Models and Decision with Financial Applications

UNIT 1: Elements of Decision under Uncertainty

• We always need to make a decision (or select from among actions, options or moves) even when there exists a lot of uncertainty.

• The outcome of your decision is not only determined by a decision maker’s action, but also determined by “the state of the world”.

• 1. There are two alternatives, which one do you prefer
   A. Get 250RMB for sure; B. With 26% chance to get 1000RMB and with 74% chance to get nothing.
   2. There are two alternatives, which one do you prefer
   A. Loose 250RMB for sure; B. With 75% chance to loose 1000RMB and with 25% chance to loose nothing.

• St. Petersburg Paradox: Suppose that a fair coin is to be tossed repeatedly until the head appears. The gambler will be paid $2^n$ if the head appears in the $n$th toss. What is the expected return from this gambling? What price will you pay for the gamble?

• The expected value does not always work! We need to use the expected utility theory.
• Let \( c(x, s) \) be the consequence resulting from a combination of a decision maker’s action \( x \) and the state of the world \( s \).

• Basic Elements in a Decision Problem:
  1. a set of actions available to the decision maker, 
     \( (1, \ldots, x, \ldots, X) \);
  2. a set of possible states available to the Mother Nature, 
     \( (1, \ldots, s, \ldots, S) \);
  3. A consequence function \( c(x, s) \) showing outcomes for all pairs of \( x \) and \( s \).
  4. A probability function \( \pi(s) \) expressing the decision maker’s beliefs of the likelihood of Nature choosing each and every state;
  5. An elementary-utility function (or preference-scaling function) \( v(c) \) measuring the desirability of the different consequences to the decision maker.

• A good decision should be derived for a specific decision maker involved!
  
  – Different decision makers often have different assignments of the probability function even when they have the same available information set. The probability function is subjective.
  
  – Different decision makers always have different preference-scaling functions over consequences even when they
have the same wealth. The utility function is subjective.

- When the knowledge is imperfect, we assume that in the decision theory there is no vagueness in assigning the subjective probability and the utility function.

- There are two types of decisions: terminal actions versus informational actions.
  - Terminal actions: Action based on the decision maker’s current knowledge that will lead to a consequence.
  - Informational actions: Decisions concerning whether and how to improve decision maker’s knowledge before making a terminal action.

- We take the notion in this study that probability is simply degree of belief.
  - In general, a high degree of subjective assurance is reflected by a relatively “tight” probability distribution over the range of possible states, while a high degree of doubt would be reflected by a wide dispersion.
  - The better the knowledge the decision maker has, the better the decision in general.
  - Greater prior doubt makes it more important to acquire additional information (take an informational action) before making a terminal action.
• The Utility Function and the Expected-Utility Rule

– Utility attaches directly to consequences and only derivatively to actions.
– Let $v(c)$ be a preference-scaling function defined over consequences and $U(x)$ be the utility function defined over actions.
– While an action in situations not involving uncertainty is to choose among different consequences, an action under uncertainty is to choose among different prospects $(c_{x1}, c_{x2}, \ldots, c_{xs})$, here $c_{xs}$ is an abbreviation of $c(x, s)$.
– “Expected-utility rule” of von Neumann and Morgenstern (1944):

\[
U(x) = \pi_1 v(c_{x1}) + \pi_2 v(c_{x2}) + \ldots + \pi_s v(c_{xs})
\]

where $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ is the state probability and $\sum_{i=1}^{s} \pi_i = 1$.

• The expected-utility rule is applicable if and only if the $v(c)$ function can be determined by the assignment of “cardinal” utilities to consequences

– A cardinal variable is one that can be measured quantitatively.
– The cardinal variable has the property that regardless of shift of zero point or unit-interval, the relative magnitudes of differences remains unchanged.
We assume the decision maker is able to give an ordinal utility: i) Let \( x \) and \( y \) be two possible consequences. The decision maker can only have one from the following three: \( x \succ y \) (DM prefers \( x \) to \( y \)), or \( x \prec y \) (DM prefers \( y \) to \( x \)), or \( x \sim y \) (DM is indifferent between \( x \) to \( y \)). ii) Let \( x \), \( y \) and \( z \) be three possible consequences. There is no cycle such that \( x \succ y \succ z \succ x \).

Given an ordinal utility scale, the question now is how to cardinalize the scale.

- Let consequences in amounts of income the decision maker might receive. Let \( m \) be the worst possible consequence and assign its utility value \( v(m) = 0 \). Let \( M \) be the best possible consequence and assign its utility value \( v(M) = 1 \).

- The reference-lottery technique to construct a preference-scaling function: For any value \( c^* \) between \( m \) and \( M \), consider a choice between \( c^* \) for certain versus a gamble yielding the best possible outcome \( M \) with probability \( \pi \) and the worst possible outcome \( m \) with probability 1 - \( \pi \). Adjust the value of \( \pi \) such that the decision maker is indifferent between these two scenarios when \( \pi \) is set at \( \pi^* \). This indifference yields

\[
v(c^*) = \pi^* v(M) + \left(1 - \pi^* \right) v(m) = \pi^*
\]

- The above Reference-Lottery Technique for Assess-
ing Utilities can be applied to situations where outcome \( c \) is general (not necessarily in a form of an income), as long as the decision maker has an ordinal preference scaling to begin with.

- The utility function provides (i) a ranking of the consequences or uncertain events and (ii) the degree of preference of one against the other. Let \( x \) and \( y \) be two possible consequence.

1. \( x \succ y \) (DM prefers \( x \) to \( y \)) if and only if \( v(x) > v(y) \).
2. \( x \prec y \) (DM prefers \( y \) to \( x \)) if and only if \( v(x) < v(y) \).
3. \( x \sim y \) (DM is indifferent between \( x \) to \( y \)) if and only if \( v(x) = v(y) \).

- If \( v(c) \) is a preference-scaling function, then \( \hat{v}(c) = \alpha + \beta v(c) \) is also a valid preference-scaling function for any constant \( \alpha \) and positive constant \( \beta \).

- Choosing the best action under uncertainty is to select the action that maximizes the expected utility function.

- Axioms in utility functions
  - Let \( \succ \) denote strong preference and \( \sim \) indifference.
  - Let \((x, z; p, 1 - p)\) denote a lottery yielding \( x \) with probability \( p \) and \( z \) with probability \( 1 - p \).
– If \( x \succ z \), then \((x, z; p_1, 1 - p_1) \succ (x, z; p_2, 1 - p_2)\) if and only if \( p_1 > p_2 \).

– No complementarity effect:

If \( x \sim y \), then \((x, z; p, 1 - p) \sim (y, z; p, 1 - p)\)

If \( x \succ y \), then \((x, z; p, 1 - p) \succ (y, z; p, 1 - p)\)

– Substitution: If \( c_1 \sim (M, m; \pi_1, 1 - \pi_1) \), then \((c_1, c_2; p, 1 - p) \sim ((M, m; \pi_1, 1 - \pi_1), c_2; p, 1 - p)\)

– Any complex lottery can be simplified, based on the substitution rule, to its simplest form with two branches, one of which leading to the best consequence and one leading to the worst.

• Expected monetary value (EMV): Assume that there is an uncertain event which pays off \( x_i \) with probability \( p_i \), \( i = 1, 2, \ldots, n \). The EMV of this uncertain event is equal to \( \Sigma_{i=1}^{n} p_i x_i \).

• Certainty Equivalent: Assume that there is an uncertain event which pays off \( x_i \) with probability \( p_i \), \( i = 1, 2, \ldots, n \). \( q \) is said to be the Certainty Equivalent of this uncertainty event if

\[
v(q) = \Sigma_{i=1}^{n} p_i v(x_i)\]

In other words, the decision maker is indifferent between obtaining \( q \) for certain and entering the uncertain event:

Expected Utility of the Uncertain Event = Utility of the Certainty Equivalent.
• The preference could change when the DM’s current wealth is different.

Example: A bettor with utility \( v(x) = \ln x \), where \( x \) is total wealth, has a choice between two alternatives:

– A: Win $10,000 with probability 0.2 and win $1,000 with probability 0.8.
– B: Win $3,000 with probability 0.9 and lose $2,000 with probability 0.1.

1. If the bettor currently has $2,500, he should prefer A to B, since

\[
U(A) = 0.2v(12,500) + 0.8v(3,500) = 8.415 \\
> U(B) = 0.9v(5,500) + 0.1v(500) = 8.373
\]

– The CE of A: \( v^{-1}(8.415) - 2500 = 2014.28 \)
– The CE of B: \( v^{-1}(8.373) - 2500 = 1828.60 \)

2. If the bettor currently has $5,000, he should prefer B to A, since

\[
U(A) = 0.2v(15,000) + 0.8v(6,000) = 8.883 \\
< U(B) = 0.9v(8,000) + 0.1v(3,000) = 8.889
\]

– The CE of A: \( v^{-1}(8.883) - 5000 = 2208.38 \)
– The CE of B: \( v^{-1}(8.889) - 5000 = 2251.76 \)

3. If the bettor currently has $10,000, he should prefer A to B, since

\[
U(A) = 0.2v(20,000) + 0.8v(11,000) = 9.4254 \\
> U(B) = 0.9v(13,000) + 0.1v(8,000) = 9.4244
\]
– The CE of $A$: $v^{-1}(9.4254) - 10,000 = 2399.36$
– The CE of $B$: $uv^{-1}(9.4244) - 10,000 = 2386.97$

What is the reason behind this pattern of choices between $A$ and $B$?

• Risk Attitudes:

1. A decision maker is risk averse if the decision maker considers no uncertain event more desirable than its EMV, i.e., the decision maker’s Certainty Equivalent is less than the EMV. (A decision maker is risk averse if the decision maker prefers a certainty consequence to any risky prospect whose mathematical expectation of consequences equals that certainty.)

2. A decision maker is risk prone if the decision maker considers every uncertain event more desirable than its EMV, i.e., the decision maker’s Certainty Equivalent is always larger than the EMV. (A decision maker is risk prone if the decision maker does not prefer a certainty consequence to any risky prospect whose mathematical expectation of consequences equals that certainty.)

3. A decision maker is risk neutral if the decision maker evaluates every uncertain event by its EMV, i.e., the decision maker’s Certainty Equivalent is always equal to the EMV. (A decision maker is risk neutral if the decision maker is indifferent between
a certainty consequence and any risky prospect whose mathematical expectation of consequences equals that certainty.)

• The relationship between the risk attitudes and utility functions:

1. A decision maker is risk averse if his/her utility function is concave (the curve opens downward).
2. A decision maker is risk prone if his/her utility function is convex (the curve opens upward).
3. A decision maker is risk neutral if his/her utility function is linear.

• “Fair gamble” is used to describe an uncertain prospect whose mathematical expectation is zero.

• A risk averse person would refuse a fair gamble; A risk prone person would accept a fair gamble; and A risk neutral person would be indifferent.

• Risk Premium = EMV - Certainty Equivalent

• Example: Assume that the utility function of an individual is given by \( v = w^{1/2} \). His total wealth is $250,000 of which $160,000 is the worth of his house. There is 10% probability that his house may be destroyed by fire. What is the maximum premium (also called insurance premium) he should be willing to pay for insurance against fire?
Decision Making Theory in The Insurance Market

- Consider two possible states of the nature, $s_1$ and $s_2$, which may occur at any time $t$, where $s_1$ is the normal state while $s_2$ is the state of accident.

- Let $w_i$ denote the wealth of the DM in state $s_i$, $i = 1,2$.

- The utility function of the DM is denoted by $v(w)$ which is state independent.

- The utility function is an increasing function of the wealth. $\rightarrow v' > 0$.

- Assume that the DM is risk averse. $\rightarrow v'' < 0$.

- The probability that state $s_1$ will occur is $p_1$.

- The expected utility function which the DM wants to maximize is

$$U = p_1 v(w_1) + (1 - p_1) v(w_2)$$

- Indifference curves on the $w_1 - w_2$ plane:

$$U = p_1 v(w_1) + (1 - p_1) v(w_2) = \text{constant}$$

The DM is indifferent between any two points on an indifference curve.

- The indifference curves have a negative slope:

$$\frac{dw_2}{dw_1} = -\frac{p_1}{1 - p_1} \frac{v'(w_1)}{v'(w_2)} < 0$$
The indifference curves are convex when the DM is risk averse,
\[
\frac{d(w_2)^2}{d^2w_1} = \frac{p_1}{1 - p_1} \left[ -\frac{v''(w_1)}{v'(w_2)} + \frac{v'(w_1)}{[v'(w_2)]^2} v''(w_2) \frac{dw_2}{dw_1} \right] > 0
\]

The 45 degree line on the \( w_1 - w_2 \) plane is called the \textit{certainty line} (No uncertainty in wealth associated with the states of the world).

Along the certainty line the indifference curves have the same slope, \(-\frac{p_1}{1-p_1}\).

Insurance premium: The maximum premium the DM is willing to get himself/herself fully covered against the risk = The money which the DM pays to move from the current \((\hat{w}_1, \hat{w}_2)\) to the point on the certainty line that has the same utility value.

Insurance policy:

- If the accident occurs, the insured claims the benefit.
- If the accident does not occur, the insured loses the money paid in premium.

Suppose an insurance firm offers \(a\) worth of benefit for each $1 paid in premium.

The insurance deal is said to be fair if the expected profit from such a contract is zero,
\[
(1 - p_1)a - p_1 = 0
\]
i.e.,

\[ a = \frac{p_1}{1 - p_1} \]

• The DM’s budget line passes through his/her current position \((\hat{w}_1, \hat{w}_2)\) with slope \(-\frac{p_1}{1 - p_1}\).

\[ w_2 = \hat{w}_2 - \frac{p_1}{1 - p_1}(w_1 - \hat{w}_1) \]

• The fair premium is less than the insurance premium which is the maximum premium that the DM is willing to pay.

**Risk-bearing: The optimum of the individual**

**Basic Analysis**

• The individual’s best action under uncertainty - the “risk-bearing optimum” - involves choosing prospects

\[ x \equiv (c, \pi) \equiv (c_1, \ldots, c_S, \pi_1, \ldots, \pi_S) \]

where the \(c_s\) are the state-distributed consequences and the \(\pi_s\) are the state probabilities.

• In situations which do not involve informational actions, \(\pi\) remain constant and so the only decision variables are the \(c_s\).

• The risk-bearing optimum: Basic analysis for situations with two states
- Indifference curves on the $c_1 - c_2$ plane:

\[ U \equiv \pi_1 v(c_1) + \pi_2 v(c_2), \quad \text{where} \quad \pi_1 + \pi_2 = 1 \]

which describes the entire set of $c_1 - c_2$ combinations that are equally preferred.

- Note that $v$ is an increasing function. Thus, as $U$ increases, the family of indifference curves moves up-right.

- The marginal rates of substitution in consumption:

\[ M(c_1, c_2) \equiv -\frac{dc_2}{dc_1} \bigg|_{U=\text{constant}} = \frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)} \]

- The indifference curves have a negative slope:

\[ \frac{dc_2}{dc_1} = -\frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)} < 0 \]

- The 45 degree line on the $c_1 - c_2$ plane is called the certainty line (No uncertainty in wealth associated with the states of the world).

- Along the certainty line the indifference curves have the same slope, i.e., $\frac{dc_2}{dc_1} = -\frac{\pi_1}{\pi_2}$.

- Since

\[ \frac{d(c_2)^2}{d^2c_1} = \frac{\pi_1}{\pi_2} \left[ -\frac{v''(c_1)}{v'(c_2)} + \frac{v'(c_1)}{[v'(c_2)]^2} v''(c_2) \frac{dc_2}{dc_1} \right] \]

the indifference curves are convex when the DM is risk averse and are concave when the DM is risk prone.
The line for $c_1 - c_2$ combinations that have the same mathematical expectation:

$$\pi_1 c_1 + \pi_2 c_2 = \hat{c} \leftarrow \text{constant}$$

When the DM is risk averse, the certainty of having income $\hat{c}$ is preferred to any other $c_1 - c_2$ combinations whose mathematical expectation is $\hat{c}$.

Let $P_i$ be the price for a unit of income at state $i$, and contingent income claims $c_i$ can be exchanged according to their price ratio.

The budget line:

$$P_1 c_1 + P_2 c_2 = \hat{P} \leftarrow \text{constant}$$

Maximizing the expected utility subject to the budget constraint leads to the following indifference-curve tangency condition:

$$\frac{\pi_1 v'(c_1)}{\pi_2 v'(c_2)} = \frac{P_1}{P_2}$$

At the individual’s risk-bearing optimum, the quantities of state-claims held are such that the ratio of the probability-weighted marginal utilities equals the ratio of the state-claim prices.

Generalization to $S$ states. The Fundamental Theorem of Risk-bearing:

$$\frac{\pi_1 v'(c_1)}{P_1} = \frac{\pi_2 v'(c_2)}{P_2} = \cdots = \frac{\pi_S v'(c_S)}{P_S}$$
At the individual’s risk-bearing optimum, the expected (probability-weighted) marginal utility per dollar of income is equal in each and every state.

- Risk-aversion (with convex indifference curves) leads to diversification. While risk-prone (with concave indifference curves) leads to a non-diversified portfolio.

- The risk-bearing optimum: Basic for asset markets

  - In their risk-bearing decisions, individuals do not typically deal directly with elementary state-claims. Rather, the DM is generally endowed with, and maybe in a position to trade assets. An asset is a more or less complicated bundle of underlying pure state-claims.

  - Suppose there are only two states of the world \( s = 1, 2 \) and just two assets \( a = 1, 2 \) with prices \( P_1^A \) and \( P_2^A \). Let the income yielded by asset \( a \) in state \( s \) be \( z_{as} \).

  - Let the budget line be

    \[
P_1^A q_1 + P_2^A q_2 = \bar{W}
    \]

    where \( q_1 \) and \( q_2 \) are the numbers of units held of two assets, respectively, and \( \bar{W} \) is the DM’s endowed wealth.

  - The contingent incomes from the portfolio:

    \[
    (c_1, c_2) = q_1(z_{11}, z_{12}) + q_2(z_{21}, z_{22}) = \kappa_1(\bar{W}/P_1^A)(z_{11}, z_{12}) + \kappa_2(\bar{W}/P_2^A)(z_{21}, z_{22})
    \]
where $\kappa_1 + \kappa_2 = 1$

- Solving

$$\max_{\kappa_1, \kappa_2} U = \pi_1 v(c_1) + \pi_2 v(c_2)$$

subject to $\kappa_1 + \kappa_2 = 1$

yields the following first-order optimum condition:

$$\frac{\sum_s \pi_s v'(c_s) z_{1s}}{P_1^A} = \frac{\sum_s \pi_s v'(c_s) z_{2s}}{P_2^A}$$

At the optimum, the DM will derive the same expected marginal utility per dollar held in each asset.

- Risk-bearing Theorem for Asset Markets:

$$\frac{\sum_s \pi_s v'(c_s) z_{1s}}{P_1^A} = \frac{\sum_s \pi_s v'(c_s) z_{2s}}{P_2^A} = \cdots = \frac{\sum_s \pi_s v'(c_s) z_{As}}{P_A^A}$$

- The risk-bearing optimum: Productive opportunities

  - In making risk-bearing decisions in markets, it is possible to respond to risks by productive adjustments.

  - Let the productive opportunity constraint be $F(y_1, y_2) = 0$. Then the optimum condition is

    $$\frac{\partial F}{\partial y_1} = -\frac{dy_2}{dy_1} \bigg|_{F=0} = -\frac{dc_2}{dc_1} \bigg|_{U=\pi_1 v'(c_1)}$$

    $$\frac{\partial F}{\partial y_2} = -\frac{dc_2}{dc_1} \bigg|_{U=\pi_2 v'(c_2)}$$

  - Condition for DM’s productive and consumptive optimum position:

    $$-\frac{dy_2}{dy_1} \bigg|_{F=0} = \frac{P_1}{P_2} = -\frac{dc_2}{dc_1} \bigg|_{U}$$
State-dependent utility

- It sometimes appears that the individual’s preference-scaling function \( v(c) \) might itself depend upon the state of the world.

- We can extend the preference-scaling function to the form of \( v(c, h) \), where \( c \) still represents the outcome and \( h \) is associated with the state.

- Risk-bearing decision under state-dependent utility

\[
\frac{\pi_1 v'(c_1, h_1)}{P_{h_1}} = \frac{\pi_2 v'(c_2, h_2)}{P_{h_2}}
\]

Choosing combinations of mean and standard deviation of income

- We have described decision-making under uncertainty as choice among actions or prospects \( x = (c_1, \ldots, c_S; \pi_1, \ldots, \pi_S) \) - probability distributions that associate an amount of contingent consumption in each state of the world with the degree of belief attaching to that state.

- There is another approach to the risk-bearing decision that has proved to be very useful in modern finance theory and its applications.

- This alternative approach postulates that, for any individual, the probability distribution associated with any prospect is effectively represented by just two summary statistical measures: the mean and the standard deviation of income.
• We assume that the DM prefers higher average income measured by the mean $\mu(c)$ and lower variability measured by the standard deviation $\sigma(c)$.

• When we convert $U = Ev(c)$ into a function of $\mu(c)$ and $\sigma(c)$ only, some information has been lost. What is the justification?

• Taylor expansion of $v(c)$ about $Ec = \mu$:

$$v(c) = v(\mu) + \frac{v'(\mu)}{1!}(c-\mu) + \frac{v''(\mu)}{2!}(c-\mu)^2 + \frac{v'''(\mu)}{3!}(c-\mu)^3 + \ldots$$

$$U = Ev(\tilde{c}) = v(\mu) + \frac{v''(\mu)}{2!} + \frac{v'''(\mu)}{3!}E(c - \mu)^3 + \ldots$$

- If $v(c) = K_0 + K_1c - \frac{1}{2}K_2c^2$, then $U = K_0 + K_1\mu - \frac{1}{2}K_2(\mu^2 + \sigma^2)$ However, a quadratic $v(c)$ has some economically unacceptable implication.

- The central limit theorem states that the distribution of the sum of any large number $N$ of random variables approaches the normal distribution as $N$ increases, provided that the variables are not perfectly correlated. Normal distribution can be fully characterized by its mean and variance. Use of the normal distribution as approximation, based on the central limit theorem, remains subject to considerable questions.

- The third moment $E(c - \mu)^3$ is a measure of skewness: Skewness is zero if the two tails of a distribution are symmetrical, positive if the probability
mass humps toward the left and the right tail is long and thin, and negative in the opposite case. A preference for positive skewness in the real world suggests that individual real-world portfolios are typically not well-diversified.

– Nevertheless, the mean-variance approach remains an eminently manageable approximation, since it only requires information of the first two moments, instead of whole distribution.

• Let $\tilde{z}_a$ be the income yield per unit of asset $a$ held. Define

$$
\mu_a = E(\tilde{z}_a) \\
\sigma_a = \left[ E(\tilde{z}_a - \mu_a)^2 \right]^{1/2} \\
\sigma_{ab} = E[(\tilde{z}_a - \mu_a)(\tilde{z}_b - \mu_b)]
$$

• For a portfolio consisting of $q_a$ units for each of assets, $a = 1, \ldots, A$, the mean and the standard deviation are given as

$$
\mu = \sum_{a=1}^{A} q_a \mu_a \\
\sigma = \left[ \sum_{a=1}^{A} \sum_{b=1}^{A} (q_a \sigma_{ab} q_b) \right]^{1/2} = \left[ \sum_{a=1}^{A} \sum_{b=1}^{A} (q_a \sigma_a \rho_{ab} \sigma_b q_b) \right]^{1/2}
$$

where $\rho_{ab} = \frac{\sigma_{ab}}{\sigma_a \sigma_b}$ is the correlation coefficient between $\tilde{z}_a$ and $\tilde{z}_b$.

• The Mutual-Fund Theorem. If individuals’ preferences are summarized by desire for large $\mu$ and small
σ, and if there exists a single riskless asset and a number of risky assets, in equilibrium the asset prices will be such that everyone will wish to purchase the risky assets in the same proportions.

**Comparative statistics of the risk-bearing optimum**

- The individual’s risk-bearing optimum depends critically upon his attitudes towards risk.
- It is crucial to take into account how attitudes toward risk vary as a function of wealth.
- Decision maker’s *absolute risk aversion*:

\[ A(c) = \frac{-v''(c)}{v'(c)} \]

- A decision maker displays increasing (decreasing) aversion to absolute wealth risks if and only if \( A(c) \) is an increasing (a decreasing) function.

**Specific Utility Functions**

- *Risk tolerance and the exponential utility function*

Assume that the decision maker believes that his/her utility can be expressed by an exponential utility function:

\[ U(x) = 1 - e^{-x/R} \]
Parameter $R$ is called the risk tolerance that determines how risk averse the utility function is. Larger values of $R$ make the exponential utility function flatter, while smaller values make it more concave or more risk averse.

− How to determine $R$? Consider the gamble: Win $Y$ with probability 0.5 and lose $Y/2$ with probability 0.5.

The risk tolerance $R$ is then approximately equal to the largest value of $Y$ for which you would prefer to take the gamble rather than not to take it.

Suppose that the expected value and variance of the payoffs are $\mu$ and $\sigma^2$, respectively. Then,

$$\text{Certainty Equivalence} \approx \mu - \frac{0.5\sigma^2}{R}$$

when the utility function is of an exponential form.

• Decreasing risk aversion and the logarithmic utility function

If an individual’s preferences show decreasing risk aversion, then the risk premium decreases if a constant amount is added to all payoffs in a gamble. In other words, decreasing risk aversion means the more money you have, the less nervous you are about a particular bet.

If $U(x) = \ln x$, then we have the following table,
### Constant risk aversion and the logarithmic utility function

An individual displays constant risk aversion if the risk premium for a gamble does not depend on the initial amount of wealth held by the decision maker. In other words, a constantly risk aversion person would be just anxious about taking a bet regardless of the amount of money available.

If an individual is constantly risk aversion, the utility function is of an exponential form:

\[
U(x) = 1 - e^{-x/R}
\]

If \( R = 35 \), we have the following table:

<table>
<thead>
<tr>
<th>50-50 Gamble Between</th>
<th>Expected Value</th>
<th>Certainty Equivalent</th>
<th>Risk Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>10, 40</td>
<td>25</td>
<td>21.88</td>
<td>3.12</td>
</tr>
<tr>
<td>20, 50</td>
<td>35</td>
<td>31.88</td>
<td>3.12</td>
</tr>
<tr>
<td>30, 60</td>
<td>45</td>
<td>41.88</td>
<td>3.12</td>
</tr>
<tr>
<td>40, 70</td>
<td>55</td>
<td>51.88</td>
<td>3.12</td>
</tr>
</tbody>
</table>

- **Endowment effects**: An increase in wealth must raise the optimum amount of contingent consumption claims.
held in at least one state $t$. $\Rightarrow$ From the Fundamental Theorem of Risk Bearing:

$$\frac{\pi_1 u'(c_1)}{P_1} = \frac{\pi_2 u'(c_2)}{P_2} = \ldots = \frac{\pi_S u'(c_S)}{P_S} = \lambda$$

$\lambda$ falls when assuming risk aversion. $\Rightarrow$ When $c_t$ increases, $c_s$ will increase for each and every other state, $s \neq t$. $\Rightarrow$ Furthermore, under the assumption of decreasing (increasing) absolute risk-aversion, the absolute difference between any pair of state-claim holdings, $|c_t^* - c_s^*|$ rises (falls).

- “Pure substitution effect”: Suppose that, $P_s$, the price of claims to consumption in state $s$, rises. It will have a negative impact on $c_s$ and a positive impact on all other $c_t$, $t \neq s$.

- Stochastic dominance: Specific conditions under which one prospect or state-distributed consumption vector is preferred over another.

  - Let two cumulative distribution functions be:
    
    $$F(c) = \text{Prob}\{\tilde{c}_1 \leq c\}$$
    $$G(c) = \text{Prob}\{\tilde{c}_2 \leq c\}$$

  - Definition. First-order stochastic dominance. If, for all $c$, $F(c) \leq G(c)$ and the inequality is strict over some interval, the distribution $F$ exhibits first-order stochastic dominance over $G$. 

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Ranking Theorem I. For all increasing, piecewise differential functions $v(c)$, if $F$ exhibits first-order stochastic dominance over $G$, then

$$E_F\{v(c) \} > E_G\{v(c)\}$$

Definition. Second-order stochastic dominance. If, for all $c$,

$$\int_{-\infty}^{c} F(r)dr \leq \int_{-\infty}^{c} G(r)dr$$

with the inequality holding strictly over some interval, the distribution $F$ exhibits second-order stochastic dominance over $G$.

Ranking Theorem II. For all increasing, concave piecewise differential functions $v(c)$, if $F$ exhibits second-order stochastic dominance over $G$, then

$$E_F\{v(c) \} > E_G\{v(c)\}$$