The Principle of Not Feeling the Boundary for the SABR Model

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The stochastic-alpha-beta-rho (SABR) model is widely used in fixed income and foreign exchange markets as a benchmark. The underlying process may hit zero with a positive probability and therefore an absorbing boundary at zero should be specified to avoid arbitrage opportunities. However, a variety of numerical methods choose to ignore the boundary condition to maintain the tractability. This paper develops a new principle of not feeling the boundary to quantify the impact of this boundary condition to the distribution of underlying prices. It shows that the probability of the SABR hitting zero decays to 0 exponentially as the time horizon shrinks. Applying this principle, we further show that conditional on the volatility process, the distribution of the underlying process can be approximated by that of a time-changed Bessel process with an exponentially negligible error. This discovery provides a theoretical justification for many almost exact simulation algorithms for the SABR model in the literature. Numerical experiments are also presented to support our results.

Keywords: SABR model, Probability of hitting zero, Principle of not feeling the boundary, Time-changed Bessel process

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1. Introduction

The stochastic-alpha-beta-rho (SABR) model introduced in Hagan et al. (2002) has become very popular with practitioners in interest rate and foreign exchange markets for valuing European-style options. It can produce analytical asymptotic expressions for implied volatility, fitting the observed smile reasonably well and capturing the correct co-movement between the smile dynamics and the underlying asset price.

The model is a special class of stochastic volatility models. In particular, its underlying process is given by a constant-elasticity-of-variance (CEV) type diffusion and its volatility process follows a geometric Brownian motion. Exactly due to this structural feature, one can show that the underlying process may hit zero with positive probability. Therefore, we have to specify an absorbing boundary condition at zero to avoid arbitrage opportunities (see, e.g. Delbaen and Shirakawa, 2002; Rebonato et al., 2009). However, some intensively used approximate formulas for the Black-model implied volatility given by SABR, such as those in Hagan et al. (2002), Oblój (2008), and Paulot (2015), were derived by simply ignoring the boundary condition for the pricing partial differential equation (PDE) system to maintain mathematical tractability. This therefore poses one interesting research problem upon us: how should we quantify the impact of the boundary condition?

To address this issue, we manage to develop a principle of not feeling the boundary in this paper. It shows that the probability of the SABR model hitting zero decays exponentially as the time horizon shrinks. Furthermore, the convergence rate of this hitting probability to zero largely depends on the modeling parameters: it becomes faster for a model with larger initial underlying price or beta (i.e. the index of the CEV component of the SABR model) or smaller initial volatility or the volatility of volatility. To the best of our knowledge, we characterize the exponentially decaying order of hitting probability for the first time.

Early works on the principle of not feeling the boundary can be traced back to Kac (1951) and Varadhan (1967), where the authors investigated the case of diffusions on Euclidean space generated by second order, uniformly elliptic operators with Hölder continuous coefficients. Hsu (1995) extended the discussion to diffusions on a more general manifold. In terms of its financial applications, Gatheral et al. (2012) use the principle to explain why the boundary behavior of local volatility models will not affect the asymptotic expansions of the transition density and the European call price written on it. This paper contributes to this literature because we are the first to present a rigorous characterization about the exponentially decaying order of the hitting probability in the case of the SABR, one important class of stochastic volatility models. We also note

\footnote{For example, if the correlation is zero and the parameter “beta” is less than 1/2 (see the SDE (1) for the SABR model), then the mass at zero is positive.}
there are several works relating to this topic. For instance, Doust (2012) computed the probability of hitting zero via Monte Carlo simulations; Bayer et al. (2013) cited this principle without any rigorous establishment to argue for the validity of their computational method in the SABR model; Gulisashvili et al. (2016) derived a formula of the hitting probability under a special case—the uncorrelated SABR model, and referred to the small hitting probability for large initial values as the principle in a numerical example, but without indicating the decaying order; using the PDE based method, Yang and Wan (2016) obtained asymptotic formulas with a polynomial error bound for the survival probability (i.e. the probability of not hitting a nonnegative lower boundary) by solving a hierarchy of PDEs. It is worth noting that in a recent literature, Hagan et al. (2014) proposed a numerical scheme to solve a simplified one-dimensional pricing PDE for the SABR model with considering the boundary conditions at zero.

Intuitively, our result implies that the specification of boundary conditions will have a limited influence to the distributional law of the SABR model for small time. That explains why a variety of numerical methods for the SABR model performs well for short-run option pricing, even they do not incorporate the boundary condition into consideration. As the second layer of contributions to the literature, the paper also develops some theoretical bounds on the bias of some almost exact SABR simulation algorithms that recently emerged in the literature (see, e.g. Chen and Liu 2011, Chen et al. 2012, Cai et al. 2017, Leitao et al. 2017). This research line of simulation stems from Islah (2009), in which the author found that the marginal distribution of the underlying price in SABR can be approximated by a noncentral chi-square distribution conditional on the volatility process. This approximation turns out to be quite accurate if we use it to compute short-term option prices. Meanwhile, the aforementioned papers also report significant approximation error in the simulated outcomes as the time horizon gets longer. Applying the principle of not feeling the boundary, we manage to identify the cause of such error — the impact of the absorbing boundary condition starts to kick in when we consider a long time horizon. Along this line, we fill the gap in the existing literature by presenting an analysis that the approximation error will be exponentially negligible as the time horizon shrinks.

The rest of this paper is organized as follows. Section 2 introduces the SABR model, followed by the main results about the principle of not feeling the boundary. Some numerical evidences are also presented to support our discovery. All the proofs are deferred to Section 3. We conclude the paper in Section 4.
2. The SABR Model and The Main Results

2.1. The SABR Model

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a filtered probability space, where \(\mathbb{P}\) is the \(T\)-forward martingale measure. Two independent standard Brownian motions \(\{B_t; 0 \leq t \leq T\}\) and \(\{W_t; 0 \leq t \leq T\}\) are defined on \((\Omega_1, \mathcal{F}_1)\) and \((\Omega_2, \mathcal{F}_2)\) with their natural filtrations \(\{\mathcal{F}_1^t\}\) and \(\{\mathcal{F}_2^t\}\), respectively. Let the sample space \(\Omega\), the \(\sigma\)-algebra \(\mathcal{F}\), and the filtration \(\mathcal{F}_t\) be \(\Omega = \Omega_1 \times \Omega_2\), \(\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2\), and \(\mathcal{F}_t = \mathcal{F}_1^t \otimes \mathcal{F}_2^t\). Denote \(F_t\) and \(A_t\) to be the forward price and volatility at time \(t \in [0, T]\), respectively. The SABR model is then defined as a solution to the following system of stochastic differential equations (SDEs):

\[
\begin{align*}
    dF_t &= A_t F_t^{\beta} [\sqrt{1 - \rho^2} dB_t + \rho dW_t], \\
    dA_t &= \nu A_t dW_t,
\end{align*}
\]

where the parameter beta \(\beta\) and the correlation correlation \(\rho\) satisfy \(\beta \in (0, 1)\) and \(\rho \in (-1, 1)\), respectively; the forward price \(F_0\), the initial volatility \(A_0\), and the volatility of volatility \(\nu\) are positive. It is a local stochastic volatility model, in which the forward price process \(\{F_t; 0 \leq t \leq T\}\) follows a CEV-type diffusion process and the dynamic of the volatility process \(\{A_t; 0 \leq t \leq T\}\) is given by a geometric Brownian motion.

We need to specify the boundary condition at \(F = 0\) for SDE (1) in order to determine the existence and uniqueness of the model, because the CEV-type dynamic specification in \(F\) allows it to hit zero with positive probability. A reflecting boundary will obviously lead to an arbitrage opportunity: one can buy the forward at zero cost when it hits 0 and sell it for profit when it reflects back to the positive region; refer to Section 3.10 of Rebonato et al. (2009) or Delbaen and Shirakawa (2002) for a detailed discussion on the issue. To rule out the arbitrage opportunity, we thus impose the following assumption on the model from now on.

**Assumption 1** 0 is an absorbing boundary of \(\{F_t; 0 \leq t \leq T\}\).

We show that the solution to (1) uniquely exists under this assumption in Lemma 3.1.

2.2. The Main Results

Let

\[ \tau^F_0 = \inf \{t \in [0, T] : F_t = 0\}, \]
the first time the forward price process hits zero. The first main result of the paper establishes a probability bound on $\mathbb{P}(\tau_0^F \leq T)$. More specifically, we have

**Theorem 2.1 (Principle of Not Feeling the Boundary)** Under Assumption 1, there exists a positive constant $C$ (depending on $\nu$, $\beta$, $A_0$, $F_0$) such that,

$$
\limsup_{T \downarrow 0} T \ln \mathbb{P}(\tau_0^F \leq T) \leq -C.
$$

In words, the theorem states that the probability of the event that the forward price $F_t$ hits 0 before $T$ will vanish exponentially as $T$ tends to zero. Since the SABR model (1) is a diffusion process changing its value continuously over time, intuitively Theorem 2.1 implies that the existence of the boundary at zero will not affect the probability law of $F_t$ for small time. In this sense, we refer to it as the principle of not feeling the boundary.

It is worthwhile pointing out that using such principle to quantify the impact of the boundary to the probability law of a diffusion process is familiar to probabilists. Kac (1951) pioneered the study for Brownian motion. Varadhan (1967) considered the case of diffusion processes in an Euclidean space generated by second order, uniformly elliptic operators. Hsu (1995) extended the principle to diffusions on a general manifold. One interesting application of the principle in option pricing appeared in Gatheral et al. (2012), in which the authors used it to obtain asymptotic expansions about the transitional probability function of a local volatility model and the associated call option price.

Turn to the implications of Theorem 2.1 on the SABR model. Define a function $g(\cdot)$ such that for $F \geq 0$,

$$
g(F) = \frac{F^{1-\beta}}{1-\beta}.
$$

Let $X_t = g(F_t)$. Applying the local Itô formula (Kallenberg 1997, Corollary 15.20) up to the stopping time $\tau_0^F$, we have

$$
X_{T \wedge \tau_0^F} = X_0 + \frac{\rho}{\nu} (A_{T \wedge \tau_0^F} - A_0) + \sqrt{1 - \rho^2} \int_0^{T \wedge \tau_0^F} A_s dB_s + \int_0^{T \wedge \tau_0^F} \frac{(1 - 2\theta)(1 - \rho^2)A_s^2}{2X_s} ds,
$$

with $X_0 = g(F_0)$ and

$$
\theta = \frac{1}{2} + \frac{\beta}{2(1 - \beta)(1 - \rho^2)}.
$$

Note that the function $g$ defined above is invertible. We have $F_T = F_{T \wedge \tau_0^F} = g^{-1}(X_{T \wedge \tau_0^F})$. To derive the probability law of $F_T$, it suffices to determine the probability law of $X_{T \wedge \tau_0^F}$.
Along the sample path that satisfies $\tau_F^G > T$, the representation of $X$ in (3) reduces to

$$X_T = X_0 + \frac{\rho}{\nu} (A_T - A_0) + \sqrt{1 - \rho^2} \int_0^T A_s dB_s + \int_0^T \frac{(1 - 2\theta)(1 - \rho^2)A^2_s}{2X_s} ds.$$  \hspace{2cm} (5)

Conditioning on the volatility process $\{A_t; 0 \leq t \leq T\}$, the distribution law of $X_T$ given by (5) should be the same as the marginal distribution of a time-changed Bessel process of parameter $(1 - 2\theta)/2$ at $T$, starting from $X_0 + \rho/\nu (A_T - A_0)$ and with the changed time clock $(1 - \rho^2) \int_0^T A^2_s ds$.

Since the probability of $\{	au_F^G \leq T\}$ is exponentially negligible for small time $T$ according to Theorem 2.1, we expect that the probability distribution of $X_T$ should provide a good approximation to the distribution of $X_{T \wedge \tau}$.

Lemma 3.4 in Section 3 presents explicitly the cumulative distribution function of a time-changed Bessel process in terms of noncentral chi-square distributions. Inspired by all the above observations, we introduce a new random variable (r.v.) $\tilde{F}_T$ whose conditional distribution, given both the values of $A_T$ and $\int_0^T A^2_s ds$, satisfies

$$\mathbb{P} \left( \tilde{F}_T \leq g(U) \bigg| \int_0^T A^2_s ds, A_0, A_T \right) = \begin{cases} 1 - Q \left( \frac{\tilde{g}^2(F_0)}{\Delta}; 2\theta, \frac{g^2(U)}{\Delta} \right), & U > 0; \\ 1 - Q \left( \frac{\tilde{g}^2(F_0)}{\Delta}; 2\theta \right), & U = 0, \end{cases}$$  \hspace{2cm} (6)

for any $U \geq 0$, where

$$\Delta = (1 - \rho^2) \int_0^T A^2_s ds$$  \hspace{2cm} (7)

and

$$\tilde{g}(F_0) := \left( X_0 + \frac{\rho}{\nu} (A_T - A_0) \right)^+ = \left( g(F_0) + \frac{\rho}{\nu} (A_T - A_0) \right)^+. \hspace{2cm} (8)$$

Here, $Q(x; \mu, \lambda)$ is the cumulative distribution function of a noncentral chi-square random variable with degree of freedom $\mu$ and noncentrality $\lambda$. $Q(x; \mu)$ is its degenerate special case when $\lambda = 0$.

The following theorem characterizes the error bound if we use the aforementioned r.v. $\tilde{F}_T$ to build up an approximation to the original model $F_T$. More precisely, we have

**THEOREM 2.2 (Approximate Conditional Marginal Distribution)** Suppose Assumption 1 holds. For any Lipschitz function $h(\cdot)$, there exists a positive constant $C$ (depending on $h$, $\nu$, $\beta$, $\rho$, $A_0$, $F_0$) such that

$$\limsup_{T \downarrow 0} T \ln \left| \mathbb{E} \left[ h(F_T) | A_0, F_0 \right] - \mathbb{E} \left[ \mathbb{E} \left[ h(\tilde{F}_T) \bigg| \int_0^T A^2_s ds, A_T \right] | A_0, F_0 \right] \right| \leq -C. \hspace{2cm} (9)$$

In (9), the first expectation $\mathbb{E} \left[ h(F_T) | A_0, F_0 \right]$ is taken with respect to the original SABR model;
the inner one in the second iterated expectations is computed from the probability distribution given in (6) and the outer one is taken with respect to the joint distribution of \( \int_0^T A_s^2 ds \) and \( A_T \). Roughly, we know from this theorem

\[
\left| \mathbb{E}[h(F_T)|A_0, F_0] - \mathbb{E}\left[ h(\tilde{F}_T) \left| \int_0^T A_s^2 ds, A_T \right. \right] \right| A_0, F_0 \right| \leq \exp\left(-\frac{C}{T}\right),
\]

i.e. the difference of these two terms will vanish exponentially as \( T \to 0 \). Specifically, if the correlation is zero, i.e. \( \rho = 0 \), then the difference of the above two expectations is zero. Because the marginal distribution of the forward price is exactly given by (6), which is an immediate consequence of the results of Islah (2009), Cai et al. (2017), and Leitao et al. (2017).

The iterated expectation in (9) can be evaluated efficiently through Monte Carlo simulation. To this end, we may use the following 3-step procedure:

**Step 1** Given \( A_0 \), simulate \( A_T \).

**Step 2** Draw a sample of \( \int_0^T A_s^2 ds \), given \( A_0 \) and \( A_T \).

**Step 3** Given \( A_0 \), \( A_T \), and \( \int_0^T A_s^2 ds \), simulate \( \tilde{F}_T \) from the distribution (6).

Several papers in the literature, including Chen and Liu (2011), Cai et al. (2017), and Leitao et al. (2017), developed different simulation schemes to materialize these steps. We include the detail of the algorithm presented in Cai et al. (2017) in Appendix A. All of these papers document accurate performance of the above approximation in numerical experiments when it is used to compute option prices written on the SABR model, especially for short-term options. But none produces any theoretical guarantees. Theorem 2.2 fills the gap by showing that the approximation error in using these Monte Carlo simulation schemes to price short-term options under the SABR model is exponentially negligible.

At the end of this subsection, we need to stress that the discussion ahead of Theorem 2.2 is not rigorous. More strict proofs of the two theorems in this subsection can be found in Section 3.

### 2.3. Numerical Evidences

In this subsection we shall provide more numerical evidences about the principle of not feeling the boundary and its implications in option pricing. Note that Eq. (2) in Theorem 2.1 implies that \( \ln(\mathbb{P}(r_0^F \leq T)) \) is in proportion to \( 1/T \). To numerically corroborate this discovery, Figure 1 displays the relationship between the logarithm of the hitting probability and the reciprocal of the maturity under different values of initial forward price \( F_0 \), beta, initial volatility \( A_0 \), and volatility of volatilities \( \nu \). The values of parameters we use as the benchmark in this experiment are \( F_0 = 0.1, A_0 = 0.2, \beta = 0.1, \nu = 0.1, \) and \( \rho = -0.5 \), respectively. We change the value of
one parameter in each subfigure while fixing the others. All the data points fall onto straight lines, strongly suggesting that the principle of not feeling the boundary holds for the SABR model under a variety of parameter values. Moreover, the slope of the lines, i.e., the constant on the right hand side of (2), characterizes how fast the probability decays as $T \to 0$. A clear pattern arises in Figure 1 that $P(\tau_0^F \leq T)$ converges to 0 faster for the SABR model with a larger initial forward price, larger beta, smaller initial volatility, or smaller volatility of volatility.

![Figure 1](image1.png)

Figure 1. The linear relation between $\log(P(\tau_0^F \leq T))$ and $1/T$ under different values of $F_0$, $A_0$, $\beta$, and $\nu$. The hitting probability of the SABR model is computed from the finite difference method, see Appendix B for more details.

Figure 2 shows the impact of the boundary on the option pricing. Using the values computed from the finite-difference method as the “true” option price, we plot the relative error in the computation of the simulation scheme presented in the last subsection. For a given $T$, the error tends to be significant as $F_0 \to 0$, $A_0 \to +\infty$, or $\nu \to +\infty$. This is consistent with the conclusions from the two theorems in the last subsection. As noted before, $P(\tau_0^F \leq T)$ becomes not negligible for a model with small $F_0$, large $A_0$ or $\nu$, and will lead to the failure of the distribution approximation presented in Theorem 2.2. In addition, through the comparison between $T = 1/2$ and $T = 1$, we can see that the boundary condition should have a smaller impact to the error of the simulation-based pricing schemes for a shorter time horizon.
Figure 2. The relative errors of the simulation under different values of $F_0$, $A_0$, and $\nu$. Consider at-the-money call option. We fix $\beta = 0.1$ and $\rho = -0.5$ in all three subfigures. The other parameter values used in this example are $A_0 = 0.3$ and $\nu = 0.1$ in (a), $F_0 = 0.1$ and $\nu = 0.1$ in (b), $F_0 = 0.1$ and $A_0 = 0.1$ in (c), respectively. We compute the benchmark price $D$ using the finite difference method in Appendix B. The simulation algorithm of Cai et al. (2017) is used to generate Monte Carlo price $S$. The relative error is defined as $|S - D|/D$. The number of simulation trials is 10,000.

3. Proofs

This section provides proofs for Theorems 2.1 and 2.2. In Section, 3.1 we present some technical lemmas for subsequent analysis. The proofs for Theorems 2.1 and 2.2 are presented in Sections 3.2, and 3.3, respectively. For convenience, we will use the following notations throughout this section.

- $C$ is a generic positive constant.
- $C(\varpi)$ is a generic positive constant depending on the parameter vector $\varpi$, which can be one or a group of $\beta$, $\nu$, $F_0$, and so on. The explicit dependence will be indicated in the following analysis.
3.1. Technical Lemmas

**Lemma 3.1** (Strong Solution up to Explosion) Under Assumption 1, the SABR model (1) has a unique strong solution up to the explosion time \( S \), where \( S = \inf\{t > 0 : F_t = 0\} \).\(^1\)

**Proof.** Under Assumption 1, Lions and Musiela (2007) and Hobson (2010) have proved that the equation system (1) admits a unique weak solution up to explosion. Moreover, the solution to the SDE (1) exists in a strong sense. Note that the diffusion coefficients are locally Lipschitz. We then have strong uniqueness for the solution by simply following the proof of Theorem 5.2.5 in Karatzas and Shreve (1991). Therefore, the SDE (1) exists a strong solution up to explosion, which is implied by the weak existence and strong uniqueness (Karatzas and Shreve 1991, Corollary 5.3.23). \( \square \)

**Lemma 3.2** Consider the volatility process \( \{A_t; 0 \leq t \leq T\} \) in the SABR model (1). Define

\[
\Omega^{X_0}_T = \left\{ \inf_{s \in [0,T]} \left( X_0 + \frac{\rho}{\nu} (A_s - A_0) \right) \leq 0 \right\}.
\]

Let \( C_a = \frac{\nu X_0}{\rho A_0} \) if \( \rho \neq 0 \). Then, for (i) \( \rho = 0 \), (ii) \( \rho > 0 \) and \( C_a \geq 1 \), we have \( \Omega^{X_0}_T = \emptyset \); otherwise,

\[
P(\Omega^{X_0}_T) \leq \frac{1}{\sqrt{1-C_a}} \frac{\nu \sqrt{T}}{\ln(1-C_a)} \exp \left( -\frac{\ln^2(1-C_a)}{2\nu^2 T} \right).
\]

**Proof.** We first consider the case \( \rho > 0 \). Note that \( A_s = A_0 \exp(-\nu s/2 + W_s) \) for all \( s \in [0,T] \).

Then,

\[
\inf_{s \in [0,T]} (X_0 + \frac{\rho}{\nu} (A_s - A_0)) \leq 0 \iff \inf_{s \in [0,T]} \exp(-\nu s/2 + W_s) \leq 1 - C_a.
\]

If \( C_a \geq 1 \), then \( \Omega^{X_0}_T = \emptyset \). Thus, the lemma holds obviously. If \( 0 < C_a < 1 \), then

\[
\Omega^{X_0}_T = \left\{ \inf_{s \in [0,T]} (-\nu s/2 + W_s) \leq \frac{\ln(1-C_a)}{\nu} \right\}.
\]

This set corresponds to the event that a drifted Brownian motion \( \{-\nu s/2 + W_s : s \geq 0\} \) hits the level \( b := \ln(1-C_a)/\nu < 0 \) before \( T \). By the distribution of first passage times of the drifted Brownian motion (Karatzas and Shreve 1991, formula (3.5.12)), we have

\[
P(\Omega^{X_0}_T) = \int_0^T \frac{-b}{\sqrt{2\pi s^3}} \exp \left( -\frac{b^2 + \nu bs + \nu^2 s^2/4}{2s} \right) ds \leq \int_0^T \frac{-b}{\sqrt{2\pi s^3}} \exp \left( -\frac{b^2}{2s} \right) \cdot \exp \left( -\frac{\nu b}{2} \right) ds,
\]

where the inequality is due to the fact that \( \exp(-\nu^2 s/8) \leq 1 \). Letting \( z = -b/\sqrt{s} \), the above

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\(^1\)\( \{F_t; 0 \leq t \leq T\} \) has a finite moment (Lions and Musiela 2007, Andersen and Piterbarg 2007) for \( \beta \in (0,1) \), thus \( S = \lim_{n \to \infty} \{t > 0 : F_t \leq 1/n, or F_t \geq n\} \).
inequality yields that

$$
P(\Omega^Y_T) \leq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 - C_a}} \int_{-b/\sqrt{T}}^{+\infty} e^{-z^2/dz} dz.
$$

By the inequality in Problem 2.9.22 of Karatzas and Shreve (1991), we have (11). A similar argument applies to the case when \( \rho \leq 0 \).

**Lemma 3.3** Consider the volatility process \( \{A_t; 0 \leq t \leq T\} \) in the SABR model (1). Recall that \( \Delta = (1 - \rho^2) \int_0^T A_t^2 ds \) in (7). For arbitrary \( C_0 > \Delta_0 \equiv (1 - \rho^2)A_0^2T \), then we have

$$
P(\Delta \geq C_0) \leq C_1(\nu\sqrt{T}) \exp\left(-\frac{C_2}{\nu^2T}\right),$$

where \( C_1 = \frac{2\sqrt{2\pi}}{\ln(C_0/\Delta_0)} \) and \( C_2 = \frac{(\ln(C_0/\Delta_0))^2}{8} \).

**Proof.** Note that \( A_t = A_0 \exp(-\nu^2t + 2\nu W_t) \) for \( t \in [0, T] \). Then, we have

$$
P(\Delta \geq C_0) = \mathbb{P}\left(\int_0^T \exp(-\nu^2t) \exp(2\nu W_t) dt \geq \frac{C_0T}{\Delta_0}\right) < \mathbb{P}\left(\int_0^T \exp(2\nu \sup_{t \in [0, T]} W_t) dt \geq \frac{C_0T}{\Delta_0}\right).
$$

Therefore, using the density of the running maximum of Brownian motion (Karatzas and Shreve 1991, formulas (2.8.3)), we have

$$
P(\Delta \geq C_0) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} W_t \geq \frac{1}{2\nu} \ln\left(\frac{c}{\Delta_0}\right)\right) = \sqrt{\frac{2}{\pi}} \int_\frac{\ln(c/\Delta_0)}{2\nu\sqrt{T}}^{+\infty} e^{-z^2/2} \, dx \leq C_1(\nu\sqrt{T}) \exp\left(-\frac{C_2}{\nu^2T}\right).
$$

The last inequality holds due to the formula (2.9.20) in Karatzas and Shreve (1991). \( \square \)

**Lemma 3.4** (Time-Changed Bessel Process) For any given deterministic positive continuous function \( \varphi : [0, T] \to (0, +\infty) \), let

$$
Y_t = Y_0 + \int_0^t \varphi(s) dB_s + \int_0^t \left(1 - 2\theta\right) \frac{\varphi^2(s)}{2Y_s} ds,
$$

(12)

where \( B_t \) is a standard Brownian motion. For any fixed \( t > 0 \), the transition density of \( Y_t \) in (12), starting from \( Y_0 > 0 \), is given as follows:

$$
p_{\varphi}(t; Y_0, y) = \begin{cases} 
Y_0^{\theta} \left(1 - \theta\right) \int_0^{Y_0} \frac{\varphi^2(s) ds}{2Y_s} \exp\left(-\frac{Y_s^2 + y^2}{2\int_0^{Y_0} \varphi^2(s) ds}\right) I_\theta\left(\frac{Y_0}{\int_0^{Y_0} \varphi^2(s) ds}\right), & y > 0; \\
\frac{1}{\Gamma(1+\theta)} \Gamma\left(\theta, \frac{Y_0^2}{2\int_0^{Y_0} \varphi^2(s) ds}\right), & y = 0;
\end{cases}
$$

(13)

where \( \Gamma(\theta) = \int_0^{+\infty} x^{\theta-1} e^{-x} dx, \Gamma(\theta, z) = \int_z^{+\infty} x^{\theta-1} e^{-x} dx, \) and \( I_\theta(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\theta}}{m! \Gamma(m+\theta+1)} \). Moreover,
$Y_t$ admits the following distribution function:

$$
P(Y_t \leq y|Y_0) = \begin{cases} 
1 - Q \left( \frac{Y_t^2}{\int_0^s \varphi^2(s)}; 2\theta, \frac{y^2}{\int_0^s \varphi^2(s)} \right), & y > 0; \\
1 - Q \left( \frac{Y_t^2}{\int_0^s \varphi^2(s)}; 2\theta \right), & y = 0,
\end{cases} \tag{14}
$$

where $Q(x; \mu, \lambda)$ is the cumulative distribution function of a noncentral chi-square random variable with degree of freedom $\mu$ and noncentrality $\lambda$. $Q(x; \mu)$ is its degenerate special case when $\lambda = 0$.

**Proof.** Given any deterministic positive continuous function $\varphi : [0, T] \to (0, +\infty)$, note that the coefficients of the process $Y$ (cf. (12)), $\varphi(s)$ and $(1 - 2\theta)\varphi^2(s)/(2Y_s)$, are locally Lipschitz in the space variable $Y_s$ when $Y_s > 0$. Mimicking the proof of Theorem 5.2.5 of Karatzas and Shreve (1991), then we can easily show the strong uniqueness of the solution to (12) up to the explosion time $\tau^Y = \{ t \geq 0 : Y_t = 0 \}$. Let

$$
\phi(t) := \int_0^t \varphi^2(\gamma) d\gamma \quad \text{and} \quad \psi(t) := \inf\{ s > 0 : \phi(s) > t \}.
$$

Since $\varphi(\cdot)$ is strictly positive and continuous, we know that $\phi(t)$ and $\psi(t)$ are both continuous and monotonously increasing. Define $M_t = \int_0^{\psi(t)} \varphi(\gamma) dB_\gamma$. Note that $\langle M, M \rangle_t = \int_0^{\psi(t)} \varphi^2(\gamma) d\gamma = t$. From Theorem 3.3.16 of Karatzas and Shreve (1991), $\{M_t\}$ is a Brownian motion with respect to the filtration $\{ \mathcal{F}_t^{\psi(t)} : t \in [0, T] \}$. Given $Y_0 > 0$, we know that

$$
Z_t = Y_0 + M_t + \int_0^t \frac{1 - 2\theta}{2Z_s} ds \tag{15}
$$

is a Bessel process with dimension $2 - 2\theta$. Therefore, it is well known that the weak solution to (15) exists up to $\tau^Z = \inf\{ t \geq 0 : Z_t = 0 \}$, and under the assumption that $Z = 0$ is an absorbing boundary for the process, its transition density should be given by

$$
p^Z(t, Z_0, Z_t) = \begin{cases} 
\frac{Y_s^2 Z_0^{1-\theta}}{t} \exp \left( -\frac{Y_s^2 + Z_0^2}{4t} \right) I_\theta \left( \frac{Y_s Z_0}{t} \right), & Z_t > 0; \\
\frac{1}{\Gamma(1+\theta)} \Gamma \left( \theta, \frac{Y_s^2}{2t} \right), & Z_t = 0.
\end{cases} \tag{16}
$$

See Borodin and Salminen (2002) for detailed discussions on the Bessel process.

Let $Y_t = Z_{\phi(t)}$ for any $t \geq 0$. It is easy to see that

$$
Y_t = Y_0 + \int_0^t \varphi(\gamma) dB_\gamma + \int_0^t (1 - 2\theta) \frac{\varphi^2(\gamma)}{2Y_\gamma} d\gamma.
$$

So far we have shown the SDE (12) admits a weak solution. Combining with the strong uniqueness, we know that the existence of strong solution to (12). Furthermore, by (16), we can also see that the transition density of $Y$ is given by (13). Then, using (13), similar to the arguments in Appendix B of Yang et al. (2017), we can derive the cumulative distribution function in (14).
In the remark below, we compare Lemma 3.4 with Section 4.3 of Yang and Wan (2016) as well as Results 2.2 and 2.4 of Chen et al. (2012).

**Remark 1** In Section 4.3 of Yang and Wan (2016), they achieve a PDE with a small perturbation parameter, in which the infinitesimal generator of a Bessel process is the leading order operator. Using the transition density of a Bessel process, Yang and Wan (2016) solve a hierarchy of PDEs to obtain the asymptotic formulas for the probability that the forward price hits zero. Result 2.2 of Chen et al. (2012) reviews the transition density for a squared Bessel process (Borodin and Salminen 2002). Then, using Result 2.2, Chen et al. (2012) arrive at Result 2.4, which is originated from Islah (2009). However, the argument of Result 2.4, especially Eq. (2.17), is not correct because they have overlooked the stopping time $\tau^F_0$ when applying Itô’s formula. Lemma 3.4 presents the probability density function and cumulative distribution function of a time-changed Bessel process. With the help of Lemma 3.4, we then show in Theorem 2.2 that Result 2.4 of Chen et al. (2012) holds with an exponential negligible error.

### 3.2. Proof of Theorem 2.1

Recall $\{X_t; 0 \leq t \leq T\}$ defined in (3). Let $\rho^\perp = \sqrt{1 - \rho^2}$, and let $\tau_0 = \inf\{t \in [0, T] : X_t = 0\}$ be the first time that the process $\{X_t; 0 \leq t \leq T\}$ hits zero. Rewriting (3), then we have

$$X_{T \wedge \tau_0} = X_0 + \rho^\perp (A_{T \wedge \tau_0} - A_0) + \rho^\perp \int_0^{T \wedge \tau_0} A_s dB_s + \int_0^{T \wedge \tau_0} \frac{(1 - 2\theta)(\rho^\perp A_s)^2}{2X_s} ds. \quad (17)$$

Given a sample path of $\{A_t(\omega) : 0 \leq t \leq T\}$ from

$$\Omega := \left\{ \omega : \inf_{s \in [0, T]} \left( X_0 + \frac{\rho}{\nu} (A_s(\omega) - A_0) \right) > \bar{X}_0 \right\}, \quad (18)$$

consider a new process $\{\bar{X}_t\}$ satisfying the following SDE

$$\bar{X}_t = \bar{X}_0 + \rho^\perp \int_0^t A_s dB_s + \int_0^t \frac{(1 - 2\theta)(\rho^\perp A_s)^2}{2\bar{X}_s} ds, \quad (19)$$

where $\bar{X}_0 \in (0, X_0)$. The strong uniqueness and existence up to $\tau_0$ for $\{X_t; 0 \leq t \leq T\}$ is presented in Lemma 3.1. If we specify an absorbing boundary at zero for the SDE (19), Lemma 3.4 indicates that $\{\bar{X}_t; 0 \leq t \leq T\}$ must exist uniquely in a strong sense up to $\bar{\tau}_0$, where

$$\bar{\tau}_0 = \inf\{t \in [0, T] : \bar{X}_t = 0\}.$$

Conditional on $\Omega$ defined in (18), the initial point $X_0 + \rho/\nu (A_{T \wedge \tau_0} - A_0)$ is larger than $\bar{X}_0$. Similar
Therefore, conditional on \( \Omega \), we have that
\[
\{ \tau_0 \leq T \} \cap \{ \nu \leq \bar{\tau} \} \cap \Omega.
\]
More precisely, on the event \( \{ \tau_0 \leq T \} \cap \Omega \), if \( \tau_0 < \bar{\tau} \) for some sample paths, then by (20), we have \( 0 = X_{\tau_0} = X_{T \wedge \tau_0} \wedge \bar{\tau} \geq \bar{X}_{T \wedge \tau_0} \geq X_{\tau_0} > 0 \). Contradiction! This implies that
\[
\{ \tau_0 \leq T \} \cap \Omega = \{ \tau_0 \leq T, \tau_0 \geq \bar{\tau}_0 \} \cap \Omega \subseteq \{ \bar{\tau}_0 \leq T \} \cap \Omega.
\]
Combining the above formula and the law of total probability, we have
\[
P(\tau_0 \leq T) = P(\{ \tau_0 \leq T \} \cap \Omega) + P(\tau_0 \leq T | \Omega^c)P(\Omega^c) \leq P(\bar{\tau}_0 \leq T) + P(\Omega^c),
\]
where \( \Omega^c \) is the complementary set of \( \Omega \) defined in (18). From Lemma 3.2, we have
\[
P(\Omega^c) \leq 1_{(\rho \neq 0)}1_{(C_a < 1)} \frac{\nu \sqrt{T}}{\sqrt{1 - C_a}} \exp \left( -\frac{\ln^2(1 - C_a)}{2\nu^2T} \right),
\]
where \( C_a = \frac{\nu(X_0 - \bar{X}_0)}{\rho A_0} \) if \( \rho \neq 0 \).

By Lemma 3.4, \( \{ \bar{X}_t; 0 \leq t \leq T \} \) defined in (19) is also a time-changed Bessel process. Moreover,
\[
P(\bar{\tau}_0 \leq T) = \frac{E[\Gamma(\theta, \bar{X}_0^2/(2\Delta))]}{\Gamma(1 + \theta)}.
\]
Note that \( \max_{x > 0} x^\alpha e^{-x} \) is bounded for \( \alpha > 0 \), then there exists a positive constant \( C \) such that
\[
\Gamma(\theta, x) = \int_x^\infty z^{(\theta - 1)/2} e^{-z/2} \frac{1}{2\pi e^z} dz < Cx^\frac{\theta}{2} e^{-Cz}.
\]
Combining the above inequality with the Cauchy-Schwartz inequality, then we have
\[
P(\bar{\tau}_0 \leq T) < \frac{E\left[ C\Delta^{1/2}/\bar{X}_0 \cdot \exp(-C\bar{X}_0^2/\Delta) \right]}{\Gamma(1 + \theta)}
\]
\[
\leq C(\beta)/\bar{X}_0 \sqrt{E[\Delta]} \sqrt{E[\exp(-C\bar{X}_0^2/\Delta)]}
\]
\[
< C(\beta)/\bar{X}_0 \left( A_0 \sqrt{T} e^{\nu^2T/2} \left( E[\exp(-C\bar{X}_0^2/\Delta)(1_{\{\Delta > 2\Delta_0\}} + 1_{\{\Delta \leq 2\Delta_0\}})] \right) \right)^{1/2}.
\]
Furthermore, taking \( \bar{X}_0 = X_0/2 \) and applying Lemma 3.3, then we have
\[
P(\bar{\tau}_0 \leq T) < C(\beta)/X_0 \left( A_0 \sqrt{T} e^{\nu^2T/2} \left( E[1_{\{\Delta > 2\Delta_0\}}] + \exp(-C\bar{X}_0^2(A_0^2T)^{-1}) \right)^{1/2}
\]
\[
< C(\beta)/X_0 \left( A_0 \sqrt{T} e^{\nu^2T/2} \left( \nu \sqrt{T} \exp(-C(\nu \sqrt{T})^{-2}) + \exp(-C\bar{X}_0^2(A_0^2T)^{-1}) \right)^{1/2}. \quad (23)
\]
Therefore, combining (21), (22), and (23), we have
\[
P(\tau_0 \leq T) \leq C_1 \sqrt{T} (1 + e^{\nu^2 T/2} (1 + \nu \sqrt{T})^{1/2}) \exp(-C_2/T),
\]
where \( C_1 = \max \{ 1_{\rho=0} 1_{C_{\alpha}<1}^{\nu}, C(\beta)A_0 / \sqrt{1-C_{\alpha}}, \} \), \( C = \min \{ \ln(1-C_{\alpha})/4 \nu^2, C_2 A_0^{1/2} \}, \) and \( C_2 = \frac{\nu X_0}{\sqrt{2A_0}} 1_{\rho \neq 0} \).

Finally, taking logarithm on both sides of (24), we have
\[
\lim_{T \downarrow 0} \sup T \ln P(\tau_0 \leq T) \leq -C_1,
\]
where \( C_1 = C(\nu, \beta, A_0, F_0) \) is a positive constant. The proof completes. \( \square \)

**Remark 2** If \( \beta = 0 \), going through the above proof, we find the derivations are still valid. Thus the conclusion of Theorem 2.1 still holds for the case \( \beta = 0 \). It is worth to note that the last term on the right hand side (17) disappears, and Lemma 3.4 still holds without the drift term in (12) in this case.

### 3.3. Proof of Theorem 2.2

Recall the process \( X_t = g(F_t) \) defined in (17). Given a path of \( \{ A_t(\omega) : t \in [0, T]\} \), consider a new process \( \tilde{X}_t; 0 \leq t \leq T \) on the probability space \( (\Omega, \mathcal{F}) \) to approximate \( X_{\tau_0} \).

\[
\tilde{X}_T = \left( X_0 + \frac{\rho}{\nu} (A_T - A_0) \right)^+ + (1 - \rho^2) \int_0^T A_s dB_s + \int_0^T \frac{(1 - 2\theta)(1 - \rho^2) A_s^2}{2X_s} ds.
\]

Note that our construction is feasible because the driving Brownian motion \( W_t \) of the volatility process is independent of \( B_t \). Define \( \tilde{F}_T := g^{-1}(\tilde{X}_T) \). By Lemma 3.4, we know that the distribution function of \( \tilde{F}_T \) is given by (6).

We now use the distribution of \( \tilde{F}_T = g^{-1}(\tilde{X}_T) \) to approximate the distribution of \( F_T = g^{-1}(X_{\tau_0 \wedge T}) \) determined by (17). Note that the distributions of \( \tilde{F}_T \) and \( F_T \) are exactly the same if the correlation is zero (see, e.g. Islah 2009, Cai et al. 2017, Leitao et al. 2017). Therefore, we only need to prove the approximation error (9) holds when \( \rho \neq 0 \).

Let \( S_n = \inf \{ t \in [0, T] : X_t \leq 1/n \} \) or \( X_t \geq n \), \( \tilde{S}_n = \inf \{ t \in [0, T] : \tilde{X}_t \leq 1/n \} \) or \( \tilde{X}_t \geq n \}, \) and \( \sigma_n = S_n \wedge \tilde{S}_n \). Moreover, \( \lim_{n \to \infty} \sigma_n = \tau_0 \wedge \tilde{\tau}_0 \) where \( \tilde{\tau}_0 = \inf \{ t \in [0, T] : \tilde{X}_t = 0 \} \). Given a Lipschitz function \( h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+ \) and recalling \( g(\cdot) \) in (8), the composition \( h \circ g^{-1}(\cdot) \) is a locally Lipschitz function. Thus, we have

\[
\mathbb{E}[|h(F_T) - h(\tilde{F}_T)|] \leq C(\beta, h) \mathbb{E}[|X_T - \tilde{X}_T| 1_{\{\sigma_n \leq T\}}] + \mathbb{E}[|h \circ g^{-1}(X_T) - h \circ g^{-1}(\tilde{X}_T)| 1_{\{\sigma_n \leq T\}}].
\]

(26)
Note that
\[
\mathbb{E}[|X_T - \tilde{X}_T|1_{\{\sigma_n > T\}}] = \int_0^T \frac{(1 - \rho^2)A_t^2(1 - 2\theta)}{2} \mathbb{E} \left[ \left| \frac{1}{X_t} - \frac{1}{\tilde{X}_t} \right| 1_{\{\sigma_n > T\}} \right] dt \\
\leq \frac{(1 - 2\theta)(1 - \rho^2)n^2}{2} \int_0^T A_t^2 \mathbb{E} \left[ |X_t - \tilde{X}_t| 1_{\{\sigma_n > T\}} \right] dt.
\]
By the Gronwall’s inequality, we have
\[
\mathbb{E} \left[ |X_T - \tilde{X}_T| 1_{\{\sigma_n > T\}} \right] = 0.
\]
(27)

Combining (26) and (27), and letting \( n \to +\infty \), the following inequality holds
\[
\mathbb{E}[|h(F_T) - h(\tilde{F}_T)|] \leq \mathbb{E}[|h \circ g^{-1}(X_T) - h \circ g^{-1}(\tilde{X}_T)|1_{\{\tau_0 \wedge \tilde{\tau}_0 \leq T\}}].
\]
By the Lipschitz property of \( h(\cdot) \) and the definition of \( g(\cdot) \) in (8), we have
\[
\mathbb{E}[|h(F_T) - h(\tilde{F}_T)|] \leq C(h) \mathbb{E} \left[ F_T 1_{\{\tau_0 \wedge \tilde{\tau}_0 \leq T\}} \right] + C(\beta, h) \mathbb{E} \left[ X_T^{1/\beta} 1_{\{\tau_0 \wedge \tilde{\tau}_0 \leq T\}} \right].
\]
(28)
Recall the definition of \( \theta \) in (4). Define \( p := 2\theta(1 - \beta) \equiv 1 + \beta \rho^2/(1 - \rho^2) > 1 \) (\( \rho \neq 0 \)), and let \( q \) satisfy \( 1/p + 1/q = 1 \). By the Hölder inequality, we have
\[
\mathbb{E} \left[ F_T 1_{\{\tau_0 \wedge \tilde{\tau}_0 \leq T\}} \right] \leq (\mathbb{E}[F_T^p])^{1/p} \mathbb{P}(\tau_0 \wedge \tilde{\tau}_0 \leq T)^{1/q} < C(\beta, \rho, F_0) \mathbb{P}(\tau_0 \wedge \tilde{\tau}_0 \leq T)^{1/q},
\]
(29)
\[
\mathbb{E} \left[ X_T^{1/\beta} 1_{\{\tau_0 \wedge \tilde{\tau}_0 \leq T\}} \right] \leq (\mathbb{E}[X_T^{2\theta}])^{1/p} \mathbb{P}(\tau_0 \wedge \tilde{\tau}_0 \leq T)^{1/q}.
\]
(30)
where the second inequality in (29) holds because \( \mathbb{E}[F_T^p] < \infty \) (Andersen and Piterbarg 2007, Proposition 5.1). Lemma 3.4 indicates that \( \tilde{X}_T \) is a time-changed Bessel process. Letting \( \tilde{X}_0 = (X_0 + \frac{\rho}{\sqrt{\nu}}(A_T - A_0))^+ \), then we have
\[
\mathbb{E}[(\tilde{X}_T)^{2\theta}] = E \left[ \int_0^\infty \tilde{X}_0^{2\theta} \tilde{X}_T \Delta \left( \frac{\tilde{X}_0}{\tilde{X}_T} \right)^\theta \exp \left( -\frac{\tilde{X}_0^2 + \tilde{X}_T^2}{2\Delta} \right) I_\theta \left( \frac{\tilde{X}_0 \tilde{X}_T}{\Delta} \right) d\tilde{X}_T \right]
\]
\[
= E \left[ \tilde{X}_0^{2\theta} \int_0^\infty \frac{1}{2} \left( \frac{z}{\lambda} \right)^{\theta/2} \exp \left( -\frac{\lambda + z}{2} \right) I_\theta \left( \sqrt{\lambda z} \right) \right]_{\lambda = \tilde{X}_0^2/\Delta} dz
\]
\[
= \mathbb{E}[(\tilde{X}_0^{2\theta})],
\]
where the third equality holds due to the definition of a noncentral chi-square distribution’s density.
function. Furthermore, by the Minkowski inequality, we have
\[
E[\tilde{X}_T^2] = E\left[\left(\frac{\rho A_0}{\nu} + \frac{\rho}{\nu} A_T\right)^{2\theta}\right] \\
\leq \left(\left|X_0 - \frac{\rho A_0}{\nu}\right| + \left|\frac{\rho}{\nu} E[A_T^{2\theta}]\right|^{\frac{1}{2\theta}}\right)^{2\theta} < C(\nu, \beta, \rho, A_0, F_0).
\]

(31)

Therefore, combining (28), (29), (30), and (31), we have
\[
E[|h(F_T) - h(F_T)|] \leq C(\beta, \nu, A_0, F_0, h) (\mathbb{P}(\tau_0 \leq T) + \mathbb{P}(|\tilde{\tau}_0| \leq T))^{1/q}.
\]

(32)

Noting the formulas (2) and (23) in Theorem 2.1 and its proof, we have the conclusion in (9). □

Remark 3 The conclusion in Theorem 2.2 stills holds if we replace \( \tilde{g}(F_0) = (g(F_0) + \frac{\rho}{\nu}(A_T - A_0))^+ \) with \( \tilde{g}(F_0) = |g(F_0) + \frac{\rho}{\nu}(A_T - A_0)| \). This is because the probability that \( g(F_0) + \frac{\rho}{\nu}(A_T - A_0) \leq 0 \) is exponentially negligible for small \( T \) (see Lemma 3.2).

4. Conclusions

This paper develops the principle of not feeling the boundary to quantify the impact of an absorbing boundary at zero to the SABR model. More precisely, we have the probability of the SABR hitting zero decays to zero exponentially as the time horizon tends to zero. With the help of the principle, we demonstrate that the distribution of the forward price conditional on the volatility can be approximated by that of a time-changed Bessel process with an exponentially negligible error, which provides a theoretical justification for a variety of almost exact simulation algorithms recently emerged in the literature.

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References


Leitao, A., Grzelak, L. and Oosterlee, W., On an efficient multiple time step Monte Carlo simulation of the


Appendix A: Exact Simulation of the Approximate Distribution

This subsection presents a method to simulate the sample path based on the conditional approximate distribution given in Theorem 2.2. Specifically, if $\tilde{F}_T$ is determined by the density in (6) conditional on $A_0$, $A_T$, and $\Delta$, then the sample of $\tilde{F}_T$ can be generated exactly. The algorithm for the exact simulation of the approximate distribution is from Chen and Liu (2011) and Cai et al. (2017).

**Step 1. Sampling from the distribution of $A_T$, given $A_0$.** Recall (1), then, we have

$$A_T = A_0 \exp \left( -\frac{1}{2} \nu^2 T + \nu W_T \right) \overset{d}{=} A_0 \exp \left( -\frac{1}{2} \nu^2 T + \nu \sqrt{T} Z \right),$$

where $Z$ follows from the standard normal distribution. Thus, we can generate a standard normal random variable $Z \sim \mathcal{N}(0,1)$ instead of $A_T$.

**Step 2. Sampling from $\Delta$, given $A_0$ and $A_T$.** A Laplace transform inversion-based approach can be used to generate a sample from $\Delta$ conditional on $A_0$ and $A_T$ (Chen and Liu 2011, Section 2.2.2; Cai et al. 2017, Section 3.2). More precisely, let $h(x) = (1 - \rho^2)/x$ for $x > 0$. Recall $\Delta$ defined in (7). Denote

$$G_h(y) := \mathbb{P}(h(\Delta) \leq y | A_0, A_T) \equiv \mathbb{P} \left( \left( \int_0^T A_t^2 dt \right)^{-1} \leq y \mid A_0, A_T \right), \quad y \geq 0.$$
For $\xi > 0$, the Laplace transform of $G_h(\cdot)$ is given by (Matsumoto and Yor 2005, Cai et al. 2017)

$$\hat{G}_h(\xi) := \int_{\mathbb{R}_+} e^{-\xi y} G_h(y) dy = \frac{1}{\xi} \exp \left( -\phi_{\ln(A_T/A_0)}^2(\xi \nu^2/A_0^2) - \ln^2(A_T/A_0) / 2\nu^2 T \right),$$

where $\phi x(\lambda) = \arg \cosh(\lambda e^{-x} + \cosh(x))$, and $\arg \cosh(z) = \ln(z + \sqrt{z^2 - 1})$, $\cosh(z) = (e^z + e^{-z})/2$.

Therefore, we can obtain the function $G_h(\cdot)$ by numerically inverting the Laplace transform $\hat{G}_h(\cdot)$ via some algorithms such as Abate and Whitt (1992).

Generate a sample $U \sim U(0,1)$ from the standard uniform distribution. Find the root of the equation $G_h(V) = U$. Then $h^{-1}(V) = (1 - \rho^2)/V$ is a sample of $\Delta$ given $A_0$ and $A_T$.

Step 3. Sampling from the approximate distribution $\tilde{F}_T$, given $F_0$, $A_0$, $A_T$, and $\Delta$. Recall the approximate distribution of $F_T$, conditional on $F_0$, $A_0$, $A_T$, and $\Delta$, given by (6). Generate a sample $U \sim U(0,1)$; if $U \leq 1 - Q \left( \frac{\hat{c}(F_0)}{\Delta}; 2\theta \right)$, then set $\tilde{F}_T = 0$; otherwise, find $\hat{U}$ which solves

$$\mathbb{P} \left( \tilde{F}_T \leq \hat{U} \mid \Delta, A_0, A_T \right) = 1 - Q \left( \frac{\hat{c}(F_0)}{\Delta}; 2\theta, \frac{\hat{c}(\hat{U})}{\Delta} \right) = U,$$

and then set $\tilde{F}_T = \hat{U}$.

Appendix B: PDE for the Survival Probability and Call Option Price without Arbitrage

Let $\tau^F_t := \min\{s \geq t : F_s = 0\}$. Consider the following conditional expectation with a payoff function $h(\cdot)$

$$\varphi_h(t, f, a) = \mathbb{E}[h(F_T)1_{\{\tau^F_T > T\}} \mid F_t = f, A_t = a].$$

If $h(F) = (F - K)^+$, then $\varphi_h(t, f, a)$ corresponds to the price of a call without arbitrage (Yang et al. 2017, Eq. p(4)). If $h(F) = 1$, then $\varphi_h(t, f, a)$ denotes the probability that the forward price does not hit 0 before $T$, i.e. the survival probability.

Moreover, the function $\varphi_h(t, f, a)$ is the solution to the following PDE (Yang et al. 2017, Theorem 1; Yang and Wan 2016, Theorem 2.1):

$$\frac{\partial \varphi_h}{\partial t} + \frac{1}{2} \left( a^2 f^3 \beta \frac{\partial^2 \varphi_h}{\partial f^2} + 2\rho \nu a f \frac{\partial^2 \varphi_h}{\partial f \partial a} + \nu^2 a^2 \frac{\partial^2 \varphi_h}{\partial a^2} \right) = 0,$$

with boundary and terminal conditions

$$\varphi_h(t, 0, a) = 0, \quad \varphi_h(T, f, a) = h(f).$$
To obtain the benchmark for the call option price and the hitting (survival) probability, we numerically solve the PDE (B1) with boundary and terminal conditions (B2). Specifically, we use the Alternative Direction Implicit (ADI) algorithm proposed by In ’t Hout and Foulon (2010) to solve the related PDE; We truncate the region for \((F,A)\) to \([0,2] \times [0,2]\) and discretize 2500 and 200 steps for the variable \(F\) and \(A\), respectively. The number of steps for time is 500. All the numerical experiments are run in an environment of Matlab R2017b and a PC desktop with Intel(R) Core(TM)2 Quad CPU Q9400@2.66GHZ.