Lecture 4: Risk Neutral Pricing

1 Part I: The Girsanov Theorem

1.1 Change of Measure and Girsanov Theorem

- Change of measure for a single random variable:

  Theorem 1. *Let $(\Omega, \mathcal{F}, P)$ be a sample space and Z be an almost surely nonnegative random variable with $E[Z] = 1$. For $A \in \mathcal{F}$, define*

  $$\tilde{P}(A) = \int_A Z(\omega) dP(\omega).$$

  Then $\tilde{P}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

  $$\tilde{E}[X] = E[XZ].$$

  If Z is almost surely strictly positive, we also have

  $$E[Y] = \tilde{E}\left[\frac{Y}{Z}\right].$$

  Definition 1. *Let $(\Omega, \mathcal{F})$ be a sample space. Two probability measures P and $\tilde{P}$ on $(\Omega, \mathcal{F})$ are said to be* equivalent *if*

  $$P(A) = 0 \ \Leftrightarrow \ \tilde{P}(A) = 0$$

  for all such A.

  Theorem 2 (Radon-Nikodym)**.** *Let P and $\tilde{P}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable Z such that $E[Z] = 1$ and*

  $$\tilde{P}(A) = \int_A Z(\omega) dP(\omega)$$

  for $A \in \mathcal{F}$. We say Z is the Radon-Nikodym derivative *of $\tilde{P}$ with respect to P, and we write*

  $$Z = \frac{d\tilde{P}}{dP}.$$

  Example 1. *Change of measure on $\Omega = [0, 1]$.*

  Example 2. *Change of measure for normal random variables.*

- We can also perform change of measure for a whole process rather than for a single random variable. Suppose that there is a probability space $(\Omega, \mathcal{F})$ and a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$. Suppose further that $\mathcal{F}_T$-measurable Z is an almost surely positive random variable satisfying $E[Z] = 1$. We can then define the *Radon-Nikodym derivative process*

  $$Z_t = E[Z|\mathcal{F}_t], \ 0 \leq t \leq T.$$

Theorem 3. *Let t satisfying $0 \le t \le T$ be given and let Y be an $\mathcal{F}_t$-measurable random variable. Then*

$$\tilde{E}[Y] = E[YZ] = E[YZ_t].$$

Theorem 4. *Let t and s satisfying $0 \le s \le t \le T$ be given and let Y be an $\mathcal{F}_t$-measurable random variable. Then*

$$\tilde{E}[Y|\mathcal{F}_t] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s].$$

Theorem 5 (Girsanov)**.** *Let W_t, $0 \le t \le T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t$, $0 \le t \le T$, be a filtration for this Brownian motion. Let Θ_t be an adapted process. Define*

$$\begin{aligned} Z_t &= \exp\left\{-\int_0^t \Theta_u dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du\right\}, \\ \tilde{W}_t &= W_t + \int_0^t \Theta_u du, \end{aligned}$$

and assume that

$$E\left[\int_0^t \Theta_u^2 Z_u^2 du\right] < +\infty.$$

Set $Z = Z_T$. Then $E[Z] = 1$ and under the probability $\tilde{P}$ given by Theorem 5, the process $\tilde{W}$ is a Brownian motion.

2 Part II: Fundamental Theorem in Finance (Continuous-Time)

2.1 Risk Neutral Measure

- Consider again the standard market assumption that there is a stock whose price satisfies

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t.$$

In addition, suppose that we have an adapted interest rate process $r_t, t \ge 0$. The corresponding discount process follows

$$dD_t = -r_t D_t dt.$$

The discount stock price process is given by

$$d(D_t S_t) = \sigma_t D_t S_t [\Theta_t dt + dW_t]$$

where we define the *market price of risk* to be

$$\Theta_t = \frac{\alpha_t - r_t}{\sigma_t}.$$

We introduce a probability measure $\tilde{P}$ defined in Girsanov's theorem, which uses the market price of risk Θ_t. In terms of the Brownian motion $\tilde{W}_t$ of that theorem, we rewrite the discount stock price as

$$d(D_t S_t) = \sigma_t D_t S_t d\tilde{W}_t.$$

We call $\tilde{P}$ the *risk neutral measure.*

- Consider a trader who begins with initial wealth X_0 and at each time t, he buys Δ_t shares of the stock and invests or borrows at the interest rate r_t as necessary to finance this. Then, the total wealth of the trader X_t follows

  $$dX_t = r_t X_t dt + \Delta_t \sigma_t S_t [\Theta_t dt + dW_t].$$

  It implies that the discount wealth

  $$d(D_t X_t) = \Delta_t \Delta_t \sigma_t S_t [\Theta_t dt + dW_t].$$

  Under Girsanov theorem,

  $$d(D_t X_t) = \Delta_t \sigma_t S_t d\tilde{W}_t.$$

- We have shown in the last lecture that the trader can construct a portfolio in order to hedge a short position in the call position. In other words, we can find X_0 and Δ_t such that

  $$X_T = \max(S_T - K, 0).$$

  Note that $D_t X_t$ is a martingale under the risk neutral probability measure. Therefore,

  $$X_0 = \tilde{E}[D_T X_T] = \tilde{E}\left[e^{-\int_0^T r_t dt} \max(S_T - K, 0)\right].$$

- Revisit the Black-Scholes-Merton formula.

2.2 Martingale Representation Theorem

- Martingale representation theorem:

  Theorem 6. *Let $(\Omega, \mathcal{F}, P)$ be a sample space and W_t be a Brownian motion on it, and let $\mathcal{F}_t$ be the filtration generated by this Brownian motion. Let M_t be a martingale with respective to this filtration. Then there exists an adapted process Γ_t such that*

  $$M_t = M_0 + \int_0^t \Gamma_s dW_s$$

  for all $0 \le t \le T$.

  Theorem 7. *Let W_t be a Brownian motion on it, and let $\mathcal{F}_t$ be the filtration generated by this Brownian motion. Let Θ_t be an adapted process. Define*

  $$\begin{aligned} Z_t &= \exp\left\{-\int_0^t \Theta_u dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du\right\}, \\ \tilde{W}_t &= W_t + \int_0^t \Theta_u du, \end{aligned}$$

  and assume that $E[\int_0^T \Theta_u^2 Z_u^2 du] < +\infty$. Set $Z = Z_T$. Then $E[Z] = 1$ and we can define a new probability $\tilde{P}$ such that

  $$\tilde{P}(A) = E[1_A Z]$$

for all A. The process $\tilde{W}$ is a Brownian motion under $\tilde{P}$.

Now let $\tilde{M}_t$ be a martingale under $\tilde{P}$. Then there is an adapted process $\tilde{\Gamma}$ such that

$$\tilde{M}_t = \tilde{M}_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s$$

for all $0 \le t \le T$.

- Hedging with one stock. Consider the preceding market model. Let V_T be an $\mathcal{F}_T$-measurable random variable, standing for a future (uncertain) cash flow we want to hedge. Define V_t through
$$D_t V_t = \tilde{E}[D_T V_T | \mathcal{F}_t].$$
Note that $D_t V_t$ is a martingale and admits a representation
$$D_t V_t = V_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s.$$
On the other hand, since $\sigma_t > 0$ for sure, we can define
$$\Delta_t = \frac{\tilde{\Gamma}_t}{\sigma_t D_t S_t}.$$
By choosing such Δ_t and letting $X_0 = V_0$,
$$D_t X_t = X_0 + \int_0^t \Delta_u \sigma_u D_u S_u d\tilde{W}_s = V_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s = D_t V_t$$
for all $0 \le t \le T$.

2.3 Fundamental Theorem in Finance

- Throughout this section,
$$W_t = (W_t^1, \cdots, W_t^d).$$
Theorem 8 (Girsanov, Multidimension)**.** *Let T be a fixed positive time, and let $\Theta_t = (\Theta_t^1, \cdots, \Theta_t^d)$ be a d-dimensional adapted process. Define*
$$\begin{aligned} Z_t &= \exp\left\{ -\int_0^t \Theta_u \cdot dW_u - \frac{1}{2}\int_0^t \|\Theta_u\|^2 du \right\}, \\ \tilde{W}_t &= W_t + \int_0^t \Theta_u du, \end{aligned}$$
and assume that
$$E\left[\int_0^t \|\Theta_u\|^2 Z_u^2 du\right] < +\infty.$$
Set $Z = Z_T$. Then $E[Z] = 1$ and under the probability $\tilde{P}$ given by Theorem 5, the process $\tilde{W}$ is a d-dimensional Brownian motion.

Theorem 9 (Martingale Representation Theorem, Multidimension)**.** *Let T be a fixed positive time, and assume that $\mathcal{F}_t$ is the filtration generated by the d-dimensional Brownian motion $W_t, 0 \leq t \leq T$. Let M_t be a martingale with respective to this filtration. Then there exists an adapted process Γ_t such that*

$$M_t = M_0 + \int_0^t \Gamma_s \cdot dW_s$$

for all $0 \leq t \leq T$.

- We assume there are m stocks, each with SDE

$$dS_t^i = \alpha_t^i S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j, \ i = 1, \cdots, m.$$

In this market with stock price S_t^i and interest rate r_t, a trader can begin with initial wealth X_0 and choose adapted portfolio process Δ_t^i for the ith stock. Then

$$dX_t = r_t X_t dt + \sum_{i=1}^{m} \Delta_t^i (dS_t^i - r_t S_t^i dt).$$

The differential of the discounted portfolio value is

$$d(D_t X_t) = \sum_{i=1}^{m} \Delta_t^i d(D_t S_t^i).$$

- Arbitrage and fundamental theorem of asset pricing:

Definition 2. *An* arbitrage *is a portfolio value process X_t satisfying $X_0 = 0$ and also satisfying for some time $T > 0$*

$$P(X_T \geq 0) = 1, \quad P(X_T > 0) > 0.$$

Theorem 10 (First Fundamental Theorem of Asset Pricing)**.** *A market has a risk-neutral probability measure if and only it does not admit arbitrage.*

Definition 3. *A market model is* complete *if every derivative security can be hedged.*

Theorem 11 (Second Fundamental Theorem of Asset Pricing)**.** *Consider a market has a risk-neutral probability measure. The market is complete if and only if the risk-neutral probability is unique.*

- Link between continuous-time version and discrete-time version

Homework Set 4 (Due on Nov 8)

1. Consider a stock whose price differential is

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t.$$

where $r(t)$ and $\sigma(t)$ are nonrandom functions of t and W is a Brownian motion under the risk-neutral measure P. Let $T > 0$ be given, and consider a European call, whose value at time zero is

$$c(0, S(0)) = E\left[\exp\{-\int_t^T r(t)dt\}(S(T) - K)^+\right]$$

(i). Show that S_t is of the form $S(0)e^X$, where X is a normal random variable, and determine the mean and variance of X.
(ii). Let

$$BSM(T, x; K, R, \Sigma) = xN\left(\frac{1}{\Sigma\sqrt{T}}[\ln\frac{x}{K} + (R + \Sigma^2/2)T]\right) - e^{-RT}KN(\Sigma\sqrt{T}[\ln\frac{x}{K} + (R - \Sigma^2/2)T]\Big)$$

denote the value at time zero of a European call expiring at time T when the underlying stock has constant volatility Σ and the interest rate R is constant. Show that

$$c(0, S(0)) = BSM\left(S(0), T, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}\right)$$

2. Consider a model with a unique risk-neutral measure $\tilde{P}$ and constant interest rate r. According to the risk-neutral pricing formula, for $0 \le t \le T$, the price at time t of a European call expiring at time T is

$$C(t) = E\left[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)\right]$$

where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is

$$P(t) = E\left[e^{-r(T-t)}(K - S(T))^+|\mathcal{F}(t)\right].$$

Finally, because $e^{-rt}S(t)$ is a martingale under $\tilde{P}$, the price at time t of a forward contract for delivery of one share of stock at time T in exchange for a payment of K at time T is

$$\begin{aligned} F(t) &= E\left[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)\right] \\ &= e^{rt}E\left[e^{-rT}S(T)|\mathcal{F}(t)\right] - e^{-r(T-t)}K \\ &= S(t) - e^{-r(T-t)}K. \end{aligned}$$

Because

$$(S(T) - K)^+ - (K - S(T)^+ = S(T) - K,$$

we have the *put-call parity* relationship

$$C(t) - P(t) = E\left[e^{-r(T-t)}(S(T) - K)^+ - e^{-r(T-t)}(K - S(T))^+|\mathcal{F}(t)\right] \tag{1}$$

$$= E\left[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)\right] \tag{2}$$

$$= F(t). \tag{3}$$

Now consider a date t_0 between 0 and T, and consider a *chooser option*, which gives the right at time t_0 to choose to own either the call or the put.
(i) Show that at time t_0 the value of the chooser option is

$$C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)$$

(ii) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

3. Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}(t)$,$0 \le t \le T$, be the filtration generated by this Brownian motion. Let the mean rate of return $\alpha(t)$, the interest rate $R(t)$, and the volatility $\sigma(t)$ be adapted processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \qquad 0 \le t \le T.$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time t, where $c(t), 0 \le t \le T$, is an adapted process. If the agent holds $\Delta(t)$ shares of stock at each time t, then the differential of her portfolio value will be

$$dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta S(t))dt - C(t)dt.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t), 0 \le t \le T$, such that $X(T) = 0$ almost surely.

4. Exercise 4. 20 in Shreve's book.

5. On a standard probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, we define a Brownian motion $\{W_t, t \ge 0\}$. Let

$$dX_t = adt + 2\sqrt{X_t}dW_t.$$

(i) Show that

$$L_t = \exp\left(-\frac{k}{2}\int_0^t \sqrt{X_s}dW_s - \frac{k^2}{8}\int_0^t X_s dW_s\right)$$

is a martingale.
(ii) Use the above L to do measure change, i.e., define

$$\frac{dQ}{dP} = L_t.$$

What is the dynamic of process X_t under the new probability measure Q?