Optimality Conditions and Lagrangian Multipliers of Vector Optimization with Set-Valued Maps in Linear Spaces

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Abstract

Under generalized subconvexlikelihood, some Kuhn-Tucker type optimality conditions, and Lagrangian multiplier theorems for set-valued vector optimization with equality and inequality constraints are established by applying the alternative theorem in linear spaces.

Key words: Generalized subconvexlikelihood, alternative theorems, optimality conditions, Lagrangian multipliers, real linear spaces.

1. Introduction

In recent years, many authors have been interested in vector optimization of set-valued maps, and various results have been obtained. For instance, Corley [1] defined the maximization of a set-valued map with respect to a cone in possibly infinite dimensions and established an existence result of Lagrangian multipliers and Lagrangian duality theory. Li [2] obtained optimality conditions for optimization of set-valued maps by using the alternative theorem under subconvexlike set-valued maps. Hu [3] presented a number of the necessary and sufficient conditions in term of cone-weakly efficient subdifferential for the

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In this paper, we consider vector optimization of set-valued maps in real linear spaces, without any topological structure. Under assumption of generalized subconvexldness for set-valued maps, we obtain some optimality conditions and Lagrangian multiplier theorems in vector optimization of set-valued maps. Weak saddle points and Lagrange duality will be treated elsewhere.

This paper is organized as follows. In section 2, we give some notations and preliminaries. In sections 3 to 4, the main results are presented.

2. Notations and Preliminaries

Let \( D \) be an arbitrarily chosen nonempty set; \( Y \) a real linear space; the set \( Y_+ \subset Y \) a pointed convex cone. Let \( B \) be a nonempty subset in \( Y \). Denote by \( B^\circ \) the algebraic interior of \( B \). The generated cone of \( B \) is defined by \( \text{cone}(B) = \{ab | a \geq 0, b \in B\} \).

Throughout this paper, we assume that \( Y_+ \), the algebraic interior of \( Y_+ \), is nonempty. Define \( Y^* \) to consist of all linear functionals \( Y \to R \), where \( R \) is set of all real numbers. Obviously, \( Y^* \), the algebraic dual of \( Y \), is also a linear space under the pointwise addition and multiplication with real numbers. The algebraic dual cone \( Y_+^* \) of \( Y_+ \), is defined by \( Y_+^* = \{ y^* \in Y^* | (y, y^*) \geq 0, \forall y \in Y_+ \} \), where \((y, y^*)\) denotes the value of the linear functional \( y^* \) at the point \( y \). Denote by \( 0 \) the null element for every linear space.

Suppose that \( F : D \to 2^Y \) is a set-valued map form \( D \) to \( Y \), where \( 2^Y \) denotes the power set of \( Y \). Let \( F(D) = \bigcup_{x \in D} F(x) \), \( (F(x), y^*) = \{ (y, y^*) | y \in F(x) \} \), and \((F(D), y^*) = \bigcup_{x \in D} (F(x), y^*) \) For \( A \subset R, b \in R \), write \( A \geq (\leq, >, <) b \), if \( a \geq (\leq, >, <) b, \forall a \in A \).

**Definition 2.1** A set-valued map \( F : D \to 2^Y \) is called generalized \( Y_+ \) -subconvexlike (or shortly, generalized subconvexlike), if the set \( \text{cone}(F(D)) + Y_+^* \) is convex.

**Lemma 2.1** Let \( Y \) be a real linear space with pointed convex cone \( Y_+ \). Suppose \( Y_+^* \neq \emptyset \).

Then \( Y_+ + Y_+^* \subset Y_+^* \).

In fact, Lemma 2.1 is derived directly by definition of \( Y_+^* \), which was introduced in Ref.7.

**Lemma 2.2** (See Ref.6) The set-valued map \( F : D \to 2^Y \) is generalized subconvexlike, if and only if, there exists \( \theta \in Y_+^* \), such that \( \forall x_1, x_2 \in D, \forall \lambda \in (0, 1), \forall \epsilon > 0, \) satisfying

\[ \epsilon \theta + \lambda F(x_1) + (1 - \lambda)F(x_2) \subset \text{cone}(F(D)) + Y_+ \].

**Lemma 2.3** If \( y_0 \in Y_+^* \), \( y^\star \in Y_+^* \), with \( y^\star \neq 0 \), then \((y_0, y^\star) > 0\).

Indeed, the proof of Lemma 2.3 is similar to the proof of Lemma 3.1 in Ref. 2, or the proof of Lemma 2.1 in Ref. 3.
Lemma 2.4 (See Ref.6) Let $D$ be a nonempty set. Let $Y$ be a real linear space with a pointed convex cone $Y_+$ with nonempty algebraic interior $Y_+^a$. If a set-valued map $F : D \rightarrow 2^Y$ is generalized subconvexlike, then either (i) or (ii) holds:

(i) there is $x_0 \in D$ such that $-F(x_0) \cap Y_+^a \neq \emptyset$;

(ii) there is $y^* \in Y_+^a$, with $y^* \neq O$, such that $\langle F(x), y^* \rangle \geq 0, \forall x \in D$.

The two alternatives (i) and (ii) exclude each other.

3. Optimality conditions

Let $Y, Z, W$ be real linear spaces with pointed convex cones $Y_+, Z_+, W_+$ with nonempty algebraic interior, respectively. Let $F : D \rightarrow 2^Y, G : D \rightarrow 2^Z, H : D \rightarrow 2^W$ be set-valued maps from $D$ to $Y, Z,$ and $W$, respectively.

In this paper, we consider the following vector optimization problem $(P)$ with set-valued maps.

$$\min_{x \in D} F(x)$$

subject to

$$-G(x) \cap Z_+ \neq \emptyset;$$

$$O \in H(x), \ x \in D.$$

The feasible set of problem $(P)$ is defined as $K = \{x \in D | -G(x) \cap Z_+ \neq \emptyset, O \in H(x)\}$.

Definition 3.1 $x_0 \in K$ is called a weakly efficient solution of $(P)$, if there exists $y_0 \in F(x_0)$ such that $(y_0 - F(K)) \cap Y_+^a = \emptyset$. The pair $(x_0, y_0)$ is called a $Y_+ -$ weak minimizer of $(P)$.

Set $I(x) = F(x) \times G(x) \times H(x), \forall x \in D$. Obviously, $I$ is a set-valued map from $D$ to the product space $Y \times Z \times W$, which is a real linear space with a positive cone $Y_+ \times Z_+ \times W_+$ with nonempty algebraic interior. It is easy to verify that $(Y_+ \times Z_+ \times W_+)^t = Y_+^a \times Z_+^a \times W_+^a$, and that $(Y_+ \times Z_+ \times W_+)^* = Y_+^* \times Z_+^* \times W_+^*$. We say later that $I$ is generalized subconvexlike; i.e., the set cone $(I(D)) + (Y_+ \times Z_+ \times W_+)^t$ is convex.

Theorem 3.1 Suppose the following:

(i) $(x_0, y_0)$ is a $Y_+ -$ weak minimizer of $(P)$;

(ii) $(F(x) - y_0) \times G(x) \times H(x)$ is generalized subconvexlike on $D$, and $-G(D) \cap Z_+^a = \emptyset$;

(iii) $O \in H(D)^t; \exists x' \in D$, such that $O \in H(x'), -G(x') \cap Z_+^a \neq \emptyset$.

Then, there exist $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$, such that

$$\inf_{x \in D} \langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle = \langle y_0, y^* \rangle,$$

$$\inf \langle (G(x_0), z^*) + \langle H(x_0), w^* \rangle = 0,$$

$$\inf G(x_0) = 0.$$

Proof. Set $I^\#(x) = (F(x) - y_0) \times G(x) \times H(x), \forall x \in D$. Obviously, $I^\#(x) = I(x) - (y_0, O, O), \forall x \in D$. By assumption (i), we have $(y_0 - F(K)) \cap Y_+^a = \emptyset$. Since
\(-G(D\setminus K) \cap Z^+_+=\emptyset\), hence we get \(-I^R(x) \cap (Y^+_+ \times Z^+_\times W)^+ = \emptyset\), \(\forall x \in D\). Thus, according to Lemma 2.4 and assumption (ii), there exist \((y^*, z^*, w^*) \in Y^+_+ \times Z^+_\times W^+_\times \{(O, O, O)\}\), such that \((I^R(x), (y^*, z^*, w^*)) \geq 0, \forall x \in D\). i.e.,
\[
\langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq \langle y_0, y^* \rangle, \quad \forall x \in D.
\] (1)

We show \(y^* \neq O\) in the following. Assume the contrary. Then, \((z^*, w^*) \neq (O, O)\). We have two cases.

First case. \(z^* \neq O\). Then (1) can be written as
\[
\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \quad \forall x \in D.
\] (2)

It follows by assumption (iii) that \(\exists u \in G(x')\) such that \(-u \in Z^+_\). Hence, by Lemma 2.3, we have \((u, z^*) < 0\), which contradicts (2).

Second case. \(z^* = O\). Then \(w^* \neq O\), and (1) can be written as
\[
\langle H(x), w^* \rangle \geq 0, \quad \forall x \in D.
\]

Since \(O \in (H(D))^+\), then for any given \(v \in W\), there is \(\epsilon > 0\) such that \(\pm \epsilon v \in H(D)\). It follows that \((v, w^* ) = 0, \forall v \in W\). i.e., \(w^* = O\). This is a contradiction.

Therefore, the proof of \(y^* \neq O\) is complete.

Since \(x_0 \in K\), there is \(p \in G(x_0)\) such that \(-p \in Z^+_\). It follows that \((p, z^*) \leq 0\). On the other hand, taking \(x = x_0\) in (1), we obtain
\[
\langle y_0, y^* \rangle + \langle p, z^* \rangle + \langle O, w^* \rangle \geq \langle y_0, y^* \rangle.
\]

i.e., \((p, z^*) \geq 0\). Therefore, \((p, z^*) \geq 0\), which implies that
\[
\langle y_0, y^* \rangle \in \langle F(x_0), y^* \rangle + \langle G(x_0), z^* \rangle + \langle H(x_0), w^* \rangle.
\]

Hence, it follows by (1) that
\[
\inf_{x \in D} \left( \langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \right) = \langle y_0, y^* \rangle.
\]

In a similar way, we have \(\inf \langle G(x_0), z^* \rangle + \langle H(x_0), w^* \rangle = 0\), and \(\inf \langle G(x_0), z^* \rangle = 0\).

**Theorem 3.2** Suppose the following:

(i) \(x_0 \in K\);

(ii) \(\exists y_0 \in F(x_0), (y^*, z^*, w^*) \in Y^+_+ \times Z^+_\times W^+_\times, with y^* \neq O, such that \)
\[
\inf_{x \in D} \left( \langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \right) \geq \langle y_0, y^* \rangle;
\]

Then, \((x_0, y_0)\) is a \(Y^+_+ -\) weak minimizer of \((P)\).

**Proof.** According to assumption (ii), we have
\[
\langle F(x) - y_0, y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geq 0, \quad \forall x \in D.
\] (3)
Suppose that there is $z^* \in K$ such that $(y_0 - F(x^*)) \cap Y_+^* \neq \emptyset$. Then, $\exists t \in F(x^*), \exists q \in G(x^*)$ such that $y_0 - t \in Y_+^* - q \in Z_+$. It follows that

$$(t - y_0, y^*) + \langle q, z^* \rangle + \langle O, w^* \rangle < 0,$$

which contradicts (3). Therefore, we have $(y_0 - F(K)) \cap Y_+^* = \emptyset$, which implies that $(x_0, y_0)$ is a $Y_+ - weak$ minimizer of $(P)$.

**Corollary 3.2** Suppose the following:

(i). $x_0 \in K$;

(ii). $\exists y_0 \in F(x_0), (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \backslash \{(O, O, O)\}$, such that

$$\inf_{x \in D} \langle (F(x), y^*), (G(x), z^*), (H(x), w^*) \rangle \geq \langle (y_0, y^*) \rangle;$$

(iii). $O \in (H(D))^1; \exists x' \in D$, such that $O \notin H(x')$, $-G(x') \cap Z_+^* \neq \emptyset$.

Then, $(x_0, y_0)$ is a $Y_+ - weak$ minimizer of $(P)$.

### 4. Lagrangian Multipliers

In the following, we consider a scalar minimization problem $(P_{y^*})$ with set-valued maps for problem $(P)$.

$$\min_{x \in K} \langle F(x), y^* \rangle, \quad y^* \in Y_+^* \backslash \{O\}. \quad (P_{y^*})$$

**Definition 4.1** $x_0 \in K$ is called an optimal solution of $(P_{y^*})$, if there exists $y_0 \in F(x_0)$ such that $(y_0, y^*) \leq \langle F(x), y^* \rangle, \forall x \in K$.

**Lemma 4.1** (See Ref.6) Suppose that $(x_0, y_0)$ is a $Y_+ - weak$ minimizer of $(P)$, and suppose that $F(x) - y_0$ is generalized subconvexlike on $K$. Then there exists $y^* \in Y_+^*$, with $y^* \neq O$, such that $x_0$ is optimal for problem $(P_{y^*})$.

Let $L(Z, Y)$ be the set of all linear operators from $Z$ to $Y$. Define $L_+(Z, Y)$ by $L_+(Z, Y) = \{T \in L(Z, Y) | T(Z_+) \subset Y_+\}$. The meanings of $L(W, Y), L_+(W, Y)$ are respectively similar.

The Lagrangian map for $(P)$ is the set-valued map $L : D \times L_+(Z, Y) \times L(W, Y) \rightarrow 2^Y$ defined by

$$L(x, T, M) = F(x) + T(G(x)) + M(H(x)), (x, T, M) \in D \times L_+(Z, Y) \times L(W, Y).$$

Consider the following unconstrained vector optimization problem

$$(UP) \quad \min_{\pi \in D} L(x, T, M), \quad (x, T, M) \in L_+(Z, Y) \times L(W, Y).$$

**Lemma 4.2** Let $I(x)$ be generalized subconvexlike on $D$, and let $y^* \in Y_+^* \backslash \{O\}$. Then, $T(x) = (F(x), y^*) \times G(x) \times H(x)$ is generalized $R_+ \times Z_+^* \times W_+^*$-subconvexlike (or shortly, generalized subconvexlike) on $D$.

In fact, Lemma 4.1 is easily deduced by Lemma 2.2.

**Theorem 4.1** Suppose the following:

(i). $(x_0, y_0)$ is a $Y_+ - weak$ minimizer of $(P)$;
(ii). \( F(z) - y_0 \) is generalized subconvexlike on \( K \), and \(-G(D\setminus K) \cap Z_+ = \emptyset \), and \( F(z) - y_0 \) \( \times G(z) \times H(z) \) is generalized subconvexlike on \( D \);

(iii). \( O \in (H(D))^1; \exists z' \in D \) such that \( O \in H(z'), \neg \{G(z') \cap Z_+ \neq \emptyset \} \).

Then, there exists \((T, M) \in L_+(Z, Y) \times L(W, Y)\), such that \((x_0, y_0)\) is a \( Y_+\)-weak minimizer of \((UP)\), and \( O \in T(G(x_0)) \cap M(H(x_0)) \).

Proof. According to Lemma 4.1, there exists \( y^* \in Y_+^* \), with \( y^* \neq O \), such that

\[
\langle F(z) - y_0, y^* \rangle \geq 0, \quad \forall z \in K.
\]  

(4)

Set \( J(z) = \langle F(z) - y_0, y^* \rangle \times G(z) \times H(z), \forall z \in D \). Obviously, \( J(z) = \langle F(z), y^* \rangle \times G(z) \times H(z) = (\langle y_0, y^* \rangle, O, O), \forall z \in D \). It follows by Lemma 4.2 that \( J(z) \) is generalized subconvexlike on \( D \). By (4) and assumption (ii), we have

\[
-J(z) \cap (R^+_1 \times Z_+^1 \times W_+^1) = \emptyset, \quad \forall z \in D,
\]

where \( R^+_1 = \{ r \in R| r > 0 \} \).

Thus, by Lemma 2.4, \( \exists k \in R_+^1, z^* \in Z_+^1, w^* \in W_+^1 \), with \( (k, z^*, w^*) \neq O \) such that

\[
k(F(z) - y_0, y^*) + \langle G(z), z^* \rangle + \langle H(z), w^* \rangle \geq 0, \quad \forall z \in D.
\]  

(5)

Due to \( x_0 \in K \), consequently, there exists \( p \in G(x_0) \) such that \( -p \in Z_+^1 \), which implies \( \langle p, z^* \rangle \leq 0 \). Take \( z = x_0 \) in (5). Observing \( O \in H(x_0) \), we get \( \langle p, z^* \rangle \geq 0 \). Therefore, \( \langle p, z^* \rangle = 0 \). This illustrates that

\[
O \in (G(x_0), z^*).
\]  

(6)

By assumption (iii) and Lemma 2.3, we have \( k > 0 \).

Since \( Y_+^* \) is a cone, then \( ky^* \in Y_+^* \), and \( ky^* \neq O \). Thus, for any given \( y \in Y_+^* \subset Y_+^1 \), we have \( \langle y, ky^* \rangle > 0 \). Set \( y_1 = y, ky^* \rangle \in Y_+^* \). Then \( \langle y_1, ky^* \rangle = 1 \).

Define the two linear operators \( T : Z \rightarrow Y, M : W \rightarrow Y \) as

\[
T(z) = \langle z, z^* \rangle y_1, \quad \forall z \in Z; \quad M(w) = \langle w, w^* \rangle y_1, \quad \forall w \in W,
\]

respectively. It is clear that \( T \in L_+(Z, Y), M \in L_1(W, Y) \subset L(W, Y) \). So, \( T(G(x_0)) = \langle G(x_0), z^* \rangle y_1 \). By (6) we have \( O \in T(G(x_0)) \). Because of \( O \in H(x_0) \), we also have \( O \in M(H(x_0)) \). Thus, \( O \in T(G(x_0)) \cap M(H(x_0)) \).

Since \( (1/k)\langle G(z), z^* \rangle = \langle T(G(z)), y^* \rangle \) and \( (1/k)\langle H(z), w^* \rangle = \langle M(H(z)), y^* \rangle \), then (5) can be written as \( \langle F(z), y^* \rangle + \langle T(G(z)), y^* \rangle + \langle M(H(z)), y^* \rangle \geq \langle y_0, y^* \rangle , \forall z \in D \). i.e.,

\[
\langle F(z) + T(G(z)) + M(H(z)), y^* \rangle \geq \langle y_0, y^* \rangle, \quad \forall z \in D.
\]  

(7)

Thereby, \( x_0 \) is a weakly efficient solution of \((UP)\). In fact, assume the contrary. Then, due to \( y_0 \in F(z_0) + T(G(z_0)) + M(H(z_0)) = L(x_0, T, M) \), we have \( (y_0 - L(D, T, M)) \cap Y_+^1 \neq \emptyset \), where \( L(D, T, M) = \bigcup_{x \in D} L(x, T, M) \). Thus, there exists \( y^* \in L(D, T, M) \) such that \( y_0 - y^* \in Y_+^1 \).

Therefore, \( \langle y_0 - y^*, y^* \rangle > 0 \). i.e.,

\[
\langle y_0, y^* \rangle > \langle y^*, y^* \rangle,
\]
which contradicts (7). Therefore, \((x_0, y_0)\) is a \(Y_+\) weak minimizer of (UP).

**Theorem 4.2** Suppose the following:

(i). \(x_0 \in K\).

(ii). \(\exists (T, M) \in L_+ (Z, Y) \times L(W, Y), \) s.t. \( O \in T(G(x_0)) \cap M(H(x_0)); \)

(iii). \(y_0 \in F(x_0)\) such that \((x_0, y_0)\) is a \(Y_+\) weak minimizer of (UP), the pair \((T, M)\) of which is given by (ii).

Then, \(x_0\) is a weakly efficient solution of \((P)\).

**Proof.** By assumption (iii), we have

\[
(y_0 - L(x, T, M)) \cap Y_+^k = \emptyset, \quad \forall x \in D. \tag{8}
\]

Because of \(O \in T(G(x_0)) \cap M(H(x_0)),\) and \(y_0 \in F(x_0),\) we obtain \(y_0 \in F(x_0) + T(G(x_0)) + M(H(x_0))).

If \(x_0\) is not a weakly efficient solution of \((P)\), then \((y_0 - F(K)) \cap Y_+^k \neq \emptyset.\) Thus, there exists \(x' \in K\) such that

\[
(y_0 - F(x')) \cap Y_+^k \neq \emptyset. \tag{9}
\]

Then, there exists \(p \in G(x')\) such that \(-p \in Z_+.\) So, \(T(-p) \in Y_+.\) By (9), there exists \(y' \in F(x')\) such that \(y_0 - y' \in Y_+^k.\) Hence, we have \(y_0 - (y' + T(p) + M(O)) \in Y_+^k + Y_+ \subset Y_+^k.\) i.e.,

\[
(y_0 - L(x', T, M)) \cap Y_+^k \neq \emptyset,
\]

which contradicts (8).

\[\square\]

**References**


