1. Recall the definition of the self-concordant barrier function as follows:

Let \( \mathcal{K} \subseteq \mathbb{R}^n \) be a given solid, closed, convex cone, and \( F \) be a barrier function defined in int \( \mathcal{K} \). We call \( F \) to be a self-concordant function if for any \( x \in \text{int} \mathcal{K} \) and any direction \( h \in \mathbb{R}^n \) the following two properties are satisfied:

- \(|\nabla^3 F(x)[h, h, h]| \leq 2(\nabla^2 F(x)[h, h])^{3/2};\)
- \(|\nabla F(x)[h]| \leq \theta(\nabla^2 F(x)[h, h])^{1/2}.\)

Such barrier function \( F \) is called a self-concordant barrier function for the cone \( \mathcal{K} \) with the constant \( \theta \) referred to as the complexity value of the cone with respect to the barrier function \( F \).

- Show that \( F(x) = -\sum_{i=1}^n \ln x_i \) is a self-concordant barrier function for \( \mathbb{R}_+^n \) with the complexity value \( \sqrt{n} \).
- Is \( F(x) = -\ln(1 - \|x\|^2) \) is a self-concordant barrier for the unit Euclidean ball?

2. Suppose that

\(|\nabla^3 F_1(x)[h, h, h]| \leq M_1(\nabla^2 F_1(x)[h, h])^{3/2} \text{ for all } x \in \mathcal{K}_1\)

and

\(|\nabla^3 F_2(x)[h, h, h]| \leq M_2(\nabla^2 F_2(x)[h, h])^{3/2} \text{ for all } x \in \mathcal{K}_2.\)

Let \( \alpha, \beta > 0 \), and \( F(x) = \alpha F_1(x) + \beta F_2(x) \). Show that

\(|\nabla^3 F(x)[h, h, h]| \leq M(\nabla^2 F(x)[h, h])^{3/2} \text{ for all } x \in \mathcal{K}_1 \cap \mathcal{K}_2\)

where \( M = \max \left\{ \frac{M_1}{\sqrt{\alpha}}, \frac{M_2}{\sqrt{\beta}} \right\}.\)