Interconnected Balance Sheets, Market Liquidity, and the Amplification Effects in a Financial System

Nan Chen

(Joint Work with David Yao (Columbia) and Xin Liu (CUHK))

Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong

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Systemic Risk

- Systemic risk: the risk of collapse of an entire financial system due to its ill structure.
  - The risk starts from the failure of single entity or a cluster of entities
  - Interconnectedness among entities propagates such idiosyncratic risk to the whole system
- The objective of our work attempts to understand the amplification effect of the financial system, especially the contribution of two major factors to the systemic risk:
  - interconnected balance sheets of the banking system (Network Effect)
  - market liquidity (Liquidity Effect)
The Financial Network Becomes Complex: Transaction Level

- Comparison between the traditional and modern banking systems

**Traditional Banking System**
- households
- Mortgage bank
- households

  - mortgage
  - deposits

**Modern Banking System**
- households
- Mortgage pool
- CDO issuer
- Security firm
- Commercial Bank
- Money market fund

  - mortgage
  - Mortgage-backed securities
  - CDO
  - repo
  - MMF shares
  - Commercial papers
The Financial Network Becomes Complex: Interbank Market Level

- A diagram of the UK interbank exposure network (Financial Stability Report (2013))
The Financial Network Becomes Complex: Global Level


Figure: 1980  
Figure: 2007
Market Liquidity

- Market liquidity constitutes another channel transmitting the systemic risk.
  - An unexpected loss causes problem in the liquidity of a firm. It may need to start asset sales to meet its obligation.
  - However, other institutions may also be in stress at the same time and rely on the market to liquidate their assets to gain access to additional cash. And the market has only limited liquidity to absorb.
A Vicious Circle Amplified by the Market Liquidity

- The asset selling of many financial institutions creates a liquidity spiral (Brunnermeier (2009, JEP), Brunnermeier and Pedersen (2009, RFS), Adrian and Shin (2010, JFI)):
The contribution of our work are two-fold

- **Insights:**
  - Two multipliers, **network multiplier** and **liquidity amplifier**, are identified to characterize analytically the amplification effects.
  - Liquidity plays a dominating role in triggering large scale contagion under the current market environment.
  - Capital as a metric of systemic resilience.
  - Policy assessment: direct asset purchase and capital injection

- **Method:**
  - Equilibrium-constrained optimization formulation
  - Partition algorithm
  - Sensitivity analysis
Literature Review on Financial Networks and Contagion

- Some early literature: Economic origin of financial contagion

- Network effect:

- Liquidity and fire sale:

- Some related work:
  - Cifuentes, Ferrucci, and Shin (2005) and Nier et al. (2007): simulation
  - Amini, Filipovic and Minca (2013): CCP
Review of Eisenberg and Noe (2001)

Timeline:
- Time 0: An interbank lending-and-borrowing system is established
- Time 1 (Armageddon, maybe?): All debts mature; the system reaches a market-clear repayment equilibrium
There are $N$ banks, indexed by $1, 2, \cdots, N$, and a non-leveraged sector in the economy.

For bank $i$, its balance sheet at time 0 is given by

<table>
<thead>
<tr>
<th>The Balance Sheet of Bank $i$ at Date 0</th>
</tr>
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<tbody>
<tr>
<td><strong>Assets</strong></td>
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<tr>
<td>End-user Loan: $\bar{y}_i$</td>
</tr>
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<td>Interbank Loans: $\bar{x}_{i,k}$ (some $k \neq i$)</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Let $l_i$ indicate the nominal amount of total liabilities of bank $i$:

$$l_i = \sum_{j} \bar{x}_{j,i} + \bar{x}^E_i.$$

Let $\Pi$ denote the relative liability matrix in which $\pi_{ji} = \bar{x}_{ij}/l_j$. 
Review of E-N: Illustration

Date 0

Banking system

Bank 1

Bank 2

Bank 3

Non-leveraged Sector

Date 1

Banking system

Bank 1

Bank 2

Bank 3

Non-leveraged Sector

\( \hat{y}_3 \)
Review of E-N: Repayment Rules at Time 1

- Loan repayment at time 1 obeys the following rules:
  A. Residual claimant: debt claims have priority than equity.
  B. Limited liabilities: each banks pays at most its available assets.
  C. In the default of a bank, all the interbank loans and the external debt claims are with the same seniority.

- Given the loan repayments from the external borrowers are \( \hat{y} = (\hat{y}_1, \cdots, \hat{y}_N) \), \( \hat{y}_i \leq \bar{y}_i \), the interbank repayments in equilibrium for each bank is then determined by

\[
p_i = \min \left\{ l_i, \hat{y}_i + \sum_{j \neq i} p_j \pi_{ji} \right\}.
\]

- **Repayment from nonleverage sector**
- **Repayment from other banks**
In a vectorial form, the equilibrium repayment $p$ is the solution to the following fixed-point problem:

$$p = \min \{l, \hat{y} + p\Pi\}.$$  

Equilibrium existence and multiplicity:
In light of this self-fulfilling complexity, we focus on the largest equilibrium in this talk. Equivalently, we are interested in the following optimization problem:

\[
\text{max } \sum_i p_i \\
\text{s.t. } p = \min\{l, \hat{y} + p\Pi\}
\]

The above problem can be further reduced to a linear programming:

\[
\begin{align*}
\text{Primal:} & \quad \max \ p \mathbf{1} \\
& \quad \text{s.t. } p \leq l (\eta) \iff (l - \Pi)\delta + \eta \geq 1 \\
& \quad p \leq \hat{y} + p\Pi (\delta) \\
\text{Dual:} & \quad \min \ \hat{y}\delta + l\eta \\
& \quad \text{s.t. } \delta \geq 0, \ \eta \geq 0
\end{align*}
\]
The optimal solution to the LP takes the following form: there exist a partition of the set of banks \( \{1, 2, \cdots, n\} \) into two complementary subsets \( \mathcal{D} \) and \( \mathcal{N} \), \( \mathcal{D} = \{i : p_i < \ell_i\} \), \( \mathcal{N} = \{i : p_i = \ell_i\} \), such that

**Primal sol:** \( \mathbf{p}_\mathcal{N} = \mathbf{l}_\mathcal{N}; \quad \mathbf{p}_\mathcal{D} = (\hat{\mathbf{y}}_\mathcal{D} + \mathbf{l}_\mathcal{N} \Pi_{\mathcal{N}, \mathcal{D}})(I_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1} \).

**Dual sol:** \( \delta_\mathcal{D} = (I_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1} \mathbf{1}_\mathcal{D}; \quad \delta_\mathcal{N} = \mathbf{0}_\mathcal{N}; \)
\( \eta_\mathcal{D} = \mathbf{0}_\mathcal{D}; \quad \eta_\mathcal{N} = \mathbf{1}_\mathcal{N} + \Pi_{\mathcal{N}, \mathcal{D}}(I_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1} \mathbf{1}_\mathcal{D} \)

**Network multiplier:**

\( \delta_\mathcal{D} = (I_{\mathcal{D}} - \Pi_{\mathcal{D}})^{-1} \mathbf{1}_\mathcal{D} = (I_{\mathcal{D}} + \Pi_{\mathcal{D}} + \Pi^2_{\mathcal{D}} + \cdots) \mathbf{1}_\mathcal{D} \)
Partition Algorithm

Round 1

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \emptyset$
- Equilibrium Candidate: $p = 1$
- Feasibility Checking: $l \leq \hat{y} + l\Pi$?
  (Equity := $\hat{y} + l\Pi - l \geq 0$?)
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Partition Algorithm

Round 2

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I}$
- Equilibrium Candidate: $p_\mathcal{N} = I_{\mathcal{N}}$, $p_\mathcal{D} = \hat{y}_\mathcal{D} + I_{\mathcal{N}} \Pi_{\mathcal{N},\mathcal{D}} + p_\mathcal{D} \Pi_\mathcal{D}$
- Feasibility Checking: $I_{\mathcal{N}} \leq \hat{y}_\mathcal{N} + I_{\mathcal{N}} \Pi_{\mathcal{N}} + p_\mathcal{D} \Pi_{\mathcal{D},\mathcal{N}}$?
  
  (Equity $:= \hat{y}_\mathcal{N} + I_{\mathcal{N}} \Pi_{\mathcal{N}} + p_\mathcal{D} \Pi_{\mathcal{D},\mathcal{N}} - I_{\mathcal{N}} \geq 0$?)
  
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Partition Algorithm

Round 3

- **Partition Candidate:** $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I} + \mathcal{II}$
- **Equilibrium Candidate:** $p_{\mathcal{N}} = l_{\mathcal{N}}$, $p_{\mathcal{D}} = \hat{y}_{\mathcal{D}} + l_{\mathcal{N}}\Pi_{\mathcal{N},\mathcal{D}} + p_{\mathcal{D}}\Pi_{\mathcal{D}}$
- **Feasibility Checking:**
  $l_{\mathcal{N}} \leq \hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}}$?  
  (Equity := $\hat{y}_{\mathcal{N}} + l_{\mathcal{N}}\Pi_{\mathcal{N}} + p_{\mathcal{D}}\Pi_{\mathcal{D},\mathcal{N}} - l_{\mathcal{N}} \geq 0$? )
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
An Extended Model with Market Liquidity

▶ Now we consider an extended model to incorporate the influence of market liquidity. For bank $i$, its balance sheet at time 0 is given by

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<td>Liquid Assets: $\bar{a}_i$</td>
</tr>
<tr>
<td>Illiquid Assets: $\bar{s}_i$</td>
</tr>
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</table>

▶ The selling price of illiquid assets is influenced by the market demand. Its inverse demand function is

$$q = Q\left(\sum_{i=1}^{N} s_i\right),$$

where $q$ is the market price of one unit illiquid asset, $\sum_{i=1}^{N} s_i$ is the aggregate supply, and $Q(0) = 1$, $Q(\cdot)$ monotonically decreasing.

▶ Example: $Q(y) = \exp(-\gamma y)$, $\gamma$: market depth

▶ Liquid asset can be sold at its face value.
Given the realization of external repayment $\hat{y} = (\hat{y}_1, \cdots, \hat{y}_N)$, the liquidation amounts of liquid and illiquid assets $a = (a_1, \cdots, a_N)$ and $s = (s_1, \cdots, s_N)$, the income for each bank is

$$\hat{y}_i + \sum_{j \neq i} p_j \pi_{ji} + (a_i + s_i q)$$

where $q = Q(\sum_{i=1}^{N} s_i)$.
Market Equilibrium

Definition
A quadruplet \((p^*, a^*, s^*, q^*)\) is called a market equilibrium if for all \(i\),

1. (Limited liability) \(p_i = \min\{l_i, \hat{y}_i + \sum_{j \neq i} p_{j}^* \pi_{ji} + a_{i}^* + s_{i}^* q^*\}\);
2. (No short sale of assets) \(a_{i}^* \in [0, \bar{\alpha}_i]\) and \(s_{i}^* \in [0, \bar{s}_i]\);
3. (Sale of liquid asset occurs first)

\[
\begin{align*}
  a_{i}^* &= \min \left\{ \max \left\{ p_{i}^* - \left( \hat{y}_i + \sum_{j \neq i} p_{j}^* \pi_{ji} \right), 0 \right\}, \bar{\alpha}_i \right\} \\
  \text{and} \\
  s_{i}^* &= \min \left\{ \frac{\max \left\{ p_{i}^* - \left( \hat{y}_i + \sum_{j \neq i} p_{j}^* \pi_{ji} \right) - l_i, 0 \right\}}{q^*}, \bar{s}_i \right\}.
\end{align*}
\]

4. (Market clearing) \(q^* = Q(\sum_{i=1}^{N} s_{i}^*)\).
Main Result I: Existence and Uniqueness of the Equilibria

- Multiple equilibria caused by limited liquidity
- We can formulate the following optimization problem with equilibrium constraints to solve for the largest equilibrium \((p^*, a^*, s^*, q^*)\):

\[
\begin{align*}
\max & \quad q \\
\text{s.t.} & \quad (p, a, s, q) \text{ satisfy constraints 1-4.}
\end{align*}
\]

- An extended partition algorithm is developed to solve the above optimization problem. We have, \(p^* \geq p\), \(a^* \leq a\), \(s^* \leq s\), and \(q^* \geq q\), for any other equilibrium \((p, a, s, q)\).
Revisiting Partition Algorithm in the Presence of Liquidity

Round 1

- Partition Candidate: $\mathcal{N} = D^c$, $\mathcal{D} = \emptyset$
- Equilibrium Candidate: $p = l$, $q^1 = 1$
- Feasibility Checking:
  
  $\text{Equity} := \hat{y} + \bar{a} + \bar{s} \cdot q^1 + l\Pi - l \geq 0$?

  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Round 2

- **Partition Candidate:** $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I}$
- **Equilibrium Candidate:** $p_\mathcal{N}, p_\mathcal{D}, q^2$
- **Feasibility Checking:**
  
  Equity $:= $
  
  $\hat{y}_\mathcal{N} + \bar{a}_\mathcal{N} + q^2 \bar{s}_\mathcal{N} + l_\mathcal{N} \cdot \Pi_\mathcal{N} +$
  $p_\mathcal{D} \cdot \Pi_{\mathcal{D},\mathcal{N}} - l_\mathcal{N} \geq 0$

  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Round 3

- Partition Candidate: $\mathcal{N} = \mathcal{D}^c$, $\mathcal{D} = \mathcal{I} + \mathcal{II}$
- Equilibrium Candidate: $p_{N}, p_{D}, q^3$
- Feasibility Checking: 
  Equity :=
  $$\hat{y}_N + \bar{a}_N + q^3\bar{s}_N + l_N\Pi_N + p_{D}\Pi_{D,N} - l_N \geq 0?$$
  - Yes: terminate
  - No: Include the violators into $\mathcal{D}$
Define a shock hits bank 1, and its size is \( \Delta y_1 = \bar{y}_1 - \hat{y}_1 \).

▶ Round 1:

\[
P(\text{Bank 1 Defaults}) \geq P(\Delta y_1 \geq e_1^{(1)})
\]

where \( e_1^{(1)} \) is the capital market value of bank 1 before this round:

\[
e_1^{(1)} := \hat{y}_1 + \bar{a}_1 + \bar{s}_1 + \sum_{k=1}^{n} l_k \pi_{k1} - l_1.
\]

▶ Round 2:

\[
E(\# \text{ of Defaults in this round} | \text{Bank 1 default}) \geq \sum_{j \sim 1} P(\Delta y_1 \geq e_1^{(2)} + \frac{e_j^{(2)}}{\pi_{1j}}),
\]

where \( e_j^{(2)} \) is the capital market value of bank j before this round:

\[
e_j^{(2)} := \hat{y}_j + \bar{a}_j + \bar{s}_j q^2 + \sum_{j=1}^{n} l_k \pi_{k1} - l_1.
\]
Contagion Estimation and Capital

A generic round:

\[ E(\text{# of Defaults in this round}|\text{All the banks in } \mathcal{D} \text{ default}) \geq \sum_{j \in N(\mathcal{D})} P \left( \Delta y_1 > \frac{e_D(l_D - P_D)^{-1}P_{D,j}}{e_1(l_D - P_D)^{-1}P_{D,j}} + \frac{e_j}{e_1(l_D - P_D)^{-1}P_{D,j}} \right). \]
Main Result II: Structure of the Largest Equilibrium

- The algorithm can help us classify the whole banking system in equilibrium into four subsets:
  - \( \mathcal{N} = \{ i : p_i = l_i, \ a_i = 0, \ s_i = 0 \} \) (Banks which can repay their liabilities without liquidating assets).
  - \( \mathcal{M} = \{ i : p_i = l_i, \ 0 < a_i < \bar{a}_i, \ s_i = 0 \} \) (Banks which have to liquidate part of their liquid asset to repay the liabilities).
  - \( \mathcal{L} = \{ i : p_i = l_i, \ a_i = \bar{a}_i, \ 0 < s_i < \bar{s}_i \} \) (Banks which have to liquidate part of their illiquid asset to repay the liabilities).
  - \( \mathcal{D} = \{ i : p_i < l_i \} \) (Banks in default)
- The repayment vectors of these three subsets of banks satisfy the following equations:

\[
\begin{align*}
\mathbf{p}^*_\mathcal{D} &= [\hat{\mathbf{y}}_{\mathcal{D}} + \mathbf{\bar{a}}_{\mathcal{D}} + q\bar{s}_{\mathcal{D}} + \mathbf{l}_{\mathcal{D}c} \Pi_{\mathcal{D}c,\mathcal{D}}][\mathbf{l}_{\mathcal{D}} - \Pi_{\mathcal{D}}]^{-1}, \\
\mathbf{p}^*_\mathcal{L} &= \hat{\mathbf{y}}_{\mathcal{L}} + \mathbf{\bar{a}}_{\mathcal{L}} + q\bar{s}_{\mathcal{L}} + \mathbf{p}_{\mathcal{D}} \Pi_{\mathcal{D},\mathcal{L}} + \mathbf{l}_{\mathcal{D}c} \Pi_{\mathcal{D}c,\mathcal{L}} = \mathbf{l}_{\mathcal{L}}, \\
\mathbf{p}^*_\mathcal{M} &= \hat{\mathbf{y}}_{\mathcal{L}} + \mathbf{a}_{\mathcal{L}} + \mathbf{p}_{\mathcal{D}} \Pi_{\mathcal{D},\mathcal{M}} + \mathbf{l}_{\mathcal{D}c} \Pi_{\mathcal{D}c,\mathcal{M}} = \mathbf{l}_{\mathcal{M}}, \\
\mathbf{p}^*_\mathcal{N} &= \mathbf{l}_{\mathcal{N}}.
\end{align*}
\]
Main Result III: Liquidity Amplifier

For $i \in L$, 

$$
\frac{\partial q^*}{\partial \hat{y}_i} = \frac{\gamma}{1 - \gamma(s_L \mathbf{1} + \bar{s}_D[l_D - \Pi_D]^{-1}\Pi_{D,L}\mathbf{1})}
$$

We refer to the above quantity as the **liquidity amplifier**. To understand this, note 

$$
\frac{\gamma}{1 - \gamma S} = \gamma + \gamma(\gamma S) + \gamma(\gamma(\gamma S))S + \cdots .
$$
Main Result IV: Policy Assessment

Take two intervention policies as an example:

- Direct asset purchase:
- Capital injection

**Figure**: Direct Asset Purchase

**Figure**: Capital Injection
Main Result IV: Policy Assessment (Con’d)

▶ Asset Purchase at Market price

\[
\frac{\partial q^*}{\partial \Delta} \bigg|_{\text{Market}} = \frac{\gamma}{1 - \gamma(s_L 1 + \bar{s}_D [l_D - \Pi_D]^{-1}\Pi_D, L 1)},
\]

\[
\frac{\partial p^*}{\partial \Delta} \bigg|_{\text{Market}} = LA \cdot \bar{s}_D^*(l_D^* - P_{D^*})^{-1}.
\]

▶ Capital Injection:

\[
\frac{\partial q^*}{\partial \Delta} \bigg|_{\text{Capital}} = LA \cdot e_i(l_D^* - P_{D^*})^{-1}P_{D^*}, L^* 1,
\]

\[
\frac{\partial p^*}{\partial \Delta} \bigg|_{\text{Market}} = e_i(l_D^* - P_{D^*})^{-1} + LA \cdot e_i(l_D^* - P_{D^*})^{-1}P_{D^*}, L^* 1 \cdot \bar{s}_D^*(l_D^* - P_{D^*})^{-1}.
\]

▶ Asset Purchase at Face Value
Main Result IV: Policy Assessment (Con’d)

- **Policy implication:**
  - **Asset purchase focuses on liquidity improvement**
  - **Total asset value**
    
    \[
    \text{Total Asset Value (TAV) After Purchase} = \text{TAV Before Purchase} + \Delta - q^* \cdot \frac{\Delta}{q^*}
    \]
    
    \[
    = \text{TAV Before Purchase}.
    \]

- **Liquidity ratio:**
  
  \[
  \text{LR After Purchase} = \frac{\text{Liquidity} + \Delta}{\text{TAV}}
  \]

- **Capital injection eyes more on network effect**
  - **Total asset value increases by \(\Delta\)**
  - **Liquidity ratio:**
    
    \[
    \text{LR After Purchase} = \frac{\text{Liquidity} + \Delta}{\text{TAV} + \Delta}
    \]
Numerical Example: Data

- We use the data set from the European Banking Authority’s 2011 stress test.

<table>
<thead>
<tr>
<th>Bank Number and Name</th>
<th>Total Asset</th>
<th>Interbank Asset</th>
<th>Non-Interbank Asset</th>
<th>Equity</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE017 DEUTSCHE BANK AG</td>
<td>1905630</td>
<td>194399</td>
<td>1711231</td>
<td>30361</td>
<td>1875269</td>
</tr>
<tr>
<td>DE018 COMMERZBANK AG</td>
<td>771201</td>
<td>138190</td>
<td>633011</td>
<td>26728</td>
<td>744473</td>
</tr>
<tr>
<td>DE019 LANDESBank BADEN</td>
<td>374413</td>
<td>133906</td>
<td>240507</td>
<td>9838</td>
<td>364575</td>
</tr>
<tr>
<td>DE020 DZ BANK AG DT. ZEN</td>
<td>323578</td>
<td>135860</td>
<td>187718</td>
<td>7299</td>
<td>316279</td>
</tr>
<tr>
<td>DE021 BAYERISCHE LANDES</td>
<td>316354</td>
<td>97336</td>
<td>219018</td>
<td>11501</td>
<td>304853</td>
</tr>
<tr>
<td>DE022 NORDDEUTSCHE LANDE</td>
<td>228586</td>
<td>91217</td>
<td>137369</td>
<td>3974</td>
<td>224612</td>
</tr>
<tr>
<td>DE023 HYPO REAL ESTATE</td>
<td>328119</td>
<td>29084</td>
<td>299035</td>
<td>5539</td>
<td>322580</td>
</tr>
<tr>
<td>DE024 WESTLB AG, DüSSELD</td>
<td>191523</td>
<td>58128</td>
<td>133395</td>
<td>4218</td>
<td>187305</td>
</tr>
<tr>
<td>DE025 HSH NORDBANK AG, H</td>
<td>150930</td>
<td>9532</td>
<td>141398</td>
<td>4434</td>
<td>146496</td>
</tr>
<tr>
<td>DE027 LANDESBank BERLIN</td>
<td>133861</td>
<td>49253</td>
<td>84608</td>
<td>5162</td>
<td>128699</td>
</tr>
<tr>
<td>DE028 DEKABANK DEUTSCHE</td>
<td>130304</td>
<td>41255</td>
<td>89049</td>
<td>3359</td>
<td>126945</td>
</tr>
</tbody>
</table>
Numerical Example: Three Network Configuration

- The detailed specification of network structures are not available in the raw data.
- We recover it using the relative-entropy based algorithm in Upper and Worms (2004, EER).
  - To find the bilateral exposures between banks, let
    \[
    L = \begin{bmatrix}
      0 & \cdots & l_{1j} & \cdots & l_{1N} \\
      \vdots & \ddots & \vdots & \ddots & \vdots \\
      l_{i1} & \cdots & 0 & \cdots & l_{iN} \\
      \vdots & \ddots & \vdots & \ddots & \vdots \\
      l_{N1} & \cdots & l_{Nj} & \cdots & 0
    \end{bmatrix}.
    \]
  - We solve the following optimization problem:
    \[
    \min_{i,j} \sum_{i,j} l_{ij} \ln(l_{ij}/x_{ij})
    \]
    subject to \( \sum_j l_{ij} = \) observed interbank asset and \( \sum_i l_{ij} = \) observed interbank liabilities.
Numerical Example: Three Network Configurations (Con’d)

- Three kinds of network configurations for the core banks are examined.

**Figure:** Complete Network

**Figure:** Ring-Like Network

**Figure:** Core-Periphery Network
Numerical Example: Market Environment

- $Q(s) = \exp(-\gamma s)$.
- Baseline case:
  - $\gamma = 1 \times 10^{-7}$
  - 30% of the total asset of each bank is illiquid

Under this choice, the market price of the illiquid asset will drop to $0.864 from its face value $1$ when every bank in the system depletes its holdings.
Numerical Example: Liquidity Contagion

We let one of the banks receive shocks on its asset in the following experiment, i.e., $\hat{y}_1 = (1 - \theta)\bar{y}_1$, $\theta \in [0, 1]$.  

\[
\begin{align*}
\hat{y}(1\%, \text{yr}) &= (1 - \theta)\bar{y}(1\%, \text{yr}) \\
\hat{y}(2\%, \text{yr}) &= (1 - \theta)\bar{y}(2\%, \text{yr}) \\
\hat{y}(5\%, \text{yr}) &= (1 - \theta)\bar{y}(5\%, \text{yr})
\end{align*}
\]
Numerical Example: Liquidity Contagion (Con’d)

- No significant contagion in the absence of market liquidity.
- Introducing illiquid asset into the banking system amplifies the contagion effect significantly.
- Network structure matters when contagion kicks off.
- Our results echo the concern of Andrea Enria, Chairperson of the European Banking Authority, in his opening statement for Publication of the 2011 EU-wide Stress Test Results:

> Where a core tier one ratio falls above the benchmark but close to 5pc and a bank has sizeable exposures to sovereigns under stress, the supervisor must oversee that it takes specific steps to strengthen its capital position, including where necessary restrictions on dividends, deleveraging, issuance of fresh capital or conversion of lower quality instruments into core tier 1 capital.
Numerical Examples: Sensitivity of Market Depth $\gamma$

We change the market-depth parameter $\gamma$ in the system in the following picture.
Numerical Examples: Sensitivity of Market Price relative to the Shock Size

We consider the market-clearing price $q$ and $\partial q / \partial \theta$
Concluding Remarks

- We present an equilibrium-constrained optimization formulation to model the systemic risk.
- Our research identifies network and liquidity multipliers explicitly and points out different roles played by these two factors in systemic risk development.
Numerical Example: Three Network Configuration (Con’d)

Liability matrices under different configurations:

<table>
<thead>
<tr>
<th>Network Structure</th>
<th>Liabilities</th>
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Liability matrices:

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Liability matrix under complete network structure
Liability matrix under star network structure
Liability matrix under ring network structure
Multiple Equilibria under Limited Liquidity

- Two bank system
  - Balance sheets at time 1
    
    | Bank 1                          | Bank 2                          |
    |--------------------------------|--------------------------------|
    | External investments $\hat{y}_1 = \$0.1$ | External debt $b_1 = \$1$         |
    | Illiquid Asset $\bar{s}_1 = 1$     | Equity $e_1$                      |
    |                                | External debt $b_2 = \$1$         |
    |                                | Equity: $e_2$                      |
    |                                | Illiquid Asset $\bar{s}_2 = 2$     |

- $Q(s) = \exp(-s)$
- Equilibrium equations:
  
  $$s_1 = 1 \wedge \frac{0.9}{q}, \quad s_2 = 2 \wedge \frac{0.1}{q}, \quad \text{and} \quad q = \exp(-(s_1 + s_2))$$

- Two equilibria: $(s_1, s_2) = (1, 0.4092)$ and $(s_1, s_2) = (1, 2)$