 \right] + \cdots
\]

counts for the ultimate impact to the market price \( q^* \) initiated by this $1 reduction in \( \beta_i \). This series gives rise to the expression of \( \partial q / \partial \beta_i \) for \( i \in D^* \) in the theorem. Figure 1 visualizes the idea behind the aforementioned self-reinforcing effect. Many literatures, including Brunnermeier [11], Brunnermeier and Pedersen [12], and Adrian and Shin [2], confirm empirically that this loss spiral contributes significantly to the severity of the US credit crisis. Here we develop a metric to quantify these empirical observations.

The preceding discussion underscores the importance of the asset price as an additional channel for the systemic risk to transmit. In particular, we refer to

\[
LA := \frac{\gamma}{1 - \gamma \left( |s^*_L| + \bar{s}_D \left( I_{D^*} - P_{D^*} \right)^{-1} P_{D^*,L^*} \right)}
\]

(32)
as the liquidity amplifier below. It serves as an essential building block in a variety of sensitivities to reflect analytically the liquidity effect as we will see soon. The price elasticity \( \gamma \), whose reciprocal measures the market depth, determines largely the magnitude of (32): holding everything else unchanged, an increase in \( \gamma \) leads to a more illiquid market and raises the amplification value of (32). Furthermore, noting that it contains the network multiplier \( \left( I_{D^*} - P_{D^*} \right)^{-1} \), pointing to a fact that the two channels stimulate inextricably over the course of the systemic risk formation.

### 3.3 Intervention Policies

We now use the above sensitivity analysis to examine the effectiveness of policy intervention. Two policies are studied for the purpose of idea illustration: (1) direct purchase of the illiquid asset by an external player, e.g., the government; (2) capital injection. In practice, central banks undertook both to alleviate
the negative impacts of financial crises. For instance, the Federal Reserve purchased nearly $1.25 trillion of mortgage-backed securities from August 2007 to August 2009, while the US Treasury injected $205 billion in the form of preferred stock to the financial industry through a capital purchase program.

The consequences of the above two policies can be modelled in our setting as modifications on the original structure of the banks’ balance sheets (cf. Table 1). Suppose that the government injects $\Delta$ to one of the banks to mitigate its systemic impact. If it uses a direct asset purchase program, it pays cash in exchange for some amount of illiquid securities held by bank $i$. Two possibilities arise under this category: the government may choose to pay either the face value or market price of the asset. As Figure 2 demonstrates, this policy will result in an increment for the amount of liquid holding of bank $i$ from $\bar{y}_i$ to $\bar{y}_i + \Delta$, and meanwhile decrease its illiquid assets from $\bar{s}_i$ to $\bar{s}_i - \Delta$ or $\bar{s}_i - \Delta/q^*$, depending on whether the government pays the face value or market price. As for the capital injection, we assume that the bank uses $\Delta$, infused by the government in the form of equity capital, to scale up its liquid holding from $\bar{y}_i$ to $\bar{y}_i + \Delta$.

Figure 2: Changes on a bank’s balance sheet caused by two intervention policies. The left plot shows the direct asset purchase leads to an increase of $\Delta$ in the part of liquid holdings of the bank and a decrease of $\Delta$ in its illiquid assets when the transaction is done under the face value. The policy of capital injection in the right plot enhances the equity base of a bank by $\Delta$.

Holding $\beta_i$, the value of external investments for the banks, unchanged, we compute the value change in equilibrium, in particular, in terms of the clearing price $q^*_{\text{Policy}}(\Delta)$ and repayment $x^*_{\text{Policy}}(\Delta)$, caused by the aforementioned balance sheet modification. Let

$$PE^I_{\text{Policy}} := \lim_{\Delta \to 0} \frac{q^*_{\text{Policy}}(\Delta) - q^*}{\Delta}$$
and

$$PE^{II}_{\text{Policy}} := \lim_{\Delta \to 0} \frac{x^*_{\text{Policy}}(\Delta) - x^*}{\Delta}$$

be two gauges of policy effectiveness. We compare them in the context of different intervention schemes in Theorem 10.

**Theorem 10.** Fix $i \in D^*$. The policy effectiveness under direct asset purchase using the market price and capital injection are given by

$$PE^I_{\text{DAP, Market}} = LA, \quad PE^{II}_{\text{DAP, Market}} = LA \cdot \bar{s}_{D^*} \cdot (I_{D^*} - P_{D^*})^{-1},$$
and

\[
PE_{I, \text{Capital}} = LA \cdot e_i(I_{D^*} - P_{D^*})^{-1}P_{D^*} \cdot e_i \cdot 1,
\]

\[
PE_{II, \text{Capital}} = e_i(I_{D^*} - P_{D^*})^{-1} + PE_{I, \text{Capital}} \cdot s_{D^*}(I_{D^*} - P_{D^*})^{-1},
\]

respectively, where \(LA\) refers to the liquidity amplifier defined in (32). The effectiveness of the face-value purchase is a weighted average of the above two, namely,

\[
PE_{I/II, \text{DAP, Face}} = q^* PE_{I/II, \text{DAP, Market}} + (1 - q^*) PE_{I/II, \text{Capital}}.
\]

All the three quantities are positive, indicating that these policies can indeed influence the market equilibrium in a desirable direction, in terms of stabilizing the market price of the illiquid asset and halting the spillover of the systemic risk.

More interestingly, the theorem also reveals that different policies may have different focuses. Note that we can show \(e_i(I_{D^*} - P_{D^*})^{-1}P_{D^*} \cdot e_i \leq 1\). Hence, \(PE_{I, \text{DAP, Market}} \geq PE_{I, \text{Capital}}\), or in words, the effect of the direct asset purchase program on the market price surpasses capital injection. As for the repayment improvement, there is no clear-cut conclusion about the comparison between \(PE_{I, \text{DAP, Market}}\) and \(PE_{II, \text{Capital}}\). We find that the latter one is greater when \(e_i(I_{D^*} - P_{D^*})^{-1}\), the network multiplier, is sufficiently large. This observation points out that the policy of capital injection will be very effective to intervene the transmission of systemic risk in a highly leveraged banking system in which the corresponding quantity of \((I_{D^*} - P_{D^*})^{-1}\) tends to be large.

The comparison indicates that the asset purchase program mainly utilizes the liquidity channel to propagate its impact, whereas the capital injection program focuses more on the network channel. We propose the following intuition to interpret the difference. Calculate the value change in the bank’s total assets under the policy of asset purchase. We find that

\[
\text{Total Asset Value (TAV) After Purchase} = \frac{TAV \text{ Before Purchase} + \Delta - q^* \cdot \frac{\Delta}{q^*}}{TAV \text{ Before Purchase}};
\]

the policy does not change the bank’s asset value and thereby will not result in a reduction in the default probability for the recipient bank. In contrast, the capital injection program increases the total asset value of the recipient bank by \(\Delta\), making it less likely to default. In a highly leveraged banking system, the asset values of every bank affect the total repayments in the ultimate equilibrium. Therefore, the improvement effect on \(x^*\) will be more significant if we use capital injection.

The price impact is more related to the relative composition of liquid and illiquid assets for the banks. To capture it, consider the liquidity ratio of a bank, which is defined as the ratio of a bank’s liquid holdings over its total asset value. The direct asset purchase program increases the liquidity ratio of bank \(i\) from \(\bar{y}_i/TAV\) to \((\bar{y}_i + \Delta)/TAV\), noting that the asset value does not change by the policy. The improvement on this ratio under capital inject is only from \(\bar{y}_i/TAV\) to

\[
(\bar{y}_i + \Delta)/(TAV + \Delta) < (\bar{y}_i + \Delta)/TAV.
\]
That explains why the liquidity improvement of this policy should be less obvious than the former.

Of course, the previous discussion concerns only the benefits of these policies. We do not intend to make any claims here as to the optimality of policy selection. To provide a more comprehensive assessment for the purpose of policy recommendation, one should count in the costs of these intervention, which is absent so far in our sensitivity analysis.

4 Net Worth, Systemic Resilience, and Market Liquidity

In this section we shall examine the probability of contagion caused by the failure of a single bank, emphasizing the role of the equity capital value/net worth of a bank as a cushion against financial contagion. In the presence of the liquidity channel, the asset market value of each bank is driven down as more liquidations are set off by defaults cascade. That in turn will impair the market value of each bank’s capital bases, making them more susceptible to the contagion. Therefore, we base the estimates of contagion probability on the market value of the net worth, a more accurate measure of a bank’s financial strength as opposed to the book value, attempting to quantify the influence of market liquidity on the resilience of a banking system.

To start with, consider a banking system satisfying the following initial condition:

$$\beta_i + \bar{y}_i + \bar{s}_i + \sum_{j=1}^{n} \ell_j p_{ji} > \ell_i, \text{ for all } i. \quad (33)$$

In words, every bank in this system at the beginning has sufficient liquid funds to meet their liabilities. No default occurs in this circumstance. Using the currently prevailing market price of the illiquid security \( q^{(0)} = 1 \), we define the net worth of bank \( i \) as

$$e^{(0)}_i := \beta_i + \bar{y}_i + \bar{s}_i + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i, \text{ for all } i. \quad (34)$$

Under (33), all the banks have positive net worth \( e^{(0)}_i > 0 \).

Then, suppose that a shock hits bank 1, so that \( \beta_1 \) becomes \( \beta_1 - Y_1 \), where \( Y_1 \in [0, \beta_1] \) is a random variable. According to the partition algorithm, we start with \( D = \emptyset \) and \( N = \{1, 2, \ldots, n\} \) by assuming first that all other banks will repay fully their liabilities. Under (33) and with this tentative partition, when

$$Y_1 > (\beta_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{j1}) - \ell_1,$$

bank 1 starts to tap into its illiquid holding to make up the shortfall between its liability and liquid funds. We can use Eqs. (28-31) to calculate how much amount of illiquid security should be sold. Denote \( s^{(1)}_1 \) to be the obtained solution and \( q^{(1)} \) to be the market price of the illiquid asset after such liquidation. Observe that, if

$$\beta_1 - Y_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{ji} + s^{(1)}_1 q^{(1)} < \ell_1, \quad (35)$$
bank 1 violates the surplus constraint in (24). The algorithm will then put it in the augmented default set \( D \), meaning that it will be in default in the ultimate equilibrium.

Use the initial net worth \( e_1^{(0)} \) to derive a sufficient condition for (35). Noting that \( s_1^{(1)} \leq \bar{s}_1 \) and \( q^{(1)} \leq Q(s_1^{(1)}) \leq Q(0) = 1 \), the condition \( Y_1 > e_1^{(0)} \) implies that

\[
Y_1 > \beta_1 + \bar{y}_1 + \ell_1 + \bar{s}_1 - \ell_1 \geq \beta_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{ji} + s_1^{(1)} q^{(1)} - \ell_1;
\]

the inequality (35) follows. The preceding simple analysis indicates us

**Proposition 11.**

\[
P(\text{Bank 1 defaults}) \geq P(Y_1 > e_1^{(0)}).\]

This proposition has a clear economic interpretation: the net worth of bank 1 serves as the first line of defence of the system, since it is the sole bank receiving the hit. Once \( Y_1 \) is sufficiently large, the shock will exhaust all the capital the bank possesses and cause it insolvent. This kind of insolvency is fundamental in the sense that bank 1 fails not because of its systemic risk exposure, but because of its insufficient liquidity relative to the size of external shock.

The failure of bank 1 may be contagious to a subset of other banks. We intend to study how deep the cascading default can go in the following theorem. To this end, define

\[
e^{(1)}_i := (\beta_i + \bar{y}_i + \bar{s}_i Q(\bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i) \vee 0, \quad \text{for all } i. \tag{36}
\]

Namely, \( e^{(1)}_i \) represents the net worth value of bank \( i \), using the market price of the illiquid security after bank 1 depletes all its holdings. We have

**Theorem 12.** The probability that the shock on bank 1 causes bank \( j, j \neq 1 \), to default satisfies

\[
P(\text{Bank } j \text{ defaults} | \text{Bank 1 defaults}) \geq P\left(Y_1 - e^{(1)}_1 > \frac{1}{z_{1j}} \sum_{i=2}^{n} e^{(1)}_i z_{ij}\right),
\]

where \( Z = (z_{ij})_{i,j \in \{1, \ldots, n\}} = (I - P)^{-1} \). Moreover, the expected number of the infected banks by the shock is at least

\[
E(\# \text{ of infected banks} | \text{Bank 1 defaults}) \geq \sum_{j \neq 1} P\left(Y_1 - e^{(1)}_1 > \frac{1}{z_{1j}} \sum_{i=2}^{n} e^{(1)}_i z_{ij}\right).
\]

In fact, we establish a more general result about an estimate of contagion size caused by multiple failures in the technical appendix. A special case with a single failure is presented here for ease of idea exposition. We call

\[
\sum_{i=2}^{n} e^{(1)}_i z_{ij} / z_{1j} \tag{37}
\]
the resilience index of bank $j$. Recall the expansion that $(I - P)^{-1} = I + P + P^2 + \cdots$, whereby we can further represent the index (37) as

$$\left[ e^{(1)} + e^{(1)}P + e^{(1)}P^2 + \cdots \right]_j \left[ I + P + P^2 + \cdots \right]_1$$

When its value is low relative to $Y_1 - e^{(1)}_1$, the residual part of the shock transmitted out after it is partly absorbed by the net worth of bank 1, bank $j$ will be infected with a high probability according to Theorem 12.

The index incorporates the effects of the two crucial risk transmission channels. To appreciate the liquidity impact, note that we use $e^{(1)}$, the market value of each individual bank’s capital marked to the security price after the entire liquidation of bank 1, to define the index. Comparing it with $e^{(0)}$, the book value of the banks’ net worth which reflect their original capital level before the shock, we have

$$e^{(0)}_i - e^{(1)}_i = \bar{s}_i(1 - Q(\bar{s}_1)) > 0, \quad \text{for all } i \neq 1. \quad (38)$$

From the comparison (38), the market values of all the other banks’ net worths in the system will be significantly weakened as bank 1 defaults and liquidates, if a considerable portion in the asset sides of both banks 1 and $i$ happens to be illiquid, i.e., $\bar{s}_1$ and $\bar{s}_i$ are large relative to the market depth. By the non-negativeness of $P$,

$$\frac{[e^{(1)} + e^{(1)}P + e^{(1)}P^2 + \cdots]_j}{[I + P + P^2 + \cdots]_1} \leq \frac{[e^{(0)} + e^{(0)}P + e^{(0)}P^2 + \cdots]_j}{[I + P + P^2 + \cdots]_1}.$$ 

Therefore, the resilience of a bank would be seriously inflated if we use $e^{(0)}$ to calculate it, which will lead to an underestimation of the contagion probability. Theorem 12 suggests that the market value of banks capital should be a more accurate indicator in a highly illiquid market to capture the systemic impact of bank 1 to the entire system.

The resilience index depicts the network effect as well. Bearing the notion of the network multiplier in mind, we interpret the denominator part of the index as a measure of an “economic distance” on the liability network between banks 1 and $j$, taking into account all the possible risk-transmission paths from bank 1 to $j$. It points out how large a residual shock will be sent through the network towards bank $j$ if bank 1 suffers a $1$ loss. A large quantity for $z_{1j}$ means a large systemic exposure of bank $j$ to the failure of bank 1.

The numerator reflects the financial strength of bank $j$ in light of the interconnected balance sheets of the banking system. In addition to $e^{(1)}_j$, the net worth owned by bank $j$ itself, the bank also has claims on the assets of the other banks. Essentially, each dollar increment in the net worth held by bank $i, i \neq j$, will be used to absorb the shock passing through $i$ to bank $j$. Hence, it should be counted partially so as to calculate the effective capital buffer of bank $j$ against external shocks. In this sense, the infinite sum in the numerator, $e^{(1)} + e^{(1)}P + e^{(1)}P^2 + \cdots$, can be regarded as the integrated capital values of the banking system. A number of literature, including [10], [24], and [22], identify similar expressions for the ultimate value of an organization to the entire economy when they analyze the effects of equity cross holding among
business groups on market capitalization and the related estimation distortion issues. Here we find that this quantity constitutes a crucial component of the systemic resilience in the context of a financial system with interconnected liability exposures.

5 Numerical Experiments

We undertake some numerical experiments on the data of the 11 Germany banks participated in the 2011 EU-wide stress test. In light of incomplete information disclosed from our dataset, we see these experiments more like illustrative of the aforementioned methodologies and notions. However, we do find the important role of market liquidity in our hypothetical setting: it may become potentially a prevailing force to trigger a massive contagion under the current market environment.

5.1 The Data and Network Reconstruction

Ninety banks in 21 countries were involved in the exercise of the 2011 stress test organized by the European Banking Authority (see EBA [21]). For each, the authority reports the total assets and core tier 1 capital after the effects of mandatory restructuring plans publicly announced and fully committed before 31 December 2010. In addition, the EBA also reports each bank’s total claims (exposure at default, EAD) on domestic and foreign institutions, corporates, retail customers, and commercial real estates. Table 3 contains some relevant information extracted from the EBA’s report.

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<th>Bank Code</th>
<th>Bank Name</th>
<th>Total Asset (A)</th>
<th>Capital</th>
<th>Domestic Interbank EAD (E)</th>
<th>EAD/ Total Assets (E/A%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE017</td>
<td>DEUTSCHE BANK AG</td>
<td>1,905,630</td>
<td>30,361</td>
<td>47,102</td>
<td>2.47%</td>
</tr>
<tr>
<td>DE018</td>
<td>COMMERZBANK AG</td>
<td>771,201</td>
<td>26,728</td>
<td>49,871</td>
<td>6.47%</td>
</tr>
<tr>
<td>DE019</td>
<td>LANDES BANK B-W</td>
<td>374,413</td>
<td>9,838</td>
<td>91,201</td>
<td>24.36%</td>
</tr>
<tr>
<td>DE020</td>
<td>DZ BANK AG DT.Z-G</td>
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<td>7,299</td>
<td>100,099</td>
<td>30.94%</td>
</tr>
<tr>
<td>DE021</td>
<td>BAYERISCHE LANDES BANK</td>
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<td>11,501</td>
<td>66,535</td>
<td>21.03%</td>
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<tr>
<td>DE022</td>
<td>NORDDEUTSCHE LANDES BANK - GZ</td>
<td>228,586</td>
<td>3,974</td>
<td>54,921</td>
<td>24.03%</td>
</tr>
<tr>
<td>DE023</td>
<td>HYPO REAL ESTATE HOLDING AG</td>
<td>328,119</td>
<td>5,539</td>
<td>7,956</td>
<td>2.42%</td>
</tr>
<tr>
<td>DE024</td>
<td>WESTLB AG DUSSELDORF</td>
<td>191,523</td>
<td>4,218</td>
<td>24,007</td>
<td>12.53%</td>
</tr>
<tr>
<td>DE025</td>
<td>HSH NORDBANK AG HAMBURG</td>
<td>150,930</td>
<td>4,434</td>
<td>4,645</td>
<td>3.08%</td>
</tr>
<tr>
<td>DE027</td>
<td>LANDES BANK BERLIN AG</td>
<td>133,861</td>
<td>5,162</td>
<td>27,707</td>
<td>20.70%</td>
</tr>
<tr>
<td>DE028</td>
<td>DEKABANK DEUTSCHE GIROZENTRALE</td>
<td>130,304</td>
<td>3,359</td>
<td>30,937</td>
<td>23.74%</td>
</tr>
</tbody>
</table>

Table 3: Data of German banks from the 2011 EBA Stress Test Report. All the quantities are in million euros.

As one may notice, the EBA data contain only aggregate information about the banks’ assets and capital. The detailed breakup about bilateral interbank exposures and illiquid asset holding for each participant bank
is not available. For the purpose of numerical experiments, we need to reconstruct banking system models in a way consistent with the above aggregate-level data. To this end, assume that each bank’s interbank liabilities equal its interbank assets, and the domestic interbank EAD of each one is held by some banks else in the table. These assumptions concentrate the interbank liabilities within these 11 banks, leaving us a closed system to be constructed. In so doing, we actually bias the experiments in the favor of network-caused contagion.

The reconstruction amounts to finding an appropriate liability matrix $L = (L_{ij})$, $L_{ii} = 0$, such that its row and column sums match the observed interbank liabilities and assets respectively for all the banks. A number of candidates can satisfy this requirement. Therefore, to examine what impact the network topology will generate to systemic risk development, we consider the following three stereotypes of structures:

A. Complete. Every bank has bilateral exposures with every other banks in the system.

B. Ring-like. Every bank concentrates its exposure to its neighboring banks.

C. Core-periphery. The 11 banks are divided into two groups: core and periphery. The core banks connect widely with all the other banks in the system, whereas the banks in the periphery have exposure to the core banks only.

The conventional wisdom in the literature (e.g., Allen and Gale [3]) is that incomplete networks are prone to large scaled contagion than complete networks, as the latter structure helps diversify away the loss caused by the failure of one single or a cluster of banks. Structural type A and B represent the two opposite extremes in the spectrum of completeness. We intend to use them as examples to assess the influence of network completeness in the presence of market liquidity. Structural type C may resemble more closely the reality of the banking industry. As reported by some empirical studies, such as Upper and Worms [41] and Craig and von Peter [15], interbank markets are typically tiered in the sense that most banks do not lend to each other directly but through money center banks acting as intermediaries. In terms of completeness, type C sits between A and B. Figure 3 shows the reconstruction results under all the three structure types. All the details about the recovery are deferred to the technical appendix part.

To incorporate the liquidity effect, suppose that the inverse demand function of the illiquid asset takes an exponential form $Q(s) = \exp(-\gamma s)$. Take a baseline case that $\gamma = 1 \times 10^{-7}$ and $\theta = 30\%$ of the total assets (i.e., the first column of Table 3) is illiquid. To let readers have a better idea about how large this choice of $\theta$ and $\gamma$ is, the market price of the illiquid asset will drop to $0.864$ from its face value $1$ when every bank in the system sells out all its illiquid holdings. Subtracting the interbank EAD and the illiquid asset from the total asset leads to $\beta_i + \bar{y}_i$, the total value of external investments and liquid asset for each bank. Obviously, the magnitude of contagion in the equilibrium highly relies on $\theta$ and $\gamma$. Therefore, we also perform experiments on a variety of other combinations of $\theta$ and $\gamma$ to assess the impact of market liquidity to the systemic risk.
5.2 Contagion via Market Liquidity

In this subsection, we compare the effects of network and liquidity as two important channels for the systemic risk to transmit. Our numerical experiments point out that the market liquidity plays a leading role in triggering a massive contagion. Given the fact that the interbank lending just accounts for a relative small portion in the total asset of each bank as disclosed by the EBA data, we find that absent the liquidity channel, the failure of one bank hardly infects the others. A significant contagion effect can be observed once the market is sufficiently illiquid for asset sale or the banks hold too high illiquid assets.

Figure 4 illustrates the number of defaults in equilibrium under different sizes of the external shock on the external projects. In this set of experiments, we assume that Bank DE017 loses the value of its external investment $\beta$ by $Y$. As the shock size of $Y$ increases, the bank fails and its failure will spill over to affect more other banks. Compare the plots in the first row and column of the figure with the others in the southeast corner. When $\theta$ or $\gamma$ is small, no notable contagion effect occurs under any structured networks even for a large shock size: at most 2 or 3 banks default in the equilibrium caused by the failure of DE017. In this regard, our experiments corroborate the findings made in the recent empirical and simulation studies on the network stress testing that only the interbank liabilities can barely generate contagion. However, the experiments further point out that the liquidity risk should not be neglected when we study the systemic risk. The severity of contagion becomes acute as we increase the value of $\theta$ or $\gamma$, i.e., the market gets illiquid. A relatively small external shock to the asset of DE017 can trigger a great number of banks to fail.

The minor role that the interbank liabilities play in developing a contagion can be attributed to the
Figure 4: The number of defaults under different shock sizes. The vertical axis is the default number in the repayment equilibrium. The percentages in the horizontal axis is the relative size of an external shock $Y$ to the total asset of Bank DE017. In the first, second, and third rows, we specify the illiquid asset ratios as $\theta = 10\%$, $30\%$, and $60\%$, respectively, while in the first, second, and third columns, we use the market depth as $\gamma = 0.2 \times 10^{-7}$, $1 \times 10^{-7}$, and $4 \times 10^{-7}$.
small size of the interbank EAD relative to the total assets of each bank (cf. the last column of Table 3). When one bank fails, the loss it sends to its interbank debt holders will be limited to its total amount of the bilateral debt exposure. Therefore, the impact from this channel to the system is not significant for this given dataset. Glasserman and Young [29] derive some estimates of network-triggered contagion probability only based on the information of the banks’ aggregate liability levels. In contrast, the liquidity infects every participants through the asset market price, not necessarily confined to those banks with direct exposures to the defaulting one. Hence, we can expect that the liquidity spillover should have a greater systemic influence than the liability networks, especially when the illiquid assets are widely held in the banking system, or when the market depth to absorb a massive asset sale vanishes.

Once a contagion is triggered, the network configuration will affect the ultimate outcome of repayment equilibrium, although its impact is somehow mild compared with the liquidity effect. Take the baseline case (the central plot in Figure 4) for an illustration. We find a “phase transition” phenomenon; that is, the number of defaults in the system of type A structure is smallest under small shocks, whereas it changes to the least stable network among the three as the shock size becomes large. Our explanation to this observation is that the well-interconnectedness in complete networks indeed has a double-edged effect. The banks may utilize its diversification effect to divert small external shocks; on the other hand, it will serve as an efficient conduit to transmit losses when the shock size is sufficiently large. Acemoglu, Ozdaglar, and Tahbaz-Salehi [1] theoretically identify two shock regimes in which the complete network is the most and least stable. Their analysis is mainly based symmetric networks. Our numerical discovery shows that this robust-yet-fragile property of highly interconnected financial networks may still hold even in the presence of asset liquidation and asymmetric liability network.

At the end of this subsection, we perform more experiments by changing the shock recipient to another bank DE020 to further highlight the role of liquidity contagion. In contrast to DE017, DE020 owns the largest amount of interbank EAD among the 11 banks. Nonetheless, as indicated by Figure 5, the interbank EAD is apparently not so contagious as the market liquidity. Similar as Figure 4, we cannot observe any massive systemic contagion when the market is sufficiently liquid (cf. the left plot in Figure 5). Meanwhile, the numbers of defaults in this set of experiments are significantly less than the second row of Figure 4, which uses the same combination of $\theta$ and $\gamma$. Since we assume the banks hold 30% of illiquid asset universally in the system, the amount of illiquid holdings of DE017 is larger than that of DE020 merely due to its larger size of total assets. The comparison between this panel of plots and the second row of Figure 4 shows that the failure of an institution holding large amount of illiquid assets may pose a greater threat to the system stability than one with large interbank exposures.

5.3 Equity Capital as an Indicator of Systemic Resilience

We proceed to illustrate the importance of equity capital, especially its market value, as a gauge of systemic resilience in the presence of the liquidity channel. Use the baseline case with the type-A structured network
as an example. Assume that an external shock bursts on DE017 such that 20% of its total asset is lost. Running the partition algorithm, we obtain an equilibrium in which seven banks ultimately default. As the sole shock recipient, the failure of Bank DE017 is fundamental: the size of external shock on it is given by $Y_1 = 20\% \times 1,905,630 = 381,126$, exceeding its net worth $e^{(0)}_1 = 30,361$. According to the analysis in Section 4, the bank will be identified by the partition algorithm into the default set $D$ in the first iteration. We refer to its failure as the 0-order default. The bankruptcy of DE017 will cause three more rounds of cascade (i.e., three more rounds of augmentations in the algorithm execution) through the system. Table 4 illustrates the hierarchy of contagion under the shock.

<table>
<thead>
<tr>
<th>Default order</th>
<th>Banks Failing in Each Round</th>
<th>Cumulative Failures up to the Round</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-order</td>
<td>DE017</td>
<td>DE017</td>
</tr>
<tr>
<td>1st-order</td>
<td>DE022, 023</td>
<td>DE017, DE022, DE023</td>
</tr>
<tr>
<td>2nd-order</td>
<td>DE020, 024</td>
<td>DE017, DE022, DE023, DE020, DE024</td>
</tr>
</tbody>
</table>
| 3rd-order     | DE019, 028                 | DE017, DE022, DE023, DE020, DE024,
|               |                            | DE019, DE028                      |

Table 4: Hierarchy of cascades under a 20% shock to the asset of DE017. The liability network is type A. The liquidity parameters are assumed to be $\theta = 30\%$ and $\gamma = 1 \times 10^{-7}$.

One interesting observation is that DE023 defaults at so early a stage, although its interbank exposure to DE017 is relatively small compared with the other banks; refer to the technical appendix for the detailed liability matrix. To understand this, we compare in Table 5 the net worths of every bank before and after the failure of bank 1. Under $\gamma = 1 \times 10^{-7}$, an illiquid market environment, the impacts that the failure of DE017 exerts on the net worths of the other banks are prominent. In particular, DE022 and 023 would lose 95.88% and 98.75% of their net worth values, respectively. Note that the resilience indices for these two banks decline to 243, 209 and 339, 477, far less than the magnitude of the shock $Y_1$. Using Theorem 12, we can infer that DE022 and 023 will default immediately after DE017 fails.

This experiment serves as another strong supportive evidence that the liquidity contagion should not be neglected in the study of financial systemic risk. It shows that the price decline due to the asset liquidation
<table>
<thead>
<tr>
<th>Banks</th>
<th>$e^{(0)}_i$</th>
<th>$e^{(1)}_i$</th>
<th>Loss ratio</th>
<th>Resilience</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE017</td>
<td>30,361</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>DE018</td>
<td>26,728</td>
<td>13,872</td>
<td>48.10%</td>
<td>4,288,565</td>
</tr>
<tr>
<td>DE019</td>
<td>9,838</td>
<td>3,597</td>
<td>63.44%</td>
<td>716,274</td>
</tr>
<tr>
<td>DE020</td>
<td>7,299</td>
<td>1,905</td>
<td>73.90%</td>
<td>441,150</td>
</tr>
<tr>
<td>DE021</td>
<td>11,501</td>
<td>6,227</td>
<td>45.85%</td>
<td>1,519,962</td>
</tr>
<tr>
<td>DE022</td>
<td>3,974</td>
<td>164</td>
<td>95.88%</td>
<td>243,209</td>
</tr>
<tr>
<td>DE023</td>
<td>5,839</td>
<td>69</td>
<td>98.75%</td>
<td>339,477</td>
</tr>
<tr>
<td>DE024</td>
<td>4,218</td>
<td>1,025</td>
<td>75.69%</td>
<td>863,219</td>
</tr>
<tr>
<td>DE025</td>
<td>4,434</td>
<td>1,918</td>
<td>56.74%</td>
<td>6,845,625</td>
</tr>
<tr>
<td>DE027</td>
<td>5,162</td>
<td>2,931</td>
<td>43.23%</td>
<td>1,830,495</td>
</tr>
<tr>
<td>DE028</td>
<td>3,359</td>
<td>1,187</td>
<td>64.67%</td>
<td>787,555</td>
</tr>
</tbody>
</table>

Table 5: Net worth changes caused by the default of DE017. The parameters used are the same as those in Table 4. The net worths $e^{(0)}_i$ and $e^{(1)}_i$ for each bank are computed according to (34) and (36), respectively. The loss ratio is defined as $(e^{(0)}_i - e^{(1)}_i) / e^{(0)}_i$. We use (37) to compute the resilience for the banks.

of DE017 weakens the capital bases of the two banks to so considerable an extent that they will not have sufficient cushion to sustain even a moderate shock propagated from DE017. Meanwhile, it also demonstrates that the market value of equity capital should be more accurate to reflect systemic resilience of a banking system. To see this, we compute the resilience indices of DE022 and 023 again, with replacing $e^{(1)}_i$ by $e^{(0)}_i$. The values are 1,569,848 and 11,658,341, respectively, much stronger than the size of the external shock. In this case, using the book value of net worth actually underestimates the true systemic danger and can be misleading in the presence of the liquidity channel.

It is worth mentioning that the EBA was acutely aware that the banks’ equity capital reported in the stress test result (i.e., $e^{(0)}_i$ appearing in the 3rd column of Table 3) might change considerably due to the deteriorating sovereign debt situation at the time of its publication. Therefore, the authority recommended national supervisors oversee the banks with sizeable exposures to sovereigns to take specific steps to strengthen their capital positions, although they had passed the 5% benchmark of core tier one capital in the test. One underlying concern was that a sharp changes in investors’ risk appetite for the debts of the sovereigns under stress would create severe liquidity problems for such debts, and in turn would hurt those banks’ capital bases. The above experiments justify this concern.

5.4 Liquidity Amplifier

In this subsection, we examine two multipliers, especially the liquidity amplifier, under different network structure and different scales of the external shock. Figure 6 shows that, within the same network framework, the amplifier value increases in $\gamma$. It indicates clearly that the liquidity channel has a greater impact in a market with less liquidity, which is captured by a larger $\gamma$. We also can see this conclusion from Eq. (32): fixing everything else the same, the ratio in it is an increasing function with respect to $\gamma$. Furthermore, we find that the amplifier values under different structures are at a commensurate level for the same $\gamma$, and they vary little across the three plots. This is consistent with the observations in the previous two subsections:
the network structure does not play a dominating role in the current dataset in comparison of the market liquidity.

6 Conclusions

This paper develops an equilibrium-constrained optimization approach to model the systemic risk in a banking system. In the literature of social networks and epidemiology, contagion of rumors, deceases, and so on, typically follows a diffusion process through the local neighborhood structure of a network; refer to, e.g., Easley and Kleinberg [17]. Hence, the network effect there is the predominant force. A distinct feature of the financial system is that two banks may not have any counter-party relationship at all, but they are still connected through a global channel, the market. Our formulation can incorporate both two important channels, the network and market liquidity, for the transmission of financial systemic risk. We present a partition algorithm to solve the equilibrium, by which we unify and extend the fixed-point-based approaches proposed in some major literatures about the study of financial networks. The numerical experiments in the paper reveals that, as the on-going de-leveraging practice in financial institutions has already significantly shrinked their mutual liability exposure, the market effect may overtake the former to become a dominant force to trigger large scaled financial contagion.

Several directions are of special interest for us to pursue in the future. First, we assume that the entire liability network, captured by the matrix $P$, is known for the purpose of solving for the ultimate equilibrium. However, the data observable from the market typically contain at best incomplete information about it. This calls for a necessity of developing some methodologies in handling systemic risk modeling with uncertain data. In this aspect, our optimization formulation provides a very appealing platform, because it can be easily extended to accommodate data uncertainty with the help of a rapid burgeoning literature of robust optimization (see, e.g., Ben-Tal, El Ghaoui, and Nemirovski [7], Bertimas, Brown, and Caramanis [9] and the references therein).

The second research direction is to endogenize the decision of network formation and illiquid asset holding in the banking system towards building a dynamic model, as opposed to the static model presented in the
paper. Such kind of models would shed more insights on the problem of how to monitor the accumulation of systemic risk within the system.

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References


A Proof of Main Results

A.1 Proofs for the Results in Section 2

Proof. Proof of Proposition 1. As noted in the discussion prior to the Partition Algorithm in Section 2.2, the algorithm terminates when there is no new default identified. That implies, \((\mathcal{D}^*, \mathcal{N}^*)\) should satisfy all the constraints in (5-6). Therefore, the vector \(x^*\), defined as in (8) with \(\mathcal{D} = \mathcal{D}^*\) and \(\mathcal{N} = \mathcal{N}^*\), is a feasible solution to (4).

Let \((\mathcal{D}, \mathcal{N})\) be a partition produced in any intermediate step and define \(x = (x_D, x_N)\) as in (8). The proof is done if we can establish that \(|x| \geq |x'|\) for any \(x'\) satisfying the constraints in (4). Obviously, \(x_N' \leq \ell_N = x_N\) because \(x'\) satisfies the constraint \(x' \leq \ell\). For the set \(\mathcal{D}\), we have

\[
x_D' \leq \alpha_D + x_D'P_D + x_N'P_{N,D} \leq \alpha_D + x_D'P_D + \ell_NP_{N,D},
\]

where the first inequality uses the fact that \(x'(I - P) \leq \alpha\) and the second inequality holds because \(x_N' \leq \ell_N\) and \(P_{N,D}\) is a nonnegative matrix. Hence,

\[
x_D'(I_D - P_D) \leq \alpha_D + \ell_NP_{N,D}.
\]

Multiplying \((I_D - P_D)^{-1}\) on both sides of the above inequality and using its non-negativeness, we have

\[
x_D' \leq (\alpha_D + \ell_NP_{N,D})(I_D - P_D)^{-1} = x_D.
\]

Consequently, \(|x| = |x_D| + |x_N| \geq |x_D'| + |x_N'| = |x'|. □

Proof. Proof of Proposition 3. We finish the proof in two steps.

Step 1. Suppose that \(P\) is an irreducible matrix, i.e., there exists a positive integer \(m\) such that \(P^m\) is a strictly positive matrix (c.f. Theorem 2.1 of [8]).

To show Part (i), consider a stochastic \(P\), i.e., all its row sums equal 1. We know that its spectral radius \(\rho(P)\) must be 1. From the classical Perron-Frobenius theorem (c.f. Theorem 8.3.13 of [8]), \(P\) has a positive left-eigenvector \(\pi\) such that \(\pi = \pi P\). Suppose, to the contrary, that all banks default for an optimal solution \(x\) to the LP (4), i.e., \(x(I - P) \leq \alpha\), and \(x < \ell\) (meaning strict inequality holds for every component). Let \(x' = x + c'\pi\), where \(c' = \min_i \{(\ell_i - x_i)/\pi_i\}\). Then, we have \(x'(I - P) \leq \alpha\), and \(x' \leq \ell\). Since \(|x'| > |x|\), it contradicts the assumption that \(x\) is the maximizer of the LP in (4). Also, the inequality \(x' \leq \ell\) holds with at least one component — say \(j\), the arg min that defines \(c'\) — holding as equality. Hence, at least \(j\) will not default.

Turn to prove Part (ii). There are two cases according to the magnitude of \(\rho(P)\). When \(\rho(P) < 1\), we know that \(\rho(P_{D^*}) < 1\) because \(\rho(P_{D^*}) \leq \rho(P)\). Therefore, \(P_{D^*}\) cannot be a stochastic matrix and Part (ii) follows. When \(\rho(P) = 1\), the proof of Part (i) in the last paragraph implies that \(\mathcal{D}^*\) must be a proper subset of \(\{1, 2, \ldots, n\}\). Suppose that \(\rho(P_{D^*}) = 1\), i.e., \(\rho(P)\) is an eigenvalue of a proper submatrix of \(P\). Invoking
Corollary 2.1.6 of [8], \( P \) must be reducible, contradicting to our assumption. This implies that \( \rho(P_D) < 1 \).

Once again, \( P_D^* \) cannot be stochastic.

Step 2. Now, consider the case of \( P \) being reducible. The theory of nonnegative matrix (e.g. Chapter 2.3 of [8]) indicates that we can break the set \( \{1,\ldots,n\} \) into several disjoint subclasses, each one forming an equivalent (or communicating) class. By a suitable permutation, \( P \) can be reduced to triangular block form

\[
P = \begin{pmatrix}
A_{11} & 0 & \cdots & 0 \\
A_{21} & A_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{s1} & A_{s2} & \cdots & A_{ss}
\end{pmatrix}.
\]

(39)

All of the diagonal blocks are irreducible or \( 1 \times 1 \) null matrix. Some (at least one) of them are basic (c.f. Definition 2.3.8 of [8]), i.e., \( \rho(A_{ii}) = \rho(P) \) for some \( i, \ 1 \leq i \leq s \). Without loss of generality, we assume that \( A_{11} \) possesses this property.

Assume that \( P \) is stochastic. We know that \( \rho(A_{11}) = \rho(P) = 1 \). Applying the arguments in Step 1 when \( P \) is irreducible, replacing the full set \( \{1,\ldots,n\} \) by the class corresponding to \( A_{11} \), at least one bank in the class will not default. Part (i) is proved.

Let us consider Part (ii). The only way for \( P_D^* \) to be a stochastic matrix is that \( D^* \) includes (at least) one communicating class represented by the diagonal blocks and that block matrix is also stochastic. But then, following the proof of Step 1 and again replacing the full set by this class, we can construct a contradiction that there exists a new solution which dominates the maximizer of (13). Therefore, we prove part (ii). \( \square \)

Proof. Proof of Proposition 4. The proof of this proposition follows similar arguments as what we used in establishing Proposition 1. The algorithm ensures that the partition \( (D^*,N^*) \) it yields when terminated must be feasible, satisfying all the constraints in (14-15). Therefore, it suffices to show that the tentative \( \tilde{x} \) obtained in every intermediate step is greater than any \( x' \) satisfying (13).

Apparently, \( \tilde{x}_N = \tilde{\ell}_N \geq x'_N \) because \( x' \), as a feasible solution to the problem (13), must satisfy \( x' \leq \ell \).

On the other hand, due to the first constraint in (13), we have

\[
x'_D \leq (\tilde{\alpha}_D + x'_N P_{N,D} + x'_D P_D)\] \(+ \leq (\tilde{\alpha}_D + \tilde{\ell}_N P_{N,D} + x'_D P_D)\] \(+ \).

In other words, \( x'_D \leq G(x'_D) \). In addition, the definition of \( D \) implies that \( G(\tilde{\ell}_D) < \ell_D \). In conjunction with the monotonicity of \( G(\cdot) \), we know that \( x'_D \leq G(x'_D) \leq G(\tilde{\ell}_D) < \ell_D \). From this, \( x'_D \leq y^1 = G(\tilde{\ell}_D) \).

Continuing to apply \( G(\cdot) \) on both sides of the above inequality, we can derive that \( x'_D \leq y^i \), thanks to the monotonicity of \( G(\cdot) \). Note that the sequence \( \{y^i : i \geq 0\} \) converges to \( \tilde{x}_D \). Hence, \( x'_D \leq \tilde{x}_D \). In summary, the norms of \( \tilde{x} \) and \( x' \) satisfy \( ||\tilde{x}|| \geq ||x'|| \). \( \square \)

A.2 Proofs for the Results in Section 3

Recall the definition of \( H \) and the sequence of \( \{(u^i, v^i, r^i, t^i), i \geq 1\} \) generated by it in Section 3.1. Endow a weak order \( \succcurlyeq \) on \( \mathcal{R} \) in the sense that \((u, v, r, t) \succcurlyeq (u', v', r', t') \) if \( u \geq u', v \leq v', r \leq r', \) and \( t \geq t' \).
Lemma 13. (i) Function $H$ is increasing on $\mathcal{R}$ relative the weak order $\succsim$.
(ii) The sequence $\{(u^i, v^i, r^i, t^i), i \geq 1\}$ is decreasing in the sense of the weak order $\succsim$. Therefore, the limits $x_D := \lim_{i \to +\infty} u^i$, $y_N := \lim_{i \to +\infty} v^i$, $s_N := \lim_{i \to +\infty} r^i$, and $q := \lim_{i \to +\infty} t^i$ exist.
(iii) For any other fixed-point of function $H$, $(u, v, r, t)$, we have $(x_D, y_N, s_N, q) \succsim (u, v, r, t)$.

Proof. (i) Suppose that we have $(u, v, r, t), (u', v', r', t') \in \mathcal{R}$, $(u, v, r, t) \succsim (u', v', r', t')$. Since $u \geq u'$, $uP_D \geq u'P_D$ and $uP_{D,N} \geq u'P_{D,N}$ due to the non-negativeness of the matrices $P_D$ and $P_{D,N}$. Hence,

$$x = (\beta_D + \bar{y}_D + \bar{s}_D t + uP_D + \ell_N P_{N,D}) \land \ell_D \geq (\beta_D + \bar{y}_D + \bar{s}_D t' + u'P_D + \ell_N P_{N,D}) \land \ell_D = x'$$

and

$$y = \bar{y}_N \land [\ell_N - (\beta_N + \ell_N P_N + uP_{D,N})]^{+} \leq \bar{y}_N \land [\ell_N - (\beta_N + \ell_N P_N + u'P_{D,N})]^{+} = y'.$$

The assumption that $v \geq v'$ implies that

$$s = \bar{s}_N \land \frac{[\ell_N - (\beta_N + \ell_N P_N + uP_{D,N}) - v]}{t} \leq \bar{s}_N \land \frac{[\ell_N - (\beta_N + \ell_N P_N + u'P_{D,N}) - v']}{t} = s'.$$

Furthermore, we can obtain

$$q = Q(|\bar{s}_D| + |r|) \geq Q(|\bar{s}_D| + |r'|) = q'$$

from the monotonicity of $Q$ and $r \leq r'$. So far, we have shown $H(u, v, r, t) \succsim H(u', v', r', t')$.

(ii) Notice that $(u^0, v^0, r^0, t^0) = (\ell_D, 0_N, 0_N, 1)$. According to the definition of $H$, $(u^0, v^0, r^0, t^0) \succsim (u^1, v^1, r^1, t^1)$. By the monotonicity of $H$ established in Part (i), we have $(u^1, v^1, r^1, t^1) = H(u^0, v^0, r^0, t^0) \succsim H(u^1, v^1, r^1, t^1) = (u^2, v^2, r^2, t^2)$. Repeatedly applying $H$, we can show that the sequence $\{(u^i, v^i, r^i, t^i), i \geq 1\}$ must be decreasing.

(iii) Consider any fixed point of $H$ denoted by $(u, v, r, t)$. We know that $(u^0, v^0, r^0, t^0) \succsim (u, v, r, t)$. Invoking the same arguments used in Part (ii), $(u^i, v^i, r^i, t^i) \succsim (u, v, r, t)$ for all $i \geq 1$. As the limits of the sequence, we know that $(x_D, y_N, s_N, q) \succsim (u, v, r, t)$. 

Lemma 13 establishes that $(x_D, y_N, s_N, q)$ must be a maximal fixed point of $H$. To justify that this quadruple also solves the equation system (28-31), we have to show

Lemma 14. For any partition generated by the algorithm, we have $x_D < \ell_D$.

Proof. Use induction. The statement of the lemma is trivially true for the initial partition the algorithm starts with because $D = \emptyset$. To complete the inductive arguments, we assume that $x_D < \ell_D$ for some intermediate partition $(D, N)$ and its related fixed point $(x_D, y_N, s_N, q)$. Denote $(D', N')$ to be the next partition we obtain after the feasibility check in Step 2. Note that $D'$ includes new banks for which the surplus constraint (cf. the second one in (24)) is violated under $(x_D, y_N, s_N, q)$, namely,

$$D' = D \cup \left\{ i : i \in N, \ell_i > \beta_i + y_i + s_i q + \sum_{j \in D} x_{ji} p_{ji} + \sum_{j \in N} \ell_j p_{ji} \right\}.$$
Certainly we have $D \subseteq D', \; N \supseteq N'$, and $N = N' \cup (D' \setminus D)$. Denote $(x_D', y_N', s_N', q')$ to be the greatest fixed point under this new partition obtained as Part (ii) of Lemma 13 instructs.

We show first that $x_D' < \ell_D$. To this end, consider the following parameterized function $\bar{H}$ on $\bar{R} := \prod_{i \in D}[0, \ell_i] \otimes \prod_{i \in D'}[0, \bar{s}_i] \otimes \prod_{i \in N'}[0, \bar{s}_i] \otimes [0, 1]$, where $(x, y, s, q) = \bar{H}(u, v, r, t)$ such that

$$x = (\theta_1 + \bar{s}_D t + u P_D) \land \ell_D, \quad y = \bar{y}_{N'} \lor [\ell_{N'} - (\theta_2 + u P_{D,N'})] +, \quad s = \bar{s}_{N'} \lor [\ell_{N'} - (\theta_2 + u P_{D,N'}) - v] +$$

and $q = Q(|s_D| + |r|)$ for some parameters $\theta_1$ and $\theta_2$. Notice that $(x_D, y_N', s_{N'}, q)$ and $(x_D', y_N', s_{N'}, q)$ are the maximal fixed point of $\bar{H}$ with the parameter sets being

$$(\theta_1, \theta_2) = (\beta_D + \bar{y}_D + \ell_{N'} P_{N',D}, \; \beta_{N'} + \ell_{N'} P_{N',N'}) \tag{40}$$

and

$$(\theta_1', \theta_2') = (\beta_D + \bar{y}_D + x_D' P_{D\setminus D,\ell_D} + \ell_{N'} P_{N',D}, \; \beta_{N'} + x_D' P_{D\setminus D,\ell_D} + \ell_{N'} P_{N',N'}) \tag{41}$$

respectively.

It is easy to see that the function $\bar{H}$ is increasing in the parameter $\theta_1$ and $\theta_2$, i.e., for any $\theta_1 \geq \theta_1'$ and $\theta_2 \geq \theta_2'$, we have $\bar{H}(u, v, r, t; \theta_1, \theta_2) \succeq \bar{H}(u, v, r, t; \theta_1', \theta_2')$ for any $(u, v, r, t) \in R$. By the celebrated Tarski’s theorem (see, e.g., Corollary 2.5.2 of [42]), we know that the maximal fixed point of $\bar{H}$ should be increasing in $(\theta_1, \theta_2)$. On the other hand, $(\theta_1, \theta_2)$ and $(\theta_1', \theta_2')$ defined as above satisfy

$$\begin{align*}
\theta_1 &= \beta_D + \bar{y}_D + \ell_{N'} P_{N',D} = \beta_D + \bar{y}_D + \ell_{D\setminus D} P_{D\setminus D,D} + \ell_{N'} P_{N',D} \\
&\geq \beta_D + \bar{y}_D + x_D' P_{D\setminus D,D} + \ell_{N'} P_{N',D} = \theta_1'
\end{align*}$$

and

$$\begin{align*}
\theta_2 &= \beta_{N'} + \ell_{N'} P_{N',N'} = \beta_{N'} + \ell_{D\setminus D} P_{D\setminus D,N'} + \ell_{N'} P_{N'} \\
&\geq \beta_{N'} + x_D' P_{D\setminus D,N'} + \ell_{N'} P_{N'} = \theta_2',
\end{align*}$$

where we use the fact that $\ell_{D\setminus D} \geq x_D' P_{D\setminus D}$. Therefore, $(x_D, y_N', s_{N'}, q) \succeq (x_D', y_N', s_{N'}, q')$. From this, we have $x_D' \leq x_D < \ell_D$. It is immediate to see that $x_D' < \ell_i$ for $i \in D\setminus D$. From the definition of set $D'$, we know that

$$\ell_i > \beta_i + y_i + s_i q + \sum_{j \in D} x_{j} p_{j} + \sum_{j \in N} \ell_j p_{ji} \tag{42}$$

for such $i$. It implies that, for $i \in D\setminus D$,

$$\left[\ell_i - (\beta_i + \sum_{j \in N} \ell_j p_{ji} + \sum_{j \in D} x_{j} p_{ji}) - y_i\right]^+ > s_i q \geq 0;$$

hence, $s_i = \bar{s}_i$ and $y_i = \bar{y}_i$. Consequently, by (42),

$$\begin{align*}
\ell_{D\setminus D} &> \beta_{D\setminus D} + \bar{y}_{D\setminus D} + \bar{s}_{D\setminus D} q + \ell_{N'} P_{N',D\setminus D} + x_D P_{D,D\setminus D} \\
&= \beta_{D\setminus D} + \bar{y}_{D\setminus D} + \bar{s}_{D\setminus D} q + \ell_{D\setminus D} P_{D\setminus D} + \ell_{N'} P_{N',D\setminus D} + x_D P_{D,D\setminus D} \tag{43}
\end{align*}$$
where we split the index set $N$ into a union of $D' \setminus D$ and $N'$ in the second equality. Using the facts that $x_D \geq x'_D$, $q \geq q'$, the right hand side of (43) is greater than

$$\beta_{D' \setminus D} + \bar{g}_{D' \setminus D} + \bar{s}_{D' \setminus D} q' + \ell_{D' \setminus D} P_{D' \setminus D} + \ell_{N'} P_{N', D' \setminus D} + x'_D P_{D, D' \setminus D}.$$  

Then,

$$\ell_{D' \setminus D} (I_{D' \setminus D} - P_{D' \setminus D}) > \beta_{D' \setminus D} + \bar{g}_{D' \setminus D} + \bar{s}_{D' \setminus D} q' + \ell_{N'} P_{N', D' \setminus D} + x'_D P_{D, D' \setminus D}.$$  

Multiplying a nonnegative matrix $(I_{D' \setminus D} - P_{D' \setminus D})^{-1}$ on both sides of the above inequality will yield that

$$\ell_{D' \setminus D} > (\alpha_{D' \setminus D} + \bar{s}_{D' \setminus D} q' + \ell_{N'} P_{N', D' \setminus D} + x'_D P_{D, D' \setminus D})(I_{D' \setminus D} - P_{D' \setminus D})^{-1}.$$  

Note that the right hand side of the above inequality equals to $x'_D$. That implies $x'_D < \ell_{D' \setminus D}$. In summary, we have $x'_D < \ell_{D' \setminus D}$. \hfill \Box

**Proof.** Proof of Theorem 8. In light of Lemmas 13 and 14, to complete the proof, what we need to establish is that, for any partition $(D, N)$ generated sequentially from the algorithm, the corresponding equilibrium $(x, y, s, q)$ associated with it is greater than any optimal solution to the problem (23). Therefore, when the algorithm terminates at a primal feasible partition, this partition should be optimal.

Use induction again to show the above claim. Denote $(\bar{x}, \bar{y}, \bar{s}, \bar{q})$ to be any market-clearing repayment equilibrium satisfying (23). It defines a partition for $\{1, 2, \ldots, n\}$ as follows: $\bar{D} := \{i : \bar{x}_i < l_i\}$ and $\bar{N} := \{i : \bar{x}_i = l_i\}$. The algorithm starts with a partition such that $D = \emptyset$ and $N = \{\infty, \epsilon, \ldots, \}\}$. The corresponding $x = x_N = \ell$ obviously dominates $\bar{x}$, i.e., $x \geq \bar{x}$. Take the notations in the proof of Lemma 14. Suppose that for an intermediate partition $(D, N)$, the corresponding $(x, y, s, q) \succ (\bar{x}, \bar{y}, \bar{s}, \bar{q})$. Following the algorithm instructions, we identify some new defaults and augment the default set from $D$ to $D'$. To accomplish the inductive step, we need to show that $(x', y', s', q')$, the equilibrium corresponding to the new partition $(D', N')$ dominates $(\bar{x}, \bar{y}, \bar{s}, \bar{q})$ in the sense of the weak order $\succ$.

From the inductive assumption that $x_D \geq x_D$, we know that $D \subseteq \bar{D}$; hence $N = (\bar{D} \setminus D) \cup \bar{N}$. For any $i \in D' \setminus D$, the inequality (42) in the Proof of Lemma 14 holds. Therefore,

$$\ell_i > \beta_i + \bar{y}_i + \bar{s}_i q + \sum_{j \in N} \ell_{j p_{j i}} + \sum_{j \in D} x_{j p_{j i}}$$

$$= \beta_i + \bar{y}_i + \bar{s}_i q + \sum_{j \in N} \ell_{j p_{j i}} + \sum_{j \in D} \ell_{j p_{j i}} + \sum_{j \in D} x_{j p_{j i}}, \quad (44)$$

where we split the sum across the set $N$ into two using the observation that $N = (\bar{D} \setminus D) \cup \bar{N}$. Because $\ell_{\bar{D} \setminus D} \geq \bar{x}_{\bar{D} \setminus D}$, $x_D \geq \bar{x}_D$, and $q \geq \bar{q}$, the right hand side of (44) will be greater than

$$\beta_i + \bar{y}_i + \bar{s}_i \bar{q} + \sum_{j \in N} \ell_{j p_{j i}} + \sum_{j \in D} \bar{x}_{j p_{j i}}.$$  

The above quantity in turn is larger than $\bar{x}_i$ according to the first constraint in (23) which $\bar{x}$ satisfies. Consequently, $D' \setminus D \subseteq \bar{D}$ and we have $D' \subseteq \bar{D}$ in conjunction with $D \subseteq \bar{D}$. As the complement set of $D'$, $N' \supseteq \bar{N}$.  

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With this relationship in mind, we can use the Tarski’s fixed point theorem again to to compare $(x'_D, y'_N, s'_N, q')$ and $(\tilde{x}_D, \tilde{y}_N, \tilde{s}_N, \tilde{q})$. More specifically, consider $H'$ on $R' := \prod_{i\in D'}[0, \ell_i] \times \prod_{i\in N'}[0, \tilde{y}_i] \times \prod_{i\in \tilde{N}'}[0, \tilde{s}_i] \times [0, 1]$, where $(x, y, s, q) = H'(u, v, r, t)$ such that

\[
x = (\theta_1 + u_{D'}t + u_{D'}v) \land \ell_{D'}, \quad y = \tilde{y}_N \land [\ell_N - (\theta_2 + u_{D'_N}v)]^+, \quad s = \tilde{s}_N \land \left[\frac{\ell_N - (\theta_2 + u_{D'_N}v) - v}{t}\right]^+
\]

and $q = Q(|\tilde{s}_{D'}| + |r|)$ for some parameters $\theta_1$ and $\theta_2$. $(x'_D, y'_N, s'_N, q')$ is its fixed point with the parameter

\[
(\theta'_1, \theta'_2) = (\beta_{D'} + \tilde{y}_{D'} + \ell_N P_{N', D'}, \beta_N + \tilde{y}_N + \ell_N P_N + \tilde{x}_{D'} D'_{D'}, \tilde{P}_{D'_{D'}, N'}).
\]

whereas $(\tilde{x}_D, \tilde{y}_N, \tilde{s}_N, \tilde{q})$ is a fixed point of $H$ with the parameters

\[
(\tilde{\theta}_1, \tilde{\theta}_2) = (\beta_{D'} + y_{D'} + \ell_N P_{N', D'}, \beta_N + y_N + \ell_N P_N + \tilde{x}_{D'} D'_{D'}, \tilde{P}_{D'_{D'}, N'}).
\]

From $\theta'_i \geq \tilde{\theta}_i$, $i = 1, 2$, we know that $(x'_D, y'_N, s'_N, q')$, as the greatest fixed point of $H'$ with parameter $(\theta'_1, \theta'_2)$, dominates the greatest fixed point of $H'$ with parameter $(\tilde{\theta}_1, \tilde{\theta}_2)$. Therefore, $(x'_D, y'_N, s'_N, q') \succeq (\tilde{x}_D, \tilde{y}_N, \tilde{s}_N, \tilde{q})$. For $i \in N'$, $x'_i = \ell_i \geq \tilde{x}_i$. For $i \in D$, $\tilde{y}_i \geq \tilde{y}_i$ and $\tilde{s}_i \geq \tilde{s}_i$. Hence, $(x', y', s', q') \succeq (\tilde{x}, \tilde{y}, \tilde{s}, \tilde{q})$. We have finished the inductive step. The theorem is proved.

**Proof.** Proof of Theorem 9. Recall that in the equilibrium the banking system is divided into three subgroups: $D^*$, $L^*$, and $N^* \setminus L^*$. The amounts of asset liquidation from $D^*$ and $L^*$ are $\tilde{s}_{D^*}$ and $s^*_{L^*}$, respectively, whereas the banks in $N^* \setminus L^*$ do not need to sell any assets to raise funds. We have

\[
q^* = Q(|s^*|) = Q(|s^*_{L^*}| + |\tilde{s}_{D^*}|).
\]

Therefore, for $i \in L^*$,

\[
\frac{\partial q^*}{\partial \beta^*_i} = q^* \sum_{k \in L^*} \frac{\partial s^*_k}{\partial \beta^*_i} = q^* \frac{\partial s^*_k}{\partial \beta^*_i} 1. \tag{45}
\]

Furthermore, since $s^*_k$ satisfies $s^*_k = d^*_k / q^*$ for $k \in L^*$, we know that

\[
q^* s^*_k = d^*_k = \ell_k - (\beta_k + \bar{y}_k + \sum_{h \in D^*} x^*_hp_{hk} + \sum_{j \in N^*} \ell_j p_{jk}). \tag{46}
\]

Taking derivatives with respect to $\beta^*_i$ on both sides,

\[
q^* \frac{\partial s^*_k}{\partial \beta^*_i} + s^*_k \frac{\partial q^*}{\partial \beta^*_i} = - \sum_{h \in D^*} \frac{\partial x^*_h}{\partial \beta^*_i} p_{hk} = - \frac{\partial x^*_h}{\partial \beta^*_i} P_{D^*, h}.
\]

Sum the above equalities over $k \in L^*$. We obtain

\[
q^* \frac{\partial s^*_L}{\partial \beta^*_i} 1 + \frac{\partial q^*}{\partial \beta^*_i} |s^*_{L^*}| = - \frac{\partial x^*_h}{\partial \beta^*_i} P_{D^*, L^*} 1. \tag{47}
\]

From (45) and (47), we can solve

\[
\frac{\partial q^*}{\partial \beta^*_i} = \left(- \frac{q^*}{q^*} - |s^*_{L^*}|\right)^{-1} \frac{\partial x^*_h}{\partial \beta^*_i} P_{D^*, L^*} 1. \tag{48}
\]
On the other hand, the repayment vector $x^*_D$, in the largest equilibrium admits the following representation:

$$x^*_D = (\beta_D^* + \bar{y}_D^* \tilde{s}_D^* q^* + x^*_N, P^* - P^* \cdot (I - P^*)^{-1}.$$

Taking partial derivative with respect to $\beta_i$ on the expression of $x^*_D$, we have

$$\frac{\partial x^*_D}{\partial \beta_i} = \left( e_i + \frac{\partial q^*}{\partial \beta_i} \tilde{s}_D^* \right) (I - P^* \cdot (I - P^*)^{-1}.$$

Finally, substituting (49) into (48) and recollecting terms will lead us to the first equality in the theorem statement.

Next we proceed to derive $\frac{\partial q^*}{\partial \beta_i}$ for $i \in \mathcal{L}$. Similarly, we start from (45). Taking derivatives with respect to $\beta$ on both sides of (46) for $i \in \mathcal{L}$ will lead to

$$q^* \frac{\partial s^*_k}{\partial \beta_i} + s^*_k \frac{\partial q^*}{\partial \beta_i} = -e_i - \frac{\partial x^*_D}{\partial \beta_i} P^*_{D^* \cdot k}, \quad \forall k \in \mathcal{L}.$$

From it, we have

$$q^* \frac{\partial s^*_L}{\partial \beta_i} 1 + \frac{\partial q^*}{\partial \beta_i} |s^*_L| = -1 - \frac{\partial x^*_D}{\partial \beta_i} P^*_{D^* \cdot 1}.$$

Note the difference between (50) and (47). Following similar arguments as the proof of the first half from now on, we can show the second equality in the theorem.

Proof. Proof of Theorem 10. Notice that

$$PE^i_{DAP, Market} = \frac{\partial q^*}{\partial \bar{y}_i} - \frac{1}{q^*} \frac{\partial q^*}{\partial \bar{s}_i}, \quad PE^i_{DAP, Market} = \frac{\partial x^*_D}{\partial \bar{y}_i} - \frac{1}{q^*} \frac{\partial x^*_D}{\partial \bar{s}_i};$$

$$PE^i_{Capital} = \frac{\partial q^*}{\partial \bar{y}_i}, \quad PE^i_{Capital} = \frac{\partial x^*_D}{\partial \bar{y}_i};$$

and

$$PE^i_{DAP, Market} = \frac{\partial q^*}{\partial \bar{y}_i} - \frac{\partial q^*}{\partial \bar{s}_i}, \quad PE^i_{DAP, Market} = \frac{\partial x^*_D}{\partial \bar{y}_i} - \frac{\partial x^*_D}{\partial \bar{s}_i}.$$

Therefore it suffices to derive $\frac{\partial q^*}{\partial \bar{y}_i}, \frac{\partial x^*_D}{\partial \bar{y}_i}, \frac{\partial q^*}{\partial \bar{s}_i}, \frac{\partial x^*_D}{\partial \bar{s}_i}$, and $\frac{\partial x^*_D}{\partial \bar{y}_i}$. Since the proof is highly similar as Theorem 9, we only present the calculation related to the first two sensitivities here for the interest of space.

On one hand, we can obtain

$$\frac{\partial q^*}{\partial \bar{y}_i} = \left( -\frac{q^*}{q^*} - |s^*_L| \right)^{-1} \frac{\partial x^*_D}{\partial \bar{y}_i} P^*_{D^* \cdot 1},$$

invoking the same arguments leading to (48). On the other hand,

$$\frac{\partial x^*_D}{\partial \bar{y}_i} = \left( e_i + \frac{\partial q^*}{\partial \bar{y}_i} \tilde{s}_D^* \right) (I - P^* \cdot (I - P^*)^{-1}.$$

from the expression of $x^*_D$. Then we can solve for $\frac{\partial q^*}{\partial \bar{y}_i}$ and $\frac{\partial x^*_D}{\partial \bar{y}_i}$ by substituting (52) into (51). □
A.3 Proofs for the Results in Section 4

Indeed, we can establish a more general result about the contagion estimates than Theorem 12. Assume that a multiple of failures are already caused by the shock on bank 1 in the banking system. Denote $\mathcal{D}$ to be a collection of all those defaulting banks. $1 \notin \mathcal{D}$. Let $e_i^D$ be the equity value of bank $i$, after all the banks in $\mathcal{D} \cup \{1\}$ sell out the illiquid holdings, i.e.,

$$
e_i^D = \left( \beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i \right) \vee 0.
$$

Let $Z^D = (z_{ij})_{i,j \in \mathcal{D}^c} = (I_{\mathcal{D}^c} - P_{\mathcal{D}^c})^{-1}$. We have

**Theorem 15.** For any $j \notin \mathcal{D} \cup \{1\}$,

$$
P(\text{Bank } j \text{ defaults} | \mathcal{D} \cup \{1\} \text{ default}) \geq P \left( Y_1 - e_1^D > \frac{\sum_{i \notin \mathcal{D}, i \neq 1} e_i^D z_{ij}}{z_{1j}} \right).
$$

Moreover,

$$
E(\# \text{ of infected defaults} | \mathcal{D} \cup \{1\} \text{ default}) \geq \sum_{j \notin \mathcal{D}} P \left( Y_1 - e_1^D > \frac{\sum_{i \notin \mathcal{D}, i \neq 1} e_i^D z_{ij}}{z_{1j}} \right).
$$

It is easy to see that Theorem 12 is a special case of the above theorem by taking $\mathcal{D} = \emptyset$.

**Proof.** Proof of Theorem 15. Given that the banks in $\mathcal{D} \cup \{1\}$ default in the ultimate equilibrium, their liquidation amounts of the illiquid security should be $s_i = \bar{s}_i$ for $i \in \mathcal{D} \cup \{1\}$. Consider any $k \neq 1$, the equilibrium repayment of bank $k$ when bank 1 receives a shock of size $Y_1$, satisfies

$$
x_k \leq \beta_k + y_k + s_k Q(|s_\mathcal{D}| + s_1 + |s_{\mathcal{D} \setminus \{1\}}|) + \sum_{j \in \mathcal{D}} x_j p_{jk} + \sum_{j \in \mathcal{D}^c} x_j p_{jk}
$$

$$
\leq \beta_k + \bar{y}_k + \bar{s}_k Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \sum_{j \in \mathcal{D}} \ell_j p_{jk} + \sum_{j \in \mathcal{D}^c} x_j p_{jk},
$$

where the first inequality is due to the limited liability condition in the definition for market clearing equilibrium, and the second inequality uses the facts that $s_i = \bar{s}_i$ for $i \in \mathcal{D} \cup \{1\}$, $y_k \leq \bar{y}_k$, $s_k \leq \bar{s}_k$, and $Q(\cdot)$ is decreasing. Meanwhile, the repayment of bank 1

$$
x_1 \leq \beta_1 - Y_1 + \bar{y}_1 + \bar{s}_1 Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \sum_{j \in \mathcal{D}} \ell_j p_{j1} + \sum_{j \in \mathcal{D}^c} x_j p_{j1}.
$$

Rewriting (53) and (54) in a matrix form, we have

$$
x_{\mathcal{D}^c} \leq \beta_{\mathcal{D}^c} - Y_1 e_1 + \bar{y}_{\mathcal{D}^c} + \bar{s}_{\mathcal{D}^c} Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \ell_{\mathcal{D}} P_{\mathcal{D}, \mathcal{D}^c} + x_{\mathcal{D}^c} P_{\mathcal{D}^c, \mathcal{D}^c}.
$$

From the definition of $e^D$,

$$
e_i^D = \left( \beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i \right) \vee \beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_\mathcal{D}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i,
$$

48
which implies, for any \( i \in D^c \),

\[
\beta_i + \bar{y}_i + \hat{s}_i Q(|\bar{s}_D| + s_1) \leq c^P_i + \ell_i - \sum_{j \in D} \ell_j p_{ji} - \sum_{j \in D^c} \ell_j p_{ji}.
\]

Substituting the above inequality into (55), we have

\[
x_{D^c} \leq e^{P^c}_{D} - Y_1 e_1 + \ell_{D^c} (I_{D^c} - P_{D^c}) + x_{D^c} P_{D^c}.
\]

(56)

In junction of the non-negativeness of the matrix \( Z^D = (I_{D^c} - P_{D^c})^{-1} \), the inequality (56) implies that

\[
x_{D^c} \leq (e^{P^c}_{D} - Y_1 e_1) Z^D + \ell_{D^c}.
\]

In particular, for any \( j \notin D \cup \{1\} \),

\[
x_j \leq e^P_j z_{1j} + \sum_{i \in D^c, i \neq 1} e^P_i z_{ij} - Y_1 z_{1j} + \ell_j.
\]

(57)

On the other hand, note that the condition

\[
Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j}
\]

implies that

\[
(Y_1 - e^P_1) z_{1j} > \sum_{i \notin D, i \neq 1} e^P_i z_{ij}.
\]

(58)

Combining (57) and (58) will lead to \( x_j < \ell_j \), in other words, bank \( j \) defaults. So far we have established that

\[
\{\text{Banks in } D \cup \{1\} \text{ default}\} \cap \left\{ Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j} \right\} \subseteq \{\text{Bank } j \text{ defaults}\}.
\]

Therefore,

\[
P(\text{Bank } j \text{ defaults}) \geq P(\text{Banks in } D \cup \{1\} \text{ default, } Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j})
\]

(59)

It is easy to show that the events,

\[
\{\text{Banks in } D \cup \{1\} \text{ default}\} \text{ and } \left\{ Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j} \right\}
\]

are positively correlated in the sense that

\[
P \left( \text{Banks in } D \cup \{1\} \text{ default, } Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j} \right) \geq P(\text{Banks in } D \cup \{1\} \text{ default}) P \left( Y_1 - e^P_1 > \sum_{i \notin D, i \neq 1} e^P_i z_{ij} z_{1j} \right).
\]

(60)

In fact, recall that we established in the proof of Theorem 8 that the equilibrium repayment \( x \) is increasing with respect to the value of \( \beta \). Therefore, for two shocks \( Y_1 \) and \( Y_1' \), \( Y_1 < Y_1' \), we have \( x \geq x' \), where \( x \) and
$x'$ are the corresponding equilibrium repayments under the two shocks respectively. If the banks in $\mathcal{D} \cup \{1\}$ default under shock $Y_1$, i.e., $x_i < \ell_i$ for all $i \in \mathcal{D} \cup \{1\}$, then $x'_i < \ell_i$, meaning that this bank will also fail under a larger shock. In this sense, the indicator function $1_{\{\text{Banks in } \mathcal{D} \cup \{1\} \text{ default}\}}$ is an increasing function of $Y_1$. Meanwhile, $1_{\{Y_1 > a\}}$ is obviously an increasing function in $Y_1$. Invoking Proposition 7.2.1 of [36], we know that the inequality (60) must be true. From (59),

$$P(\text{Bank } j \text{ defaults} | \mathcal{D} \cup \{1\} \text{ default}) \geq P \left( Y_1 - e^D_1 > \sum_{i \notin \mathcal{D}, i \neq 1} e^D_i z_{ij} \right).$$

□

B Details of Network Reconstruction

We use the entropy-minimizing estimation method developed in [41] to assign values to the liability matrix $L$ in order to recovery banking systems consistent with the EBA data.

Consider structure type A first. Define a matrix $Y = (y_{ij})$ such that $y_{ij} = l_ia_j$, the product of bank $i$’s interbank liability and bank $j$’s interbank asset, for all $i$ and $j$. Such $Y$ should be corresponding to a complete-graph structure in which interbank liabilities and assets are independently distributed among the banks. However, it may not be a feasible liability matrix consistent with the data we observe from the EU report; in other words, the matrix $Y$ may violate the following constraint: $L_{ii} = 0$,

$$l_i = \sum_{j: j \neq i} L_{ij} \quad \text{and} \quad a_j = \sum_{i: i \neq j} L_{ij}, \quad (61)$$

where both $l_i$ and $a_j$ can be read from the column of interbank EAD in Table 3. To correct this, we attempt to find a matrix $L$, which satisfies (61), as close as possible to the complete structure specified by $Y$, through minimizing the relative entropy between them. That is, we solve the minimization problem as follows:

$$\min_L \sum_{i,j} L_{ij} \ln(L_{ij}/y_{ij}) \quad (62)$$

subject to the constraint (61), $L_{ii} = 0$, and $L_{ij} \geq 0$ for all $i, j$.

To create a network of structure type B, we specify a sparse configuration for $Y$. The idea is to concentrate the liability exposures of one bank to one of its neighboring bank, as long as it does not exceed the minimum of the total amounts of the interbank liabilities and assets of both banks. This requires us to force the column/row sums of $Y$ to its super- and sub-diagonal entries as much as possible. More formally, we use a greedy algorithm as follows to define $Y$: let $y_{ii} = 0$ for all $i$ and

for $i = 1$ to $n$ do
  for $j = 1$ to $n$ do
    if $(j \neq i)$ then
      $y_{ij} \leftarrow \min\{l_i, a_j\}$;
      $l_i \leftarrow l_i - y_{ij}$ and $a_j \leftarrow a_j - y_{ij}$.

The algorithm can be viewed as a variant of the northwest corner rule in the literature of transportation problem or Monge problem. Under some regularity conditions, people use the rule to couple two random
variables with known marginal distributions so as to produce largest covariance between them; see Chapter 8 of [28] for a more comprehensive discussion. Once the matrix $Y$ is obtained, we substitute it into the optimization program (62) to find a feasible liability matrix $L$.

Finally, we proceed to present how we construct a network with a core-periphery structure, which may resemble the market reality more closely. [41] find strong evidences from the balance sheets data at the end of 1998 that the German interbank deposit market was organized in two tiers. The lower tier consisted of saving and cooperative banks, and the upper tier consists of the head institutions of two giro systems (Landesbanken and cooperative central banks) and commercial banks. The banks in the lower tier had very few direct linkages with banks in the same tier, whereas the upper tier banks maintained wide lending relationship with a variety of other banks including banks in other categories.

The information about the types of the counterparty banks in interbank lending transactions is available in their dataset so that they can identify the upper tier banks from their estimation more precisely. In contrast, our dataset provides very limited information about the banks identification. We group the 11 banks into 2 classes simply according to the sizes of their interbank EAD: DE019, DE020 and DE021 as the core and the remaining 8 as the periphery. The above specification of the core and periphery banks surely is not an accurate reflection of the real market situation. The underlying assumption is that we think the large EAD values of the specified core banks should be resulted by their wide bilateral exposures to the other banks. We also try different specifications in the experiments and find that they do not have qualitative impacts on our conclusion.

After the classification of core and periphery banks is fixed, we let $y_{ij} = 0$ if banks $i$ and $j$ both belong to the periphery class and $y_{ij} = l_i a_j$ otherwise. The program (62) is invoked again to solve for a feasible liability matrix $L$. Tables 6-8 show the reconstruction outcomes under all the three structure types.

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Table 6: The Liability matrix of the complete network.
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Table 7: The Liability matrix of the ring-link network.

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Table 8: The Liability matrix of the core-periphery network.