The Optimality Conditions for Nonconvex Vector Optimization of Set-Valued Maps

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Abstract

In this paper, we prove an alternative theorem for nearly $Y_+$—semiconvexlike set-valued maps in linear topological spaces. Using this alternative theorem, we obtain the optimality conditions for a class of nonconvex vector optimization of set-valued maps.

Key words. Set-valued Maps, Near $Y_+$—semiconvexlikeness, Theorems of the alternative, Optimality conditions.
1. INTRODUCTION

The alternative theorems play an important role in vector optimization problems. Many results of optimization problems by applying these theorems have been introduced in the literature. For instance, Illes and Kassay [1] discussed an alternative theorem for convexlike functions, and established optimality conditions for convexlike programming problems with inequality and equality considered. Li [2] gave a theorem of the alternative for subconvexlike mapping in ordered linear topological spaces, and obtained optimality conditions for the infinite dimensional differentiable vector optimization problems. Tač [3], by using a generalization of alternative theorem proved by Jeyakumar, established the optimality condition for nonsmooth and nonconvex vector mathematical programming in term of Lagrange-Kuhn-Tucker multipliers. Lin [4] generalized the Moreau-Rockafellar type theorem and the Farkas-Minkowski type theorem for set-valued maps, and established the necessary and sufficient conditions for the existence of Geoffrion efficient solution of a minimization problem, and proved the Mond-Weir type and Wolf type vector duality theorems. Li [5] presented an alternative theorem for cone-subconvexlike set-valued maps, and studied Benson proper efficiency for a vector optimization problem with set-valued maps.

In this paper, we are mainly concerned with an alternative theorem for nearly cone-semiconvexlike set-valued maps, and its application to the optimality conditions for nonconvex vector minimization problem with set-valued maps. Weak saddle points, Lagrangian duality will be treated elsewhere.

This paper is organized as follows. In section 2, we introduce some definitions and preliminaries. In section 3, we show an alternative theorem in linear topological spaces. In sections 4, we obtain the main results.

2. DEFINITIONS AND PRELIMINARIES

Throughout this paper, the scalars of linear topological spaces are always real. Denote by O the null element of every space. Let D be an arbitrarily chosen nonempty set; Y a linear topological space; \( Y_+ \subset Y \) a pointed convex cone (i.e., \( Y_+ \cap (-Y_+)=\{O\} \), \( \alpha Y_+ + \beta Y_+ \subset Y_+ \), \( \forall \alpha, \beta \geq 0 \)). Suppose that the topological interior of \( Y_+ \) is nonempty, i.e., \( \text{int} Y_+ \neq \emptyset \).

We denote by \( Y^* \) the dual of \( Y \). The dual cone \( Y^*_+ \) of \( Y^* \) is defined by \( Y^*_+=\{y^* \in Y^* | \langle y, y^* \rangle \geq 0, \forall y \in Y_+\} \), where \( \langle y, y^* \rangle \) denotes the value of the linear continuous functional \( y^* \) at the point \( y \).

Let \( C \) be a nonempty cone in \( Y \). The generated cone of \( C \) is defined by \( \text{cone}(C)=\{\alpha c | \alpha \geq 0, c \in C\} \). Obviously, the set \( \text{cone}(C) \) contains the null element and \( C \). We define \( \text{Wmin}(C, Y_+)=\{y \in C | (y-C) \cap \text{int} Y_+=\emptyset\} \); \( \text{Wmax}(C, Y_+)=\{y \in C | (C-y) \cap \text{int} Y_+=\emptyset\} \).

Denote by \( R \) set of real numbers. For \( A \subset R, b \in R \), write \( A \succeq (\leq, >, <)b \), if \( a \succeq (\leq, >, <)b, \forall a \in A \). Suppose that \( f : D \rightarrow 2^Y \) is a set-valued map, where \( 2^Y \) denotes the power set of \( Y \). Let \( f(D)=\bigcup_{x \in D} f(x); \langle f(x), y^* \rangle = \langle y, y^* \rangle | y \in f(x) \}; \langle f(D), y^* \rangle = \bigcup_{x \in D} \langle f(x), y^* \rangle \).

DEFINITION 2.1. A subset \( M \) in \( Y \) is called nearly convex, if \( \exists \alpha \in (0, 1), \forall y_1, y_2 \in M \) such that \( \alpha y_1+(1-\alpha)y_2 \in M \).

LEMMA 2.1 (See Ref.1). If \( M \subset Y \) is a nearly convex set, the \( \text{int} M \) is a convex set. (\( \text{int} M \) may be empty)

DEFINITION 2.2. A set-valued map \( f : D \rightarrow 2^Y \) is called nearly \( Y_+—\)convexlike, if \( \exists \alpha \in (0, 1), \forall x_i, \)
A set-valued map $f: D \rightarrow 2^Y$ is called nearly $Y_+-$subconvexlike, if there exist $u \in \text{int} Y_+$, $\alpha \in (0, 1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(D) + Y.$$ 

DEFINITION 2.3. A set-valued map $f: D \rightarrow 2^Y$ is called nearly generalized $Y_+-$subconvexlike, if there exist $u \in \text{int} Y_+$, $\alpha \in (0, 1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq \text{cone}(f(D)) + Y.$$ 

Remark (1). It is not difficult to verify that nearly $Y_+-$convexlike $\Rightarrow$ nearly $Y_+-$subconvexlike $\Rightarrow$ nearly generalized $Y_+-$subconvexlike. However, the converse implications are not true.

(2). If the set-valued map $f$ is nearly $Y_+-$subconvexlike on $D$, $\forall y_0 \in Y$, then $f(x) + y_0$ is also nearly $Y_+-$subconvexlike.

(3). If set-valued maps $g: D \rightarrow 2^{Y_1}$, $h: D \rightarrow 2^{Y_2}$ are nearly $Y_1-$, $Y_2-$subconvexlike, respectively, then the set-valued map $(g, h): D \rightarrow 2^{Y_1 \times Y_2}$ is not necessary to be nearly $Y_1 \times Y_2-$subconvexlike, where $Y_1$, $Y_2$ are two linear topological spaces with pointed convex cones $Y_1+$, $Y_2+$ with nonempty interiors respectively.

The following lemma will be used later.

LEMMA 2.2. Let $Y$ be a linear topological space with pointed convex cone $Y_+$ with nonempty topological interior; the dual cone of $Y^* = Y_1^* \times Y_2^*$. Let $(u, O) \neq 0$, such that $\langle u, y^*_1 \rangle + \langle u_2, y^*_2 \rangle > 0$, $\forall u \in \text{int} M$. 

3. ALTERNATIVE THEOREMS

We always suppose the following, if no special statements.

$Y_1$, $Y_2$ are two linear topological spaces with pointed convex cones $Y_1+$, $Y_2+$, respectively. The topological interior of $Y_1+$ is nonempty, but the topological interior of $Y_2+$ is not required to be nonempty. $Y_i^+$ is the dual cone of $Y_i$, $i=1, 2$. For set-valued maps $g: D \rightarrow 2^{Y_1}$, $h: D \rightarrow 2^{Y_2}$, we put $Y = Y_1 \times Y_2$, $Y_s = Y_1^+ \times Y_2^+$, $F = (g, h): D \rightarrow 2^Y$. Obviously, we have $Y^* = (Y_1 \times Y_2)^* = Y_1^* \times Y_2^*$, $Y_s^* = (Y_1^+ \times Y_2^+)^* = Y_1^* \times Y_2^*$.

DEFINITION 3.1. A set-valued map $F: D \rightarrow 2^Y$ is called nearly $Y_+-$semiconvexlike, if there exist $u \in \text{int} Y_1+$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon (u, O) + \alpha F(x_1) + (1-\alpha)F(x_2) \subseteq F(D) + Y_+.$$ 

In the following, we will discuss an alternative theorem for nearly $Y_+-$semiconvexlike set-valued map.

LEMMA 3.1. int(F(D)+Y_+)≠∅, if and only if int(F(D)+(intY_1+)×Y_2)≠∅.

LEMMA 3.2. Let $M = F(D)+Y_s$, suppose that $M$ is nearly convex, and suppose that $\text{int} M \neq \emptyset$. If $\exists y^* = (y_1^*, y_2^*) \in Y_1^+ \times Y_2^+$, $y^* \neq O$, such that $\langle u, y^* \rangle = \langle u_1, y_1^* \rangle + \langle u_2, y_2^* \rangle > 0$, $\forall u \in \text{int} M$. 

x_2 \in D$ such that

$$\alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(D) + Y_+.$$ 

DEFINITION 2.3. A set-valued map $f: D \rightarrow 2^Y$ is called nearly $Y_+-$subconvexlike, if there exist $u \in \text{int} Y_+$, $\alpha \in (0, 1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(D) + Y_+.$$ 

DEFINITION 2.3. A set-valued map $f: D \rightarrow 2^Y$ is called nearly generalized $Y_+-$subconvexlike, if there exist $u \in \text{int} Y_+$, $\alpha \in (0, 1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(D) + Y_+.$$ 

Remark (1). It is not difficult to verify that nearly $Y_+-$convexlike $\Rightarrow$ nearly $Y_+-$subconvexlike $\Rightarrow$ nearly generalized $Y_+-$subconvexlike. However, the converse implications are not true.

(2). If the set-valued map $f$ is nearly $Y_+-$subconvexlike on $D$, $\forall y_0 \in Y$, then $f(x) + y_0$ is also nearly $Y_+-$subconvexlike.

(3). If set-valued maps $g: D \rightarrow 2^{Y_1}$, $h: D \rightarrow 2^{Y_2}$ are nearly $Y_1-$, $Y_2-$subconvexlike, respectively, then the set-valued map $(g, h): D \rightarrow 2^{Y_1 \times Y_2}$ is not necessary to be nearly $Y_1 \times Y_2-$subconvexlike, where $Y_1$, $Y_2$ are two linear topological spaces with pointed convex cones $Y_1+$, $Y_2+$ with nonempty interiors respectively.

The following lemma will be used later.

LEMMA 2.2. Let $Y$ be a linear topological space with pointed convex cone $Y_+$ with nonempty topological interior; $Y_+^*$ the dual cone of $Y_+$. Let $y_0^* \in Y_+^* \setminus \{O\}$, $u \in \text{int} Y_+$. Then $\langle y_0^*, y_0 \rangle > 0$.

3. ALTERNATIVE THEOREMS

We always suppose the following, if no special statements.

$Y_1$, $Y_2$ are two linear topological spaces with pointed convex cones $Y_1+$, $Y_2+$, respectively. The topological interior of $Y_1+$ is nonempty, but the topological interior of $Y_2+$ is not required to be nonempty. $Y_i^+$ is the dual cone of $Y_i$, $i=1, 2$. For set-valued maps $g: D \rightarrow 2^{Y_1}$, $h: D \rightarrow 2^{Y_2}$, we put $Y = Y_1 \times Y_2$, $Y_s = Y_1^+ \times Y_2^+$, $F = (g, h): D \rightarrow 2^Y$. Obviously, we have $Y^* = (Y_1 \times Y_2)^* = Y_1^* \times Y_2^*$, $Y_s^* = (Y_1^+ \times Y_2^+)^* = Y_1^* \times Y_2^*$.

DEFINITION 3.1. A set-valued map $F: D \rightarrow 2^Y$ is called nearly $Y_+-$semiconvexlike, if there exist $u \in \text{int} Y_1+$, $\alpha \in (0, 1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$ such that

$$\varepsilon (u, O) + \alpha F(x_1) + (1-\alpha)F(x_2) \subseteq F(D) + Y_+.$$ 

In the following, we will discuss an alternative theorem for nearly $Y_+-$semiconvexlike set-valued map.

LEMMA 3.1. int(F(D)+Y_+)≠∅, if and only if int(F(D)+(intY_1+)×Y_2)≠∅.

LEMMA 3.2. Let $M = F(D)+(intY_1+)\times Y_+$. Suppose that $M$ is nearly convex, and suppose that $\text{int} M \neq \emptyset$. If $\exists y^* = (y_1^*, y_2^*) \in Y_1^+ \times Y_2^+$, $y^* \neq O$, such that $\langle u, y^* \rangle = \langle u_1, y_1^* \rangle + \langle u_2, y_2^* \rangle > 0$, $\forall u \in \text{int} M$. 

x_2 \in D$ such that

$$\alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(D) + Y_+.$$
Then \( \langle u, y^* \rangle \geq 0, \forall u \in M \).

The proofs of the last two lemmas are similar to the proofs of Lemma 2.2 and Theorem 3.1 of Ref. 1, respectively, when \( F \) is a single-valued map.

**LEMMA 3.3.** If \( y^*_0 = (y^*_0, y^*_2) \in Y^*_+, y^*_0 \neq O, y = (y_1, y_2) \in (\text{int}Y_1+) \times Y_2, \) then \( \langle y, y^*_0 \rangle > 0 \).

Proof. By definition of \( Y^*_+ \), we have \( \langle y, y^*_0 \rangle \geq 0 \). Suppose that \( \exists y_0 = (y_0, y_2) \in (\text{int}Y_1+) \times Y_2, \) such that \( \langle y_0, y^*_0 \rangle = 0 \), i.e., \( \langle y_0, y^*_0 \rangle + \langle y_0, y^*_0 \rangle = 0 \). Since \( y_0 \in \text{int}Y_1, \) there exists a neighborhood \( N \) of the origin of \( Y_1, \) such that \( y_0 + N \subseteq \text{int}Y_1, \) \( N \) being absorbing, we have that \( \forall v \in Y_1, \exists \varepsilon > 0, \) such that \( y_0 + \varepsilon v \in \text{int}Y_1 \). Hence, \( \langle y_0 + \varepsilon v, y^*_0 \rangle + \langle y_0, y^*_0 \rangle \geq 0, \) i.e., \( \langle y_0, y^*_0 \rangle + \langle y_0, y^*_0 \rangle \geq 0 \). Therefore, \( \langle y, y^*_0 \rangle = 0, \forall v \in Y_1, \) i.e., \( y^*_0 \neq O \), which contradicts the assumption that \( y^*_0 \neq O \).

**LEMMA 3.4.** The set-valued map \( F: D \to 2^Y \) is nearly \( Y \)-semiconvexlike, if and only if \( M = F(D) + (\text{int}Y_1+) \times Y_2, \) is nearly convex.

Proof. "Sufficiency." Since \( \text{int}Y_1, \) is nonempty, and \( M \) is nearly convex, hence, \( \exists u \in \text{int}Y_1, \) \( \exists \alpha \in (0, 1), \forall x_1, x_2 \in D, \forall \varepsilon > 0, \) such that

\[
\alpha(F(x_1) + \varepsilon(u, O)) + (1 - \alpha)(F(x_2) + \varepsilon(u, O)) \subset M \subseteq F(D) + Y_1.
\]

Therefore, \( \varepsilon(u, O) + \alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq F(D) + Y_1, \) i.e., \( F \) is nearly \( Y \)-semiconvexlike.

"Necessity." Let \( m_1, m_2 \in M, \) then \( \exists x_i \in D, y_i \in (\text{int}Y_1+) \times Y_2, \) \( i = 1, 2, \) such that \( m_i \in F(x_i) + y_i \). Since \( F \) is nearly \( Y \)-semiconvexlike, there exist \( u \in \text{int}Y_1, \) \( \alpha \in (0, 1), \) for the previous \( x_1, x_2 \in D, \) \( \forall \varepsilon > 0, \) we have

\[
\varepsilon(u, O) + \alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq F(D) + Y_1.
\]

Thus \( \varepsilon(u, O) + \alpha(m_1 - y_1) + (1 - \alpha)(m_2 - y_2) \in F(D) + Y_1. \) Because the set \( (\text{int}Y_1+) \times Y_2, \) is convex, we have \( y_0 = \alpha y_1 + (1 - \alpha) y_2 \in (\text{int}Y_1+) \times Y_2. \) Therefore,

\[
m = \alpha m_1 + (1 - \alpha)m_2 + \alpha F(x_1) + (1 - \alpha)F(x_2) + y_0.
\]

Let \( y_0 = (y_0, y_2). \) Since \( y_0 \in \text{int}Y_1, \) there is \( \varepsilon > 0 \) such that \( y_0 + \varepsilon u \in \text{int}Y_1. \) Then, \( y_0 + \varepsilon(u, O) = (y_0 + \varepsilon u, y_2) \in (\text{int}Y_1+) \times Y_2. \) It follows by (3.1) that

\[
m = \alpha F(x_1) + (1 - \alpha)F(x_2) + y_0 + \varepsilon(u, O) \subset F(D) + Y_1 + (\text{int}Y_1+) \times Y_2 = M.
\]

Therefore \( M \) is nearly convex.

In this paper, we consider the following generalized systems:

**System 1.** \( \exists x_0 \in D, \) s.t. \( -g(x_0) \cap \text{int}Y_1, -h(x_0) \cap Y_2 \neq \emptyset. \)

**System 2.** \( \exists y^* = (y^*_1, y^*_2) \in Y^*_1 \times Y^*_2, y^*_1 \neq O, \) s.t.

\[
\langle g(x), y^*_1 \rangle + \langle h(x), y^*_2 \rangle \geq 0, \forall x \in D.
\]

**THEOREM 3.1 (Alternative theorem).** Let \( F=(g, h): D \to 2^Y \) be a nearly \( Y \)-semiconvexlike set-valued map. Suppose that the topological interior of \( F(D) + Y_2 \) is nonempty. Then

(i) If System 2 has a solution \( (y^*_1, y^*_2), \) with \( y^*_1 \neq O, \) then System 1 has no solution;
(ii) If System 1 has no solution, then System 2 has a solution \((y_1^*, y_2^*)\).

Proof. (i). Suppose that System 1 has a solution \(x_0 \in D\). Then there are \(p \in g(x_0), q \in h(x_0)\), such that 
\[-p \in \text{int} Y_{1}, \quad q \in Y_{2};\]  
It follows by Lemma 3.3 that \(p, y_1^* + q, y_2^* < 0\), which contracts (3.2)  
since System 2 has a solution \((y_1^*, y_2^*) \in (Y_{1r}^* \setminus \{O\}) \times Y_{2r}^*\).

(ii). According to Lemma 3.1 and the assumption that \(\text{int}(F(D) + Y_2) \neq \emptyset\), we have \(\text{int} M \neq \emptyset\). Since \(F\) is nearly \(Y_2\)-semiconvexlike, thus by Lemma 3.4, \(M = F(D) + (\text{int} Y_{1r}) \times Y_{2r}\) is nearly convex. Hence, \(\text{int} M\) is convex. Because System 1 admits no solution, we get \(O \notin M\). By applying the separation theorem of convex sets of linear topological spaces (See Ref. 8), there is a hyperplane \(H\) properly separating \(\{O\} \) and \(\text{int} M\). i.e., \(\exists \lambda^* = (y_1^*, y_2^*) \in Y_{1r}^* \times Y_{2r}^*, y^* \neq O, a \in R\) such that
\[
\langle u, y^* \rangle \geq a \geq 0, \quad \forall u \in \text{int} M,  \tag{3.3}
\]
where the hyperplane \(H = \{y \in Y \mid \langle y, y^* \rangle = a\}\).

Next, we show that
\[
\langle u, y^* \rangle > 0, \quad \forall u \in \text{int} M.  \tag{3.4}
\]
We have two cases. First case. \(a > 0\). Then it follows by (3.3) that (3.4) holds. Second case. \(a = 0\). We have
\[
\langle u, y^* \rangle \geq 0, \quad \forall u \in \text{int} M.  \tag{3.5}
\]
Suppose that (3.4) does not hold. According to (3.5), there is \(u_0 \in \text{int} M\) such that \(\langle u_0, y^* \rangle = 0\). Let \(v \in \text{int} M\). Then \(\exists \beta > 0\) such that \(u_0 + \beta v \in \text{int} M\). Thus by (3.5), we have \(\langle u_0 + \beta v, y^* \rangle \geq 0\), i.e., \(\langle u_0, y^* \rangle \geq \epsilon(v, y^*)\). So, \(\langle v, y^* \rangle \leq 0\). On the other hand, also by (3.5) we get \(\langle v, y^* \rangle = 0, \forall v \in \text{int} M\). This is absurd since the hyperplane \(H\) separates \(\{O\}\) and \(\text{int} M\) properly. Hence the proof that (3.5) holds is complete.

By Lemma 3.2 we obtain
\[
\langle u, y^* \rangle \geq 0, \quad \forall u \in M.  \tag{3.6}
\]

In the following, we prove \((y_1^*, y_2^*) \in Y_{1r}^* \times Y_{2r}^*\). Indeed, assume \(y_1^* \notin Y_{1r}^*\). Then there is \(y_1 \in Y_{1r}\) such that \(\langle y_1, y_1^* \rangle < 0\). Thus, \(\lambda \langle y_1, y_1^* \rangle = \langle \lambda y_1, y_1^* \rangle < 0, \forall \lambda > 0\). By (3.6), for any \(x \in D\), any \(y_1^* \in \text{int} Y_{1r}\), any \(y_2^* \in Y_{2r}\), we have \(\beta = \langle p + y_1^*, y_1^* \rangle + \langle q + y_2^*, y_2^* \rangle \geq 0, \forall p \in g(x), q \in h(x)\). Since \(\lambda y_1 \in Y_{1r}\), then \(\lambda y_1 + y_1^* \in \text{int} Y_{1r}\). Again by (3.6), we have \(\langle p + \lambda y_1 + y_1^*, y_1^* \rangle + \langle q + y_2^*, y_2^* \rangle \geq 0\), i.e.,
\[
\lambda \langle y_1, y_1^* \rangle + \beta \geq 0, \quad \forall \lambda > 0.  \tag{3.7}
\]
However, (3.7) does not hold when \(\lambda > \beta / \langle y_1, y_1^* \rangle \geq 0\). The contradiction illustrates that \(y_1^* \in Y_{1r}^*\).

Similarly, we also prove \(y_2^* \in Y_{2r}^*\). Thus, \(\exists y^* = (y_1^*, y_2^*) \in Y_{1r}^* \times Y_{2r}^*, y^* \neq O, \forall \langle u, y^* \rangle \geq 0, \forall u \in M\). i.e.,
\[
\langle F(x) + t, y^* \rangle \geq 0, \quad \forall x \in D, t \in (\text{int} Y_{1r}) \times Y_{2r}.  \tag{3.8}
\]
Take \(u_0 \in (\text{int} Y_{1r}) \times Y_{2r}\), and \(\lambda_n > 0\) such that \(\lambda_n \to 0 (n \to \infty)\); then we have \(\langle F(x) + \lambda_n u_0, y^* \rangle \geq 0, \forall x \in D,\)
n=1,2,\cdots. Letting n\to\infty, we obtain
\[ (F(x), y^*) = \langle g(x), y_1^* \rangle + \langle h(x), y_2^* \rangle \geq 0, \forall x \in D. \]

The Theorem 3.1 is similar to Theorem 3.3 of Ref. 9, and also a generalization of Theorem 3.1 of Ref. 1.

In particular, if we set \( Y_2 = \{O\} \), then the following conclusion is derived directly by Theorem 3.1

\begin{corollary}
Let \( F = (g, h) : D \to 2^Y \) be a nearly \( Y_1 \)-semiconvexlike. Suppose that the topological interior of \( F(D) + Y_1 \) is nonempty. If there is no \( x \in D \) such that \(-g(x) \cap \text{int} \ Y_1 \neq \emptyset\), \( O \in h(x) \), then \( \exists (y_1^*, y_2^*) \in Y_1^* \times Y_2^* \setminus \{(O, O)\} \) such that
\[ \langle g(x), y_1^* \rangle + \langle h(x), y_2^* \rangle \geq 0, \forall x \in D. \]
\end{corollary}

\begin{lemma}
A set-valued map \( g : D \to 2^Y \) is nearly generalized \( Y_1 \)-subconvexlike, if and only if the set cone(\( f(D) \)) + \text{int} \( Y_1 \) is nearly convex.

The proof of Lemma 3.5 is analogous to the proof of Lemma 3.4, or the proof of Lemma 3.2 of Ref. 5.
\end{lemma}

\begin{corollary}
Let \( g : D \to 2^Y \) be a nearly generalized \( Y_1 \)-subconvexlike set-valued map.

Suppose that \( \text{int} (\langle g(D) \rangle + Y_1) \neq \emptyset \). Then exactly one of the following statements is true:

(i). \( \exists x_0 \in D, \text{ such that } -g(x_0) \cap \text{int} Y_1 \neq \emptyset; \)

(ii). \( \exists y_1^* \in Y_1^*, y_1^* \neq O, \text{ such that } \langle g(D), y_1^* \rangle \geq 0. \)
\end{corollary}

4. **Optimality Conditions**

Let \( f : D \to 2^Z \) be a set-valued map form \( D \) to \( Z \), where \( Z \) is a linear topological space with pointed convex cone \( Z_+ \) with nonempty topological interior. We consider the following two classes of vector optimization problems with set-valued maps:

\begin{align*}
\text{(P1)} & \quad \min f(x), \\
\text{s.t.} & \quad -g(x) \cap Y_1 \neq \emptyset, \\
& \quad -h(x) \cap Y_2 \neq \emptyset.
\end{align*}

\begin{align*}
\text{(P2)} & \quad \min f(x), \\
\text{s.t.} & \quad -g(x) \cap Y_1 \neq \emptyset, \\
& \quad O \in h(x).
\end{align*}

Since (P2) is a particular case of (P1) when \( Y_2 = \{O\} \), we are mainly concerned with Problem (P1) in this paper.

The feasible set of (P1) is defined by \( K = \{x \in D \mid -g(x) \cap Y_1 \neq \emptyset, -h(x) \cap Y_2 \neq \emptyset\} \).

\begin{definition}
\( x_0 \in K \) is called a weakly efficient solution of (P1), if \( \exists y_0 \in f(x_0) \), such that \( (y_0, f(K)) \cap \text{int} Z_+ \neq \emptyset \). \( (x_0, y_0) \) is called a weakly efficient pair of (P1).
\end{definition}

Set \( H(x) = f(x) \times g(x) \times h(x), \forall x \in D \); we write \( N = Z \times Y_1 \times Y_2, N_+ = Z_+ \times Y_1 \times Y_2, N' = Z' \times Y_1^* \times Y_2^* \), \( N'_+ = Z'_+ \times Y_1^* \times Y_2^* \). Obviously, \( H \) is a set-valued map from \( D \) to the product space \( N \).
If we consider N as the product of \((Z\times Y_1)\) and \(Y_2\), then the following definition is consistent with Definition 3.1.

**DEFINITION 4.2.** A set-valued map \(H=(f, g, h): D\rightarrow 2^N\) is called nearly \(N_s\) — semi-convexlike, if \(\exists \alpha \in (0, 1), \forall x_1, x_2 \in D, \forall \varepsilon > 0\), such that

\[
\alpha f(x_1) + (1-\alpha) f(x_2) \subseteq H(x_1) + H(x_2) \subseteq \varepsilon N.
\]

It is easy to verify that if the set-valued map \(H\) is nearly \(N_s\) — semi-convexlike on \(D\), \(y_0 \in Z\), then \((f(x), y_0)\times g(x)\times h(x)\) is also nearly \(N_s\) — semi-convexlike on \(D\).

**THEOREM 4.1.** Suppose the following:

(i) \(H=(f, g, h): D\rightarrow 2^N\) is nearly \(N_s\) — semi-convexlike on \(D\);
(ii) \((x_0, y_0)\) is a weakly efficient pair of \((P1)\);
(iii) \(\exists z_0 \in Z\), such that \((z_0, O, O) \in \text{int}(H(D) + N_s)\).

Then \(\exists (z^*, y_1^*, y_2^*) \in Z^* \times Y_1^* \times Y_2^*\), \(z^* \neq O\), such that

\[
\inf_{x \in D} ((f(x) + z^*) + (g(x), y_1^*) + (h(x), y_2^*)) = (y_0, z^*),
\]

\[
\inf_{x \in D} ((g(x_0), y_1^*) + (h(x_0), y_2^*)) = 0,
\]

\[
\inf_{x \in D} (g(x), y_1^*) = 0.
\]

Proof. Consider the set-valued map \(H^*(x) = (f(x), y_0) \times g(x) \times h(x), \forall x \in D\). Then \(H^*(x)\) is also nearly \(N_s\) — semi-convexlike. Since \((x_0, y_0)\) is a weakly efficient pair of \((P1)\), hence \((y_0, f(x)) \subseteq \text{int} Z, \forall x \in K\). For \(x \in D \setminus K\), we have \(-g(x) \cap Y_1 = \emptyset\), or \(-h(x) \cap Y_2 = \emptyset\). Therefore the following system has no solution:

\[-((f(x), y_0) \times g(x) - \text{int}(Z \times Y_1), \emptyset), -h(x) \cap Y_2 = \emptyset, \forall x \in D.\]

By assumption (ii) and Theorem 3.1, \(\exists (z^*, y_1^*, y_2^*) \in Z^* \times Y_1^* \times Y_2^*\), with \((z^*, y_1^*, y_2^*) \neq O\), such that \(\langle H^*(x), (z^*, y_1^*, y_2^*) \rangle \geq 0, \forall x \in D\), i.e.,

\[
\langle f(x), z^* \rangle + \langle g(x), y_1^* \rangle + \langle h(x), y_2^* \rangle \geq (y_0, z^*), \forall x \in D.\]  \hspace{1cm} (4.1)

Next, we prove \(z^* \neq O\). Suppose that \(z^* = O\). Then \((y_1^*, y_2^*) \neq O\), and (4.1) can be written as

\[
\langle g(x), y_1^* \rangle + \langle h(x), y_2^* \rangle \geq 0, \forall x \in D.\]  \hspace{1cm} (4.2)

Thus,

\[
\langle f(x) + z^* \rangle + \langle g(x) + Y_1, y_1^* \rangle + \langle h(x) + Y_2, y_2^* \rangle \geq 0, \forall x \in D.\]  \hspace{1cm} (4.3)

Since \((z_0, O, O) \in \text{int}(H(D) + N_s)\), there is \(\varepsilon > 0\) such that \((z_0, O, O) \neq \varepsilon (O, v_1 + k_1, v_2 + k_2) \in \text{int}(H(D) + N_s)\), where \((O, v_1 + k_1, v_2 + k_2) \in Z \times Y_1 \times Y_2\), \(x \in D\), \(v_1 \in g(x)\), \(v_2 \in h(x)\), \(k_1 \in Y_1\), and \(k_2 \in Y_2\). By (4.3), we have \((v_1 + k_1, y_1^*) + (v_2 + k_2, y_2^*) = 0\). Observing (4.2), we get

\[
\langle k_1, y_1^* \rangle + \langle k_2, y_2^* \rangle = 0, \forall k_1 \in Y_1, k_2 \in Y_2.\]  \hspace{1cm} (4.4)

Then we have two cases. First case. \(y_1^* \neq O\). Then (4.4) contradicts Lemma 3.3. Second case.
\[ y_1^* = 0. \] Then \( y_2^* \neq 0. \) By assumption (ii), there is \( \epsilon > 0 \) such that \( (z_0, O, O) \in \epsilon \langle O, O, v, \rangle \), where \( v \in Y_2. \) By (4.3), we obtain \( \langle v, \ y_2^* \rangle = 0, \ \forall v \in Y_2. \) Thus \( y_2^* = 0. \) This is a contradiction. Therefore we have \( z \neq 0. \)

Since \( x_0 \in K, \) there are \( p \in g(x_0), \ q \in h(x_0) \) such that \( -p \in Y_{1+}, \ -q \in Y_{2+}. \) It follows that \( \langle p, \ y_1^* \rangle + \langle q, \ y_2^* \rangle \leq 0. \) On the other hand, setting \( x = x_0 \) in (4.1), we get \( \langle y_0, \ y \rangle + \langle p, \ y_1^* \rangle + \langle q, \ y_2^* \rangle \geq \langle y_0, \ z \rangle. \) i.e.,

\[ \langle p, \ y_1^* \rangle + \langle q, \ y_2^* \rangle = 0 \] (4.5)

So, \( \langle y_0, \ z \rangle \in (\langle f(x), \ y_1^* \rangle + \langle g(x), \ y_2^* \rangle) = \langle y_0, \ z \rangle. \)

In a similar way, we get \( \inf (\langle g(x), \ y_1^* \rangle + \langle h(x), \ y_2^* \rangle) = 0, \) and \( \inf (\langle g(x), \ y_1^* \rangle) = 0. \)

**Theorem 4.2.** Suppose the following:

(i). \( x_0 \in K; \)

(ii). \( \exists y_0 \in f(x_0), \exists (z^*, y_1^*, y_2^*) \in Z^* \times Y_{1+} \times Y_{2+}, \) with \( (z^*, y_1^*, y_2^*) \neq 0, \) such that

\[ \inf (\langle f(x), \ z^* \rangle + \langle g(x), \ y_1^* \rangle + \langle h(x), \ y_2^* \rangle) \geq \langle y_0, \ z \rangle; \]

(iii). \( \exists x' \in D, \ -g(x') \cap \int Y_{1+} \neq \emptyset, \ -h(x') \cap Y_{2+} \neq \emptyset; \ -\int h(D) \cap Y_{2+} \neq \emptyset. \)

Then \( (x_0, y_0) \) is a weakly efficient pair of (P1).

Proof. According to assumption (ii), we have

\[ \langle f(x) - y_0, \ z^* \rangle + \langle g(x), \ y_1^* \rangle + \langle h(x), \ y_2^* \rangle \geq 0, \ \forall x \in D. \] (4.6)

In the following, we show \( z^* \neq 0. \) Suppose that \( z^* = 0. \) Then \( (y_1^*, y_2^*) \neq (0, O), \) and (4.6) can be written as

\[ \langle g(x), \ y_1^* \rangle + \langle h(x), \ y_2^* \rangle \geq 0, \ \forall x \in D. \] (4.7)

Thus we also have two cases. First case. \( y_1^* \neq 0. \) By assumption (iii), \( \exists u_1 \in g(x'), \ u_2 \in h(x'), \) such that \( -u_1 \in \int Y_{1+}, \ u_2 \in Y_{2+}. \) Hence, \( \langle u_1, \ y_1^* \rangle + \langle u_2, \ y_2^* \rangle < 0, \) which contradicts (4.7). Second case.

\( y_1^* = 0. \) Then \( y_2^* \neq 0. \) Again by assumption (iii), there is \( y' \in Y_{2+}, \) such that \( -y' \in \int h(D). \) It follows that \( y_2^* = 0. \) This is a contradiction. Therefore we have \( z^* \neq 0. \)

Next, we prove \((x_0, y_0)\) is weakly efficient pair of (P1). Otherwise, \( \exists x^* \in K \) such that
(y_0-f(x^0))\cap \text{int } Z. \neq \emptyset. \text{ Thus, } \exists t \in f(x^0) \text{ such that } y_0-t \in \text{int } Z. \text{ By Lemma 2.2, we have } \langle t-y_0, \, z^* \rangle < 0. \quad (4.8)

Since x^0 \in K, there are p \in g(x^0), q \in h(x^0) \text{ such that } -p \in Y_{1+}, -q \in Y_{2+}. \text{ Taking (4.8) into account, we get } \langle t-y_0, \, z^* \rangle + \langle p, \, y^*_1 \rangle + \langle q, \, y^*_2 \rangle < 0, \text{ which contradicts (4.6).} \quad \square

**COROLLARY 4.1.** Suppose the following:

(i). \text{H=(f, g, h): } D \rightarrow 2^N \text{ is nearly } N_+-\text{semiconvexlike on } D;

(ii). (x_0, y_0) \text{ is a weakly efficient pair of (P1)};

(iii). \exists x' \in D, -g(x') \cap \text{int } Y_{1+}, -h(x') \cap \text{int } Y_{2+} \neq \emptyset, -\text{int h(D)} \cap Y_{2+} \neq \emptyset.

Then \exists (z^*, \, y^*_1, \, y^*_2) \in Z_+^* \times Y_{1+}^* \times Y_{2+}^*, \, z^* \neq O, \text{ such that } \inf_{x \in D} ((f(x), \, z^*) + (g(x), \, y^*_1) + (h(x), \, y^*_2)) = \langle y_0, \, z^* \rangle,

\inf ((g(x_0), \, y^*_1) + (h(x_0), \, y^*_2)) = 0,

\inf (g(x_0), \, y^*_1) = 0

**COROLLARY 4.2.** Suppose the following:

(i). \text{H=(f, g, h): } D \rightarrow 2^N \text{ is nearly } N_+-\text{semiconvexlike on } D;

(ii). (x_0, y_0) \text{ is a weakly efficient pair of (P1)}.

Then \exists (z^*, \, y^*_1, \, y^*_2) \in Z_+^* \times Y_{1+}^* \times Y_{2+}^*, \, z^* \neq O, \text{ such that } \inf_{x \in D} ((f(x), \, z^*) + (g(x), \, y^*_1) + (h(x), \, y^*_2)) = \langle y_0, \, z^* \rangle,

\inf ((g(x_0), \, y^*_1) + (h(x_0), \, y^*_2)) = 0,

\inf (g(x_0), \, y^*_1) = 0

**COROLLARY 4.3.** Suppose the following:

(i). \text{x}_0 \in K;

(ii). \exists y_0 \in f(x_0), \exists (z^*, \, y^*_1, \, y^*_2) \in Z_+^* \times Y_{1+}^* \times Y_{2+}^*, \text{ with } z^* \neq O, \text{ such that } \inf_{x \in D} ((f(x), \, z^*) + (g(x), \, y^*_1) + (h(x), \, y^*_2)) \geq \langle y_0, \, z^* \rangle.

Then (x_0, y_0) \text{ is a weakly efficient pair of (P1)}.

**REFERENCES**


