Lecture 4: Risk Neutral Pricing

1 Part I: The Girsanov Theorem

1.1 Change of Measure and Girsanov Theorem

- Change of measure for a single random variable:

**Theorem 1.** Let $(\Omega, \mathcal{F}, P)$ be a sample space and $Z$ be an almost surely nonnegative random variable with $E[Z] = 1$. For $A \in \mathcal{F}$, define

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega).$$

Then $\tilde{P}$ is a probability measure. Furthermore, if $X$ is a nonnegative random variable, then

$$\tilde{E}[X] = E[XZ].$$

If $Z$ is almost surely strictly positive, we also have

$$E[Y] = \tilde{E}\left[\frac{Y}{Z}\right].$$

**Definition 1.** Let $(\Omega, \mathcal{F})$ be a sample space. Two probability measures $P$ and $\tilde{P}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if

$$P(A) = 0 \iff \tilde{P}(A) = 0$$

for all such $A$.

**Theorem 2** (Radon-Nikodym). Let $P$ and $\tilde{P}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable $Z$ such that $E[Z] = 1$ and

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega)$$

for $A \in \mathcal{F}$. We say $Z$ is the Radon-Nikodym derivative of $\tilde{P}$ with respect to $P$, and we write

$$Z = \frac{d\tilde{P}}{dP}.$$

**Example 1.** Change of measure on $\Omega = [0, 1]$.

**Example 2.** Change of measure for normal random variables.

- We can also perform change of measure for a whole process rather than for a single random variable. Suppose that there is a probability space $(\Omega, \mathcal{F})$ and a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$. Suppose further that $\mathcal{F}_T$-measurable $Z$ is an almost surely positive random variable satisfying $E[Z] = 1$. We can then define the Radon-Nikodym derivative process

$$Z_t = E[Z|\mathcal{F}_t], \ 0 \leq t \leq T.$$
Theorem 3. Let \( t \) satisfying \( 0 \leq t \leq T \) be given and let \( Y \) be an \( \mathcal{F}_t \)-measurable random variable. Then
\[
\tilde{E}[Y] = E[YZ] = E[YZ_t].
\]

Theorem 4. Let \( t \) and \( s \) satisfying \( 0 \leq s \leq t \leq T \) be given and let \( Y \) be an \( \mathcal{F}_t \)-measurable random variable. Then
\[
\tilde{E}[Y|\mathcal{F}_t] = \frac{1}{Z_s}E[YZ|\mathcal{F}_s].
\]

Theorem 5 (Girsanov). Let \( W_t, 0 \leq t \leq T \) be a Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\), and let \( \mathcal{F}_t, 0 \leq t \leq T \), be a filtration for this Brownian motion. Let \( \Theta_t \) be an adapted process. Define
\[
Z_t = \exp \left\{ -\int_{0}^{t} \Theta_u dW_u - \frac{1}{2} \int_{0}^{t} \Theta_u^2 du \right\},
\]
\[
\tilde{W}_t = W_t + \int_{0}^{t} \Theta_u du,
\]
and assume that
\[
E \left[ \int_{0}^{t} \Theta_u^2 Z_u^2 du \right] < +\infty.
\]
Set \( Z = Z_T \). Then \( E[Z] = 1 \) and under the probability \( \tilde{P} \) given by Theorem 5, the process \( \tilde{W} \) is a Brownian motion.

2 Part II: Fundamental Theorem in Finance (Continuous-Time)

2.1 Risk Neutral Measure

- Consider again the standard market assumption that there is a stock whose price satisfies
\[
\frac{dS_t}{S_t} = \alpha_t dt + \sigma_t dW_t.
\]
In addition, suppose that we have an adapted interest rate process \( r_t, t \geq 0 \). The corresponding discount process follows
\[
\frac{dD_t}{D_t} = -r_t dt.
\]
The discount stock price process is given by
\[
\frac{d(D_t S_t)}{D_t S_t} = \sigma_t D_t S_t \left( \Theta_t dt + dW_t \right)
\]
where we define the market price of risk to be
\[
\Theta_t = \frac{\alpha_t - r_t}{\sigma_t}.
\]
We introduce a probability measure \( \tilde{P} \) defined in Girsanov’s theorem, which uses the market price of risk \( \Theta_t \). In terms of the Brownian motion \( \tilde{W}_t \) of that theorem, we rewrite the discount stock price as
\[
\frac{d(D_t S_t)}{D_t S_t} = \sigma_t D_t S_t d\tilde{W}_t.
\]
We call \( \tilde{P} \) the risk neutral measure.
• Consider a trader who begins with initial wealth $X_0$ and at each time $t$, he buys $\Delta_t$ shares of the stock and invests or borrows at the interest rate $r_t$ as necessary to finance this. Then, the total wealth of the trader $X_t$ follows

$$dX_t = r_t X_t dt + \Delta_t \sigma_t S_t [\Theta_t dt + dW_t].$$

It implies that the discount wealth

$$d(D_t X_t) = \Delta_t \sigma_t S_t [\Theta_t dt + dW_t].$$

Under Girsanov theorem,

$$d(D_t X_t) = \Delta_t \sigma_t S_t d\tilde{W}_t.$$

• We have shown in the last lecture that the trader can construct a portfolio in order to hedge a short position in the call position. In other words, we can find $X_0$ and $\Delta_t$ such that

$$X_T = \max(S_T - K, 0).$$

Note that $D_t X_t$ is a martingale under the risk neutral probability measure. Therefore,

$$X_0 = \tilde{E}[D_T X_T] = \tilde{E} \left[ e^{-\int_0^T r_t dt} \max(S_T - K, 0) \right].$$

• Revisit the Black-Scholes-Merton formula.

### 2.2 Martingale Representation Theorem

• Martingale representation theorem:

**Theorem 6.** Let $(\Omega, \mathcal{F}, P)$ be a sample space and $W_t$ be a Brownian motion on it, and let $\mathcal{F}_t$ be the filtration generated by this Brownian motion. Let $M_t$ be a martingale with respective to this filtration. Then there exists an adapted process $\Gamma_t$ such that

$$M_t = M_0 + \int_0^t \Gamma_s dW_s$$

for all $0 \leq t \leq T$.

**Theorem 7.** Let $W_t$ be a Brownian motion on it, and let $\mathcal{F}_t$ be the filtration generated by this Brownian motion. Let $\Theta_t$ be an adapted process. Define

$$Z_t = \exp \left\{ -\int_0^t \Theta_u dW_u - \frac{1}{2} \int_0^t \Theta_u^2 du \right\},$$

$$\tilde{W}_t = W_t + \int_0^t \Theta_u du,$$

and assume that $E[\int_0^T \Theta_u^2 Z_u^2 du] < +\infty$. Set $Z = Z_T$. Then $E[Z] = 1$ and we can define a new probability $\tilde{P}$ such that

$$\tilde{P}(A) = E[1_A Z]$$
The process \( \tilde{W} \) is a Brownian motion under \( \tilde{P} \).

Now let \( \tilde{M}_t \) be a martingale under \( \tilde{P} \). Then there is an adapted process \( \tilde{\Gamma} \) such that

\[
\tilde{M}_t = \tilde{M}_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s
\]

for all \( 0 \leq t \leq T \).

Hedging with one stock. Consider the preceding market model. Let \( V_T \) be an \( \mathcal{F}_T \)-measurable random variable, standing for a future (uncertain) cash flow we want to hedge. Define \( V_t \) through

\[
D_t V_t = \text{E}[D_T V_T | \mathcal{F}_t].
\]

Note that \( D_t V_t \) is a martingale and admits a representation

\[
D_t V_t = V_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s.
\]

On the other hand, since \( \sigma_t > 0 \) for sure, we can define \( \Delta_t = \frac{\tilde{\Gamma}_t}{\sigma_t D_t S_t} \).

By choosing such \( \Delta_t \) and letting \( X_0 = V_0 \),

\[
D_t X_t = X_0 + \int_0^t \Delta_u \sigma_u D_u S_u d\tilde{W}_s = V_0 + \int_0^t \tilde{\Gamma}_s d\tilde{W}_s = D_t V_t
\]

for all \( 0 \leq t \leq T \).

2.3 Fundamental Theorem in Finance

Throughout this section,

\[
W_t = (W_t^1, \cdots, W_t^d).
\]

**Theorem 8** (Girsanov, Multidimension). Let \( T \) be a fixed positive time, and let \( \Theta_t = (\Theta_t^1, \cdots, \Theta_t^d) \) be a \( d \)-dimensional adapted process. Define

\[
Z_t = \exp \left\{ - \int_0^t \Theta_u \cdot dW_u - \frac{1}{2} \int_0^t \| \Theta_u \|^2 du \right\},
\]

\[
\tilde{W}_t = W_t + \int_0^t \Theta_u du,
\]

and assume that

\[
E \left[ \int_0^t \| \Theta_u \|^2 Z_u^2 du \right] < +\infty.
\]

Set \( Z = Z_T \). Then \( E[Z] = 1 \) and under the probability \( \tilde{P} \) given by Theorem 5, the process \( \tilde{W} \) is a \( d \)-dimensional Brownian motion.
**Theorem 9** (Martingale Representation Theorem, Multidimension). Let $T$ be a fixed positive time, and assume that $\mathcal{F}_t$ is the filtration generated by the $d$-dimensional Brownian motion $W_t, 0 \leq t \leq T$. Let $M_t$ be a martingale with respective to this filtration. Then there exists an adapted process $\Gamma_t$ such that

$$M_t = M_0 + \int_0^t \Gamma_s \cdot dW_s$$

for all $0 \leq t \leq T$.

- We assume there are $m$ stocks, each with SDE

$$dS^i_t = \alpha^i_t S^i_t dt + S^i_t \sum_{j=1}^d \sigma^{ij}_t dW^j_t, \quad i = 1, \cdots, m.$$  

In this market with stock price $S^i_t$ and interest rate $r_t$, a trader can begin with initial wealth $X_0$ and choose adapted portfolio process $\Delta^i_t$ for the $i$th stock. Then

$$dX_t = r_t X_t dt + \sum_{i=1}^m \Delta^i_t (dS^i_t - r_t S^i_t dt).$$

The differential of the discounted portfolio value is

$$d(D_t X_t) = \sum_{i=1}^m \Delta^i_t d(D_t S^i_t).$$

- Arbitrage and fundamental theorem of asset pricing:

**Definition 2.** An arbitrage is a portfolio value process $X_t$ satisfying $X_0 = 0$ and also satisfying for some time $T > 0$

$$P(X_T \geq 0) = 1, \quad P(X_T > 0) > 0.$$  

**Theorem 10** (First Fundamental Theorem of Asset Pricing). A market has a risk-neutral probability measure if and only it does not admit arbitrage.

**Definition 3.** A market model is complete if every derivative security can be hedged.

**Theorem 11** (Second Fundamental Theorem of Asset Pricing). Consider a market has a risk-neutral probability measure. The market is complete if and only if the risk-neutral probability is unique.

- Link between continuous-time version and discrete-time version
Homework Set 4 (Due on Nov 8)

1. Consider a stock whose price differential is
   \[ dS_t = r(t)S_t dt + \sigma(t)S_t dW_t. \]
   where \( r(t) \) and \( \sigma(t) \) are nonrandom functions of \( t \) and \( W \) is a Brownian motion under the risk-neutral measure \( P \). Let \( T > 0 \) be given, and consider a European call, whose value at time zero is
   \[ c(0, S(0)) = E \left[ \exp \left\{ - \int_0^T r(t) dt \right\} (S(T) - K)^+ \right] \]
   (i). Show that \( S_t \) is of the form \( S(0)e^X \), where \( X \) is a normal random variable, and determine the mean and variance of \( X \).
   (ii). Let
   \[ BSM(T, x; K, R, \Sigma) = xN \left( \frac{1}{\Sigma \sqrt{T}} \left[ \ln \frac{x}{K} + (R + \frac{\Sigma^2}{2}T) \right] - e^{-RT}KN(\Sigma \sqrt{T} | \ln \frac{x}{K} + (R - \frac{\Sigma^2}{2}T) |) \right) \]
   denote the value at time zero of a European call expiring at time \( T \) when the underlying stock has constant volatility \( \Sigma \) and the interest rate \( R \) is constant. Show that
   \[ c(0, S(0)) = BSM \left( S(0), T, \frac{1}{T} \int_0^T r(t) dt, \sqrt{\frac{1}{T} \int_0^T \sigma^2(t) dt} \right) \]

2. Consider a model with a unique risk-neutral measure \( \tilde{P} \) and constant interest rate \( r \). According to the risk-neutral pricing formula, for \( 0 \leq t \leq T \), the price at time \( t \) of a European call expiring at time \( T \) is
   \[ C(t) = E \left[ e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t) \right] \]
   where \( S(T) \) is the underlying asset price at time \( T \) and \( K \) is the strike price of the call. Similarly, the price at time \( t \) of a European put expiring at time \( T \) is
   \[ P(t) = E \left[ e^{-r(T-t)}(K - S(T))^+ | \mathcal{F}(t) \right]. \]
   Finally, because \( e^{-rt}S(t) \) is a martingale under \( \tilde{P} \), the price at time \( t \) of a forward contract for delivery of one share of stock at time \( T \) in exchange for a payment of \( K \) at time \( T \) is
   \[ F(t) = E \left[ e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t) \right] \]
   \[ = e^{rt}E [ e^{-rT}S(T) | \mathcal{F}(t) ] - e^{-r(T-t)}K \]
   \[ = S(t) - e^{-r(T-t)}K. \]
   Because
   \[ (S(T) - K)^+ - (K - S(T))^+ = S(T) - K, \]
   we have the put-call parity relationship
   \[ C(t) - P(t) = E \left[ e^{-r(T-t)}(S(T) - K)^+ - e^{-r(T-t)}(K - S(T))^+ | \mathcal{F}(t) \right] \]  
   \[ = E \left[ e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t) \right] 
   \[ = F(t). \]  

(1)
Now consider a date \( t_0 \) between 0 and \( T \), and consider a chooser option, which gives the right at time \( t_0 \) to choose to own either the call or the put.

(i) Show that at time \( t_0 \) the value of the chooser option is

\[
C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left( e^{-r(T-t_0)} K - S(t_0) \right)
\]

(ii) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time \( T \) with strike price \( K \) and the value of a put expiring at time \( t_0 \) with strike price \( e^{-r(T-t_0)} K \).

3. Let \( W(t) \), \( 0 \leq t \leq T \), be a Brownian motion on a probability space \( (\Omega, \mathcal{F}, P) \), and let \( \mathcal{F}(t), 0 \leq t \leq T \), be the filtration generated by this Brownian motion. Let the mean rate of return \( \alpha(t) \), the interest rate \( R(t) \), and the volatility \( \sigma(t) \) be adapted processes, and assume that \( \sigma(t) \) is never zero. Consider a stock price process whose differential is given by:

\[
dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad 0 \leq t \leq T.
\]

Suppose an agent must pay a cash flow at rate \( C(t) \) at each time \( t \), where \( c(t), 0 \leq t \leq T \), is an adapted process. If the agent holds \( \Delta(t) \) shares of stock at each time \( t \), then the differential of her portfolio value will be

\[
dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta S(t))dt - C(t)dt.
\]

Show that there is a nonrandom value of \( X(0) \) and a portfolio process \( \Delta(t) \), \( 0 \leq t \leq T \), such that \( X(T) = 0 \) almost surely.


5. On a standard probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P) \), we define a Brownian motion \( \{W_t, t \geq 0\} \). Let

\[
dx_t = adt + 2\sqrt{X_t}dW_t.
\]

(i) Show that

\[
L_t = \exp\left( -\frac{k}{2} \int_0^t \sqrt{X_s}dW_s - \frac{k^2}{8} \int_0^t X_s dW_s \right)
\]

is a martingale.

(ii) Use the above \( L \) to do measure change, i.e., define

\[
\frac{dQ}{dP} = L_t.
\]

What is the dynamic of process \( X_t \) under the new probability measure \( Q \)?