ENGG 5501: Foundations of Optimization **2023–24 First Term**

Handout 7: Optimality Conditions and Lagrangian Duality

Instructor: Anthony Man–Cho So November 27, 2023

1 Introduction

In previous lectures, we considered linear and conic linear optimization problems and derived conditions that characterize their optimal solutions. For instance, for the pair of primal–dual LPs

$$(P)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq \mathbf{0}, \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq \mathbf{0}, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given, we have shown that the solutions x^* and (y^*, s^*) are optimal for (P) and (D), respectively, if and only if they satisfy the following *optimality conditions*:

$$\begin{array}{rll} x_i^* s_i^* = 0 \quad \text{for } i = 1,\ldots,n, & & \text{(complementarity)} \\ Ax^* = b,\ x^* \geq \mathbf{0}, & & \text{(primal feasiblity)} \\ A^T y^* + s^* = c,\ s^* \geq \mathbf{0}. & & \text{(dual feasibility)} \end{array} \tag{1}$$

Such conditions are useful as they reduce the problem of finding optimal solutions to (P) and (D) to that of finding a solution to a system of equations. Now, a natural question arises whether we can find similar conditions for general nonlinear optimization problems. To motivate our discussion, let us first consider a univariate, twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$. From basic calculus, if $\bar{x} \in \mathbb{R}$ is a local minimum[1] of f, then we must have

$$\left.\frac{df(x)}{dx}\right|_{x=\bar{x}} = 0. \tag{2}$$

In other words, condition (2) is a *necessary condition* for $\bar{x}$ to be a local minimum. However, it is *not* a sufficient condition, as an $\bar{x} \in \mathbb{R}$ that satisfies (2) can be a local maximum or just a stationary point. In order to certify that $\bar{x}$ is indeed a local minimum, one could check, in addition to (2), whether

$$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=\bar{x}} > 0. \tag{3}$$

In particular, condition (3) is a *sufficient condition* for $\bar{x}$ to be a local minimum.

It turns out that conditions (2) and (3) can be generalized to *multivariate* twice continuously differentiable functions. In the following section, we shall first consider optimality conditions for *unconstrained* optimization of such functions. The main technical tool needed for establishing those conditions is the following (see, e.g., [6, Theorem 5.15]):

[1] Recall that for a generic optimization problem $\min_{x\in X\subseteq\mathbb{R}^n} f(x)$, a point $x^* \in X$ is called a **global minimum** if $f(x^*) \leq f(x)$ for all $x \in X$. On the other hand, if there exists an $\epsilon > 0$ such that the point $x^* \in X$ satisfies $f(x^*) \leq f(x)$ for all $x \in X \cap B^\circ(x^*, \epsilon)$, then it is called a **local minimum**. Here, $B^\circ(\bar{x}, \epsilon) = \{x \in \mathbb{R}^n : \|x - \bar{x}\|_2 < \epsilon\}$ denotes the *open* ball centered at $\bar{x} \in \mathbb{R}^n$ of radius $\epsilon > 0$.

Theorem 1 (Taylor's Theorem) *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $n \geq 1$ be an integer. Suppose that the function $f : [a, b] \to \mathbb{R}$ satisfies the following properties:*

1. *$f^{(n-1)}$ is continuous on $[a, b]$,*
2. *$f^{(n)}(t)$ exists for every $t \in (a, b)$.*

Let $a \leq t_1 < t_2 \leq b$, and define

$$P(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(t_1)}{j!}(t - t_1)^j.$$

Then, there exists a $t_0 \in [t_1, t_2]$ such that

$$f(t_2) = P(t_2) + \frac{f^{(n)}(t_0)}{n!}(t_2 - t_1)^n.$$

2 Unconstrained Optimization Problems

Armed with Theorem 1, we are ready to prove the following result:

Proposition 1 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If there exists a $d \in \mathbb{R}^n$ such that $\nabla f(\bar{x})^T d < 0$, then there exists an $\alpha_0 > 0$ such that $f(\bar{x} + \alpha d) < f(\bar{x})$ for all $\alpha \in (0, \alpha_0)$. In other words, d is a* **descent direction** *of f at $\bar{x}$.*

Proof Since ∇f is continuous at $\bar{x} \in \mathbb{R}^n$ and $\nabla f(\bar{x})^T d < 0$, there exists an $\alpha_0 > 0$ such that $\nabla f(\bar{x} + \alpha d)^T d < 0$ for all $\alpha \in [0, \alpha_0)$. Now, consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(\alpha) = f(\bar{x} + \alpha d)$. By the Chain Rule, we have

$$\frac{d\tilde{f}(\alpha)}{d\alpha} = \nabla f(\bar{x} + \alpha d)^T d.$$

Thus, by Theorem 1, for any $\alpha \in (0, \alpha_0)$, there exists a $t_0 \in [0, \alpha_0)$ such that

$$f(\bar{x} + \alpha d) = \tilde{f}(\alpha) = \tilde{f}(0) + \alpha \nabla f(\bar{x} + t_0 d)^T d < \tilde{f}(0) = f(\bar{x}),$$

as desired. □

Proposition 1 has the following immediate corollary, which is a generalization of (2) to multivariate differentiable functions:

Corollary 1 (First Order Necessary Condition for Unconstrained Optimization) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a local minimum, then we have $\nabla f(\bar{x}) = \mathbf{0}$. In particular, we have $\left\{ d \in \mathbb{R}^n : \nabla f(\bar{x})^T d < 0 \right\} = \emptyset$.*

Proof Suppose to the contrary that $\nabla f(\bar{x}) \neq \mathbf{0}$. Let $d = -\nabla f(\bar{x})$. Then, we have $\nabla f(\bar{x})^T d = -\|\nabla f(\bar{x})\|_2^2 < 0$. Hence, by Proposition 1, there exists an $\alpha_0 > 0$ such that $f(\bar{x} + \alpha d) < f(\bar{x})$ for all $\alpha \in (0, \alpha_0)$, which contradicts the fact that $\bar{x}$ is a local minimum. Thus, we have $\nabla f(\bar{x}) = \mathbf{0}$. This completes the proof. □

It turns out that if f is convex, then the above necessary condition is also sufficient:

Proposition 2 *Let $S \subseteq \mathbb{R}^n$ be an open convex set. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex on S and continuously differentiable at $\bar{x} \in S$. Then, $\bar{x}$ is a global minimum in S iff $\nabla f(\bar{x}) = \mathbf{0}$.*

Proof By virtue of Corollary 1, it suffices to show that if $\nabla f(\bar{x}) = \mathbf{0}$, then $\bar{x}$ is a global minimum in S. Indeed, if $\nabla f(\bar{x}) = \mathbf{0}$, then we have $\nabla f(\bar{x})^T(x - \bar{x}) = 0$ for all $x \in S$. By Theorem 14 of Handout 2, we conclude that $f(x) \geq f(\bar{x})$ for all $x \in S$, which completes the proof. □

An inspection of the proof of Proposition 2 reveals that its essential ingredient is the minorization of convex functions by affine functions (i.e., Theorem 14 of Handout 2). This suggests that the result of Proposition 2 can be extended to non–differentiable functions. One way to formalize this observation is to use the notion of subdifferentials (see Definition 10 of Handout 2).

Proposition 3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be arbitrary. Then, $\bar{x}$ is a global minimum iff $\mathbf{0} \in \partial f(\bar{x})$.*

Proof Recall that

$$\partial f(\bar{x}) = \left\{ s \in \mathbb{R}^n : f(x) \geq f(\bar{x}) + s^T(x - \bar{x}) \text{ for all } x \in \mathbb{R}^n \right\},$$

and that $\bar{x}$ is a global minimum if and only if $f(x) \geq f(\bar{x}) = f(\bar{x}) + \mathbf{0}^T(x - \bar{x})$ for all $x \in \mathbb{R}^n$. This completes the proof. □

Although Proposition 3 may seem very powerful, it is often difficult to compute ∂f for an arbitrary f. Moreover, it is important to note that even if f is differentiable at $\bar{x}$, we may not have $\nabla f(\bar{x}) \in \partial f(\bar{x})$ if f is not convex at $\bar{x}$.

Similar to the univariate case, even if $\bar{x} \in \mathbb{R}^n$ satisfies $\nabla f(\bar{x}) = \mathbf{0}$, we cannot conclude that $\bar{x}$ is a local minimum. However, if $\bar{x}$ also satisfies $\nabla^2 f(\bar{x}) \succ \mathbf{0}$, then it would indeed be a local minimum. Specifically, we have the following proposition, which generalizes conditions (2) and (3) for the univariate case:

Proposition 4 (Second Order Sufficient Condition for Unconstrained Optimization) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $\bar{x} \in \mathbb{R}^n$. If $\nabla f(\bar{x}) = \mathbf{0}$ and $\nabla^2 f(\bar{x})$ is positive definite, then $\bar{x}$ is a local minimum.*

Proof For any $d \in \mathbb{R}^n$ such that $\|d\|_2^2 = 1$, consider the function $\tilde{f}_d : \mathbb{R} \to \mathbb{R}$ given by $\tilde{f}_d(\alpha) = f(\bar{x} + \alpha d)$. By the Chain Rule, we have

$$\frac{d\tilde{f}(\alpha)}{d\alpha} = \nabla f(\bar{x} + \alpha d)^T d, \qquad \frac{d^2\tilde{f}(\alpha)}{d\alpha^2} = d^T \nabla^2 f(\bar{x} + \alpha d) d. \tag{4}$$

Since $\nabla^2 f$ is continuous at $\bar{x} \in \mathbb{R}^n$ and $\nabla^2 f(\bar{x}) \succ \mathbf{0}$, there exists an $\alpha_0 > 0$ such that *for all* unit vectors $d \in \mathbb{R}^n$ and for all $\alpha \in [0, \alpha_0)$, we have $\nabla^2 f(\bar{x} + \alpha d) \succ \mathbf{0}$. Now, suppose that $\bar{x}$ is not a local minimum. Then, there exists an $\bar{x}' \in \mathbb{R}^n$ such that $\|\bar{x}' - \bar{x}\|_2 < \alpha_0$ and $f(\bar{x}') < f(\bar{x})$. Let $d = (\bar{x}' - \bar{x})/\|\bar{x}' - \bar{x}\|_2$ and $\alpha = \|\bar{x}' - \bar{x}\|_2$. Then, by (4), Theorem 1, and the fact that $\nabla f(\bar{x}) = \mathbf{0}$, we have

$$f(\bar{x}) > f(\bar{x}') = f(\bar{x} + \alpha d) = \tilde{f}_d(\alpha) = \tilde{f}_d(0) + \frac{\alpha^2}{2} d^T \nabla^2 f(\bar{x} + t_0 d) d > \tilde{f}_d(0) = f(\bar{x})$$

for some $t_0 \in (0, \alpha_0)$, which is a contradiction. This completes the proof. □

Remarks: With slightly more effort, one can show that whenever the conditions in Proposition 4 hold, then $\bar{x}$ is a *strict* local minimum; i.e., there exists an $\epsilon > 0$ such that $f(\bar{x}) < f(x)$ for all $x \in B^\circ(\bar{x}, \epsilon) \backslash \{\bar{x}\}$. We refer the readers to [1, Theorem 4.1.4] for details.

3 Constrained Optimization Problems

In this section we turn our attention to optimization problems with both equality and inequality constraints. Specifically, let $f, g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2} : \mathbb{R}^n \to \mathbb{R}$ be functions that are continuously differentiable on the non–empty open subset X of $\mathbb{R}^n$. Consider the following class of optimization problems:

$$
\begin{array}{lll}
\inf & f(x) & \\
\text{subject to} & g_i(x) \le 0 & \text{for } i = 1, \ldots, m_1, \\
& h_j(x) = 0 & \text{for } j = 1, \ldots, m_2, \\
& x \in X. &
\end{array}
\tag{5}
$$

Let

$$S = \{x \in X : g_i(x) \le 0 \;\text{ for } i = 1, \ldots, m_1;\; h_j(x) = 0 \;\text{ for } j = 1, \ldots, m_2\}$$

be the feasible region of (5). Our primary goal in this section is to prove the following theorem:

Theorem 2 (The Fritz John Necessary Conditions) *Let $\bar{x} \in S$ be a local minimum of problem (5). Then, there exist $u \in \mathbb{R}$, $v_1, \ldots, v_{m_1} \in \mathbb{R}$, and $w_1, \ldots, w_{m_2} \in \mathbb{R}$ such that*

$$
\begin{array}{rcll}
u\nabla f(\bar{x}) + \displaystyle\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) & = & \mathbf{0}, & \\
u, v_i & \ge & 0 & \text{for } i = 1, \ldots, m_1, \\
(u, v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}) & \ne & \mathbf{0}. &
\end{array}
\tag{6}
$$

Furthermore, in every neighborhood $\mathcal{N}$ of $\bar{x}$, there exists an $x' \in \mathcal{N}$ such that $v_i g_i(x') > 0$ for all $i \in \{1, \ldots, m_1\}$ with $v_i \ne 0$, and $w_j h_j(x') > 0$ for all $j \in \{1, \ldots, m_2\}$ with $w_j \ne 0$.

REMARKS:

(a) The last statement in Theorem 2 actually implies the complementary slackness condition (i.e., $v_i g_i(\bar{x}) = 0$ for $i = 1, \ldots, m_1$), since if $v_i > 0$, then the corresponding constraint $g_i(x) \le 0$ will be violated by points arbitrarily close to $\bar{x}$. This implies that $g_i(\bar{x}) = 0$.

(b) In Theorem 2, the scalar v_i (resp. w_j) is usually called the **Lagrange multiplier** of the corresponding constraint $g_i(x) \le 0$, where $i = 1, \ldots, m_1$ (resp. $h_j(x) = 0$, where $j = 1, \ldots, m_2$). In a fashion reminiscent to the case of LP, we may summarize the Fritz John necessary conditions in (6) as follows:

$$
\begin{array}{rclll}
g_i(\bar{x}) & \le & 0 & \text{for } i = 1, \ldots, m_1, & \text{(primal feasibility I)} \\
h_j(\bar{x}) & = & 0 & \text{for } j = 1, \ldots, m_2, & \text{(primal feasibility II)} \\
u\nabla f(\bar{x}) + \displaystyle\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) & = & \mathbf{0}, & & \text{(dual feasibility I)} \\
u, v_i & \ge & 0 & \text{for } i = 1, \ldots, m_1, & \text{(dual feasibility II)} \\
(u, v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}) & \ne & \mathbf{0}, & & \text{(dual feasibility III)} \\
v_i g_i(\bar{x}) & = & 0 & \text{for } i = 1, \ldots, m_1. & \text{(complementary slackness)}
\end{array}
\tag{7}
$$

Proof We shall use the penalty function approach (see, e.g, [2]) to prove Theorem 2. The idea is to disregard the constraints in (5) while adding to the objective a high penalty for violating them. By doing so, we obtain an unconstrained optimization problem, which we could tackle using the appropriate optimality conditions. As we increase the penalty and pass to the limit, the desired optimality conditions for the original problem would then follow.

To realize the above idea, consider the following sequence of "penalized" problems, where $k = 1, 2, \ldots$:

$$\begin{aligned} \text{minimize} \quad & F^k(x) \equiv f(x) + \frac{k}{2}\sum_{i=1}^{m_1}\left(g_i^+(x)\right)^2 + \frac{k}{2}\sum_{j=1}^{m_2}(h_j(x))^2 + \frac{1}{2}\|x-\bar{x}\|_2^2 \\ \text{subject to} \quad & x \in B(\bar{x}, \epsilon). \end{aligned} \tag{8}$$

Here, $g_i^+(x) = \max\{g_i(x), 0\}$ for $i = 1, \ldots, m_1$, and $\epsilon > 0$ is such that $B(\bar{x}, \epsilon) \subseteq X$ and $f(\bar{x}) \le f(x)$ for all $x \in S \cap B(\bar{x}, \epsilon)$. Note that such an $\epsilon > 0$ exists, since X is open and $\bar{x} \in S$ is a local minimum of (5). To gain some intuition on problem (8), observe that the term $(k/2)\left(g_i^+(x)\right)^2$ can be viewed as a penalty for violating the constraint $g_i(x) \le 0$. Similarly, the term $(k/2)(h_j(x))^2$ can be viewed as a penalty for violating the constraint $h_j(x) = 0$. Finally, since we are only interested in the points that lie in a neighborhood of the local minimum $\bar{x}$, we introduce the proximity term $(1/2)\|x-\bar{x}\|_2^2$, so that points far away from $\bar{x}$ will be penalized. As we shall see, such a property will be useful in our analysis.

Let x^k be an optimal solution to (8), where $k = 1, 2, \ldots$. Note that such an x^k exists by Weierstrass' theorem. Let us first prove that the sequence $\{x^k\}_k$ converges to $\bar{x}$. By definition of F^k and the feasibility of $\bar{x}$, we have

$$F^k(x^k) = f(x^k) + \frac{k}{2}\sum_{i=1}^{m_1}\left(g_i^+(x^k)\right)^2 + \frac{k}{2}\sum_{j=1}^{m_2}\left(h_j(x^k)\right)^2 + \frac{1}{2}\|x^k-\bar{x}\|_2^2 \le F^k(\bar{x}) = f(\bar{x}). \tag{9}$$

We claim that

$$\lim_{k\to\infty}\sum_{i=1}^{m_1}\left(g_i^+(x^k)\right)^2 = 0, \quad \lim_{k\to\infty}\sum_{j=1}^{m_2}\left(h_j(x^k)\right)^2 = 0. \tag{10}$$

Indeed, upon dividing both sides of (9) by k, we see that

$$\frac{f(x^k)}{k} \le \frac{f(x^k)}{k} + \frac{1}{2}\sum_{i=1}^{m_1}\left(g_i^+(x^k)\right)^2 + \frac{1}{2}\sum_{j=1}^{m_2}\left(h_j(x^k)\right)^2 + \frac{1}{2k}\|x^k-\bar{x}\|_2^2 \le \frac{f(\bar{x})}{k}. \tag{11}$$

Since the sequence $\{f(x^k)\}_k$ is bounded over $B(\bar{x}, \epsilon)$, we have $\lim_{k\to\infty} f(x^k)/k = 0$. It follows from (11) that

$$\lim_{k\to\infty}\left(\frac{f(x^k)}{k} + \frac{1}{2}\sum_{i=1}^{m_1}\left(g_i^+(x^k)\right)^2 + \frac{1}{2}\sum_{j=1}^{m_2}\left(h_j(x^k)\right)^2 + \frac{1}{2k}\|x^k-\bar{x}\|_2^2\right) = 0.$$

Moreover, since $\|x^k - \bar{x}\|_2 \le \epsilon$ for $k = 1, 2, \ldots$, we have $\lim_{k\to\infty}\|x^k-\bar{x}\|_2^2/(2k) = 0$. It follows that (10) holds as claimed.

Now, let $\tilde{x}$ be a limit point of the sequence $\{x^k\}_k$.[2] Note that such a point exists, since the sequence $\{x^k\}_k$ belongs to the compact set $B(\bar{x}, \epsilon)$ and hence has a convergent subsequence. By (10), we see that the limit point $\tilde{x}$ satisfies $g_i(\tilde{x}) \leq 0$ for $i = 1, \ldots, m_1$ and $h_j(\tilde{x}) = 0$ for $j = 1, \ldots, m_2$. Moreover, (9) implies that

$$f(x^k) + \frac{1}{2}\|x^k - \bar{x}\|_2^2 \leq f(\bar{x})$$

for $k = 1, 2, \ldots$. Hence, upon taking $k \to \infty$ and using the continuity of f, we obtain

$$f(\tilde{x}) + \frac{1}{2}\|\tilde{x} - \bar{x}\|_2^2 \leq f(\bar{x}). \tag{12}$$

On the other hand, since $\tilde{x} \in S \cap B(\bar{x}, \epsilon)$, we have $f(\bar{x}) \leq f(\tilde{x})$, which, when combined with (12), yields $\|\tilde{x} - \bar{x}\|_2^2 = 0$. This shows that the sequence $\{x^k\}_k$ actually converges to $\bar{x}$. In particular, when k is sufficiently large, x^k is an interior point of $B(\bar{x}, \epsilon)$, which implies that x^k is an *unconstrained* local minimum of F^k. For such k, we can apply the necessary condition for unconstrained optimization (Corollary 1) and conclude that $\nabla F^k(x^k) = \mathbf{0}$. To compute $\nabla F^k(x^k)$, we need the following lemma:

Lemma 1 *Let $q : \mathbb{R} \to \mathbb{R}_+$ be the function defined by $q(x) = (\max\{0, x\})^2$. Then, q is continuously differentiable, with $dq/dx = 2\max\{0, x\}$.*

Proof We prove the statement via first principles. Specifically, let $x, t \in \mathbb{R}$. We compute

$$\begin{aligned}
\frac{q(x+t) - q(x)}{t} &= \frac{(\max\{0, x+t\} + \max\{0, x\})(\max\{0, x+t\} - \max\{0, x\})}{t} \\
&= \begin{cases} 2x + t & \text{if } x > 0 \text{ and } -|x| \leq t \leq |x|, \\ 0 & \text{if } x < 0 \text{ and } -|x| \leq t \leq |x|, \\ (\max\{0, t\})^2 / t & \text{if } x = 0. \end{cases}
\end{aligned}$$

It follows that

$$\lim_{t \to 0^+} \frac{q(x+t) - q(x)}{t} = \lim_{t \to 0^-} \frac{q(x+t) - q(x)}{t} = \begin{cases} 2x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0; \end{cases}$$

i.e., $dq/dx = 2\max\{0, x\}$. It is now clear that dq/dx is continuous. This completes the proof. □

By Lemma 1 and the Chain Rule, we see that the function $x \mapsto \left(g_i^+(x)\right)^2$ is continuously differentiable with gradient $2g_i^+(x)\nabla g_i(x)$. Thus, we conclude that

$$\mathbf{0} = \nabla F^k(x^k) = \nabla f(x^k) + \sum_{i=1}^{m_1} \left(k g_i^+(x^k)\right) \nabla g_i(x^k) + \sum_{j=1}^{m_2} \left(k h_j(x^k)\right) \nabla h_j(x^k) + x^k - \bar{x}. \tag{13}$$

Now, for $k = 1, 2, \ldots$, let

$$\delta^k = \sqrt{1 + \sum_{i=1}^{m_1} \left(k g_i^+(x^k)\right)^2 + \sum_{j=1}^{m_2} \left(k h_j(x^k)\right)^2}$$

[2] Recall that p is a **limit point** of a set S if every neighborhood of p contains a point $q \neq p$ such that $q \in S$.

and

$$u^k = \frac{1}{\delta^k} \geq 0; \quad v_i^k = \frac{k g_i^+(x^k)}{\delta^k} \geq 0 \text{ for } i = 1, \ldots, m_1; \quad w_j^k = \frac{k h_j(x^k)}{\delta^k} \text{ for } j = 1, \ldots, m_2. \tag{14}$$

Then, upon dividing both sides of (13) by δ^k, we obtain

$$u^k \nabla f(x^k) + \sum_{i=1}^{m_1} v_i^k \nabla g_i(x^k) + \sum_{j=1}^{m_2} w_j^k \nabla h_j(x^k) + \frac{1}{\delta^k}\left(x^k - \bar{x}\right) = \mathbf{0}. \tag{15}$$

Note that by construction, we have

$$\left(u^k\right)^2 + \sum_{i=1}^{m_1} \left(v_i^k\right)^2 + \sum_{j=1}^{m_2} \left(w_j^k\right)^2 = 1. \tag{16}$$

This implies that the sequence $\left\{\left(u^k, v_1^k, \ldots, v_{m_1}^k, w_1^k, \ldots, w_{m_2}^k\right)\right\}_k$ is bounded. In particular, by taking a subsequence if necessary, the sequence converges to some limit $(u, v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2})$. The FJ conditions (6) then follow from (15), (14) and (16). To prove the last statement in Theorem 2, let $I = \{i \in \{1, \ldots, m_1\} : v_i > 0\}$ and $J = \{j \in \{1, \ldots, m_2\} : w_j \neq 0\}$. Then, for all sufficiently large k, we must have $v_i v_i^k > 0$ for all $i \in I$ and $w_j w_j^k > 0$ for all $j \in J$. This, together with (14), implies that $v_i g_i(x^k) > 0$ for all $i \in I$ and $w_j h_j(x^k) > 0$ for all $j \in J$. Since every neighborhood of $\bar{x}$ must contain some x^k, the proof is completed. □

For any $\bar{x} \in \mathbb{R}^n$, if there exist Lagrange multipliers $u, \{v_i\}_{i=1}^{m_1}, \{w_j\}_{j=1}^{m_2}$ that solve system (7), then we say that $\bar{x}$ is a **Fritz John (FJ) point**. We remark that an FJ point need not be a local minimum, as the Fritz John conditions (7) are only *necessary* conditions for local optimality.

The above formulation of Fritz John's theorem is very general and can be used to derive many necessary conditions for optimization problems of the form (5). For instance, we can use it to derive the Karush–Kuhn–Tucker theorem:

Theorem 3 (The Karush–Kuhn–Tucker Necessary Conditions) *Let $\bar{x} \in S$ be a local minimum of problem (5). Let $I = \{i \in \{1, \ldots, m_1\} : g_i(\bar{x}) = 0\}$ be the index set for the active constraints. Suppose that $\bar{x}$ is* regular*; i.e., the family $\{\nabla g_i(\bar{x})\}_{i \in I} \cup \{\nabla h_j(\bar{x})\}_{j=1}^{m_2}$ of vectors is linearly independent. Then, there exist $v_1, \ldots, v_{m_1} \in \mathbb{R}$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$ such that*

$$\begin{aligned} \nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) &= \mathbf{0}, \\ v_i &\geq 0 \quad \text{for } i = 1, \ldots, m_1. \end{aligned} \tag{17}$$

Furthermore, in every neighborhood $\mathcal{N}$ of $\bar{x}$, there exists an $x' \in \mathcal{N}$ such that $v_i g_i(x') > 0$ for all $i \in \{1, \ldots, m_1\}$ with $v_i \neq 0$, and $w_j h_j(x') > 0$ for all $j \in \{1, \ldots, m_2\}$ with $w_j \neq 0$.

We leave the proof of Theorem 3 as an easy exercise to the reader. Similar to the notion of an FJ point, we say that $\bar{x} \in \mathbb{R}^n$ is a **Karush–Kuhn–Tucker (KKT) point** if (i) $\bar{x} \in S$ and (ii) there exist Lagrange multipliers $\{v_i\}_{i=1}^{m_1}, \{w_j\}_{j=1}^{m_2}$ that solve system (17).

We remark that if the gradient vectors of the active constraints are not linearly independent, then the KKT conditions are *not* necessary for local optimality, *even when the optimization problem is convex*. The following example demonstrates such possibility.

Example 1 (Failure of the KKT Conditions in the Absence of Regularity) *Consider the following problem:*

$$\begin{array}{ll} \min & x_1 \\ \text{subject to} & (x_1-1)^2+(x_2-1)^2 \le 1, \\ & (x_1-1)^2+(x_2+1)^2 \le 1. \end{array} \tag{18}$$

Since there is only one feasible solution (i.e., $(x_1, x_2) = (1, 0)$), it is automatically optimal. Besides the primal feasibility condition, the KKT conditions of (18) are given by

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2v_1 \begin{bmatrix} x_1-1 \\ x_2-1 \end{bmatrix} + 2v_2 \begin{bmatrix} x_1-1 \\ x_2+1 \end{bmatrix} &= \mathbf{0}, \\ v_1\left((x_1-1)^2+(x_2-1)^2-1\right) &= 0, \\ v_2\left((x_1-1)^2+(x_2+1)^2-1\right) &= 0, \\ v_1, v_2 &\ge 0. \end{aligned}$$

However, it is clear that there is no solution $(v_1, v_2) \ge \mathbf{0}$ to the above system when $(x_1, x_2) = (1, 0)$.

Note that in Theorem 3 we express the regularity condition in terms of the gradient vectors of the active constraints. There are other regularity conditions, a more well–known one is the following:

Theorem 4 *Consider problem (5), where $g_1, \ldots, g_{m_1}$ are convex and $h_1, \ldots, h_{m_2}$ are affine. Let $\bar{x} \in S$ be a local minimum and $I = \{i \in \{1, \ldots, m_1\} : g_i(\bar{x}) = 0\}$. Suppose that the* Slater condition *is satisfied; i.e., there exists an $x' \in S$ such that $g_i(x') < 0$ for $i \in I$. Then, $\bar{x}$ satisfies the KKT conditions (17).*

Proof Since $h_1, \ldots, h_{m_2}$ are affine, we may assume without loss that the family $\{\nabla h_j(\bar{x})\}_j$ of vectors is linearly independent. Now, by Theorem 2, we have

$$u\nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0} \tag{19}$$

for some $u, v_1, \ldots, v_{m_1} \ge 0$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$, where not all of them are zero. We claim that $u > 0$. Suppose that this is not the case. Then, we have

$$\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0}. \tag{20}$$

Since not all of $v_1, \ldots, v_{m_1}, w_1, \ldots, w_{m_2}$ are zero, we conclude that there exists an $i' \in I$ with $v_{i'} > 0$, for otherwise we would have $\sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0}$ with some $w_j \neq 0$, which contradicts the linear independence of $\{\nabla h_j(\bar{x})\}_j$.

Now, by the Slater condition and the convexity of $g_1, \ldots, g_{m_1}$, we have

$$0 > g_i(x') \ge g_i(\bar{x}) + \nabla g_i(\bar{x})^T(x' - \bar{x}) = \nabla g_i(\bar{x})^T(x' - \bar{x}) \qquad \text{for } i \in I. \tag{21}$$

Moreover, by the feasibility of x' and the affinity of $h_1, \ldots, h_{m_2}$, we have

$$0 = \nabla h_j(\bar{x})^T(x' - \bar{x}) \qquad \text{for } j = 1, \ldots, m_2. \tag{22}$$

Let $d = x' - \bar{x}$. Since $v_1, \ldots, v_{m_1} \geq 0$, $v_i = 0$ for $i \notin I$, and $v_{i'} > 0$, by (21) and (22), we have

$$\left(\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) \right)^T d = \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x})^T d \leq v_{i'} \nabla g_{i'}(\bar{x})^T d < 0,$$

which contradicts (20). It follows that $u > 0$ as claimed. Now, upon dividing both sides of (19) by u, the desired result follows. □

As the following theorem shows, the situation is even simpler when $g_1, \ldots, g_{m_1}$ are concave and $h_1, \ldots, h_{m_2}$ are affine.

Theorem 5 *Consider problem (5), where $g_1, \ldots, g_{m_1}$ are concave and $h_1, \ldots, h_{m_2}$ are affine. Let $\bar{x} \in S$ be a local minimum. Then, $\bar{x}$ satisfies the KKT conditions (17).*

Proof By Theorem 2, we have

$$u \nabla f(\bar{x}) + \sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) = \mathbf{0} \tag{23}$$

for some $u, v_1, \ldots, v_{m_1} \geq 0$ and $w_1, \ldots, w_{m_2} \in \mathbb{R}$, where not all of them are zero. We claim that $u > 0$. Suppose that this is not the case; i.e., $u = 0$. By the concavity of $g_1, \ldots, g_{m_1}$ and affinity of $h_1, \ldots, h_{m_2}$, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} g_i(x) &\leq g_i(\bar{x}) + \nabla g_i(\bar{x})^T (x - \bar{x}) && \text{for } i = 1, \ldots, m_1, \\ h_j(x) &= h_j(\bar{x}) + \nabla h_j(\bar{x})^T (x - \bar{x}) && \text{for } j = 1, \ldots, m_2. \end{aligned}$$

Since $v_i g_i(\bar{x}) = 0$ for $i = 1, \ldots, m_1$ and $h_j(\bar{x}) = 0$ for $j = 1, \ldots, m_2$, it follows that

$$\begin{aligned} & \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x) \\ \leq\ & \sum_{i=1}^{m_1} v_i g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j h_j(\bar{x}) + \left(\sum_{i=1}^{m_1} v_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} w_j \nabla h_j(\bar{x}) \right)^T (x - \bar{x}) \\ =\ & 0. \end{aligned} \tag{24}$$

Now, since $u = 0$, we either have $v_i > 0$ for some $i = 1, \ldots, m_1$ or $w_j \neq 0$ for some $j = 1, \ldots, m_2$. Thus, by Theorem 2, there exists an $x' \in \mathbb{R}^n$ such that $v_i g_i(x') > 0$ for all i with $v_i > 0$ and $w_j h_j(x') > 0$ for all j with $w_j \neq 0$. However, such an x' satisfies

$$\sum_{i=1}^{m_1} v_i g_i(x') + \sum_{j=1}^{m_2} w_j h_j(x') > 0,$$

which contradicts (24). □

In particular, Theorem 5 implies that the KKT conditions (17) are necessary for local optimality in a *linearly constrained* optimization problem.

Let us now illustrate the above results via some examples.

Example 2 (Optimality Conditions of Some Optimization Problems)

1. **Linear Programming.** *Consider the standard form LP:*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq \mathbf{0}, \end{array} \tag{25}$$

where, as usual, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ are given. Since problem (25) contains only linear constraints, the KKT conditions are necessary for optimality. Upon letting $v \in \mathbb{R}^n$ (resp. $w \in \mathbb{R}^m$) be the vector of Lagrange multipliers associated with the inequality constraint (resp. equality constraint), we may write the KKT conditions as follows:

$$\begin{aligned} \underbrace{c}_{\nabla(c^T x)} + \sum_{i=1}^n v_i \underbrace{(-e_i)}_{\nabla(-e_i^T x)} + \sum_{j=1}^m w_j \underbrace{(-a_j)}_{\nabla(b_j - a_j^T x)} &= \mathbf{0}, \\ v &\geq \mathbf{0}, \\ v_i x_i &= 0 \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

Here, $a_j \in \mathbb{R}^n$ is the j–th row of A, where $j = 1, \ldots, m$. The above can be expressed more compactly as

$$v = c - A^T w \geq \mathbf{0}, \quad v^T x = 0,$$

which, as the reader may readily recognize, correspond to the dual feasibility and complementarity conditions for LP. It follows from the results in Handout 3 that the KKT conditions are also sufficient for optimality in this case.

2. **Smallest Eigenvalue of a Symmetric Matrix.** *Let $A \in \mathcal{S}^n$ be given. Consider the following problem:*

$$\begin{array}{ll} \text{minimize} & x^T A x \\ \text{subject to} & \|x\|_2^2 = 1. \end{array} \tag{26}$$

Since the feasible set is compact and the objective function is continuous, problem (26) has an optimal solution. Moreover, since the constraint gradient $\nabla(1 - \|x\|_2^2)$ does not vanish at any feasible solution to (26), the regularity condition in Theorem 3 is satisfied. Hence, the KKT conditions are necessary for optimality. Upon letting $w \in \mathbb{R}$ be the Lagrange multiplier associated with the equality constraint, we can write the KKT condition of (26) as

$$\underbrace{2Ax}_{\nabla(x^T A x)} - w \underbrace{2x}_{\nabla(1 - \|x\|_2^2)} = \mathbf{0}.$$

This yields $Ax = wx$, which shows that x has to be an eigenvector of A with eigenvalue w. To determine the optimal value w^* of and optimal solution x^* to problem (26), note that $(x^*)^T A(x^*) = w^* \|x^*\|_2^2 = w^*$. This implies that the objective value is smallest when w^* is the smallest eigenvalue of A, and the optimal solution x^* is an eigenvector of A corresponding to the eigenvalue w^*.

3. **Optimization of a Matrix Function.** *Let $A \in \mathcal{S}^n_{++}$ and $b \in \mathbb{R}_{++}$ be given. Consider the following problem:*

$$\begin{array}{ll} \inf & -\log\det Z \\ \text{subject to} & A \bullet Z \le b, \\ & Z \in \mathcal{S}^n_{++}. \end{array} \tag{27}$$

Note that (27) is an instance of problem (5). *We claim that problem* (27) *has an optimal solution. To see this, observe that $Z = (b/\mathrm{tr}(A))I$ is feasible for* (27)*. Thus, problem* (27) *is equivalent to*

$$\inf_{Z \in \mathcal{F}} -\log\det Z, \tag{28}$$

where

$$\mathcal{F} = \left\{ Z \in \mathcal{S}^n_+ : A \bullet Z \le b,\ -\log\det Z \le -n\log(b/\mathrm{tr}(A)) \right\}.$$

Now, for any $Z \in \mathcal{F}$, we have $\lambda_{\min}(A)\mathrm{tr}(Z) \le A \bullet Z \le b$. This implies that $\mathcal{F}$ is bounded and $\lambda_i(Z) \le b/\lambda_{\min}(A)$ for $i = 1, \ldots, n$. On the other hand, for $i = 1, \ldots, n$, we have

$$-n\log(b/\mathrm{tr}(A)) \ge -\log\det Z = -\sum_{i=1}^n \log\lambda_i(Z) \ge -\log\lambda_i(Z) - (n-1)\log(b/\lambda_{\min}(A)),$$

which yields $\lambda_i(Z) \ge \exp(n\log(b/\mathrm{tr}(A)) - (n-1)\log(b/\lambda_{\min}(A)) > 0$. In particular, we see that $Z \mapsto -\log\det Z$ is continuous on $\mathcal{F}$ and hence $\mathcal{F}$ is closed. Since problem (28) *involves optimizing a continuous function over a compact set, it has an optimal solution. This implies that* (27) *has an optimal solution as claimed.*

Since problem (27) *contains only linear constraints, the KKT conditions are necessary for optimality. It is known that*

$$\nabla(-\log\det Z) = -Z^{-1}, \quad \nabla(A \bullet Z - b) = A;$$

see, e.g., [3]. Upon letting $v \in \mathbb{R}$ be the Lagrange multiplier associated with the inequality constraint, we can write the KKT conditions of (27) as

$$\begin{array}{rcl} -Z^{-1} + vA & = & \mathbf{0}, \\ v & \ge & 0, \\ v(A \bullet Z - b) & = & 0. \end{array}$$

From the first equality, we must have $v > 0$ and $Z = A^{-1}/v$. This, together with the third equality, implies that

$$b = A \bullet Z = \frac{1}{v}(A \bullet A^{-1}) = \frac{n}{v}.$$

Hence, we obtain $v = n/b$. Since the above KKT conditions admit a unique solution, we conclude that $Z^* = bA^{-1}/n$ must be the optimal solution to (27)*.*

In the case where (5) is a convex optimization problem, the KKT conditions are sufficient for optimality as well. To prove this, let us first define the **Lagrangian function** $L : X \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ associated with problem (5) by

$$L(x, v, w) = f(x) + \sum_{i=1}^{m_1} v_i g_i(x) + \sum_{j=1}^{m_2} w_j h_j(x).$$

We then have the following theorem:

Theorem 6 *Consider problem* (5)*, where* X *is open and convex,* $f, g_1, \ldots, g_{m_1}$ *are convex on* X*, and* $h_1, \ldots, h_{m_2}$ *are affine. Suppose that* $(\bar{x}, \bar{v}, \bar{w}) \in X \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ *is a solution to the KKT conditions*

$$
\begin{aligned}
g_i(\bar{x}) &\leq 0 \quad \text{for } i = 1, \ldots, m_1, && (a)\\
h_j(\bar{x}) &= 0 \quad \text{for } j = 1, \ldots, m_2, && (b)\\
\nabla f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j \nabla h_j(\bar{x}) &= \mathbf{0}, && (c)\\
\bar{v} &\geq \mathbf{0}, && (d)\\
\bar{v}_i g_i(\bar{x}) &= 0 \quad \text{for } i = 1, \ldots, m_1. && (e)
\end{aligned}
$$

Then, $\bar{x}$ *is an optimal solution to* (5).

Proof Since the function $x \mapsto L(x, \bar{v}, \bar{w}) = f(x) + \sum_{i=1}^{m_1} \bar{v}_i g_i(x) + \sum_{j=1}^{m_2} \bar{w}_j h_j(x)$ is convex on X, by condition (c) and Proposition 2, we see that $\bar{x}$ is a global minimum of $x \mapsto L(x, \bar{v}, \bar{w})$ in X. This, together with conditions (b), (d), and (e), implies that

$$
\begin{aligned}
f(\bar{x}) &= f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j h_j(\bar{x}) \\
&= \min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m_1} \bar{v}_i g_i(x) + \sum_{j=1}^{m_2} \bar{w}_j h_j(x) \right\} \\
&\leq \inf_{\substack{x \in X \\ g_i(x) \leq 0,\ i=1,\ldots,m_1 \\ h_j(x)=0,\ j=1,\ldots,m_2}} \left\{ f(x) + \sum_{i=1}^{m_1} \bar{v}_i g_i(x) + \sum_{j=1}^{m_2} \bar{w}_j h_j(x) \right\} \\
&\leq \inf_{\substack{x \in X \\ g_i(x) \leq 0,\ i=1,\ldots,m_1 \\ h_j(x)=0,\ j=1,\ldots,m_2}} f(x),
\end{aligned}
$$

which completes the proof. □

It is important to note that Theorem 6 assumes the existence of the Lagrange multipliers $\bar{v} \in \mathbb{R}^{m_1}$ and $\bar{w} \in \mathbb{R}^{m_2}$. Thus, it does not contradict the observation we made in Example 1.

The KKT conditions are often useful in gaining insights into the optimization problem at hand, and sometimes they even suggest simpler algorithms for solving the problem. As an illustration, let us consider the following example:

Example 3 (Power Allocation Optimization in Parallel AWGN Channels) *Consider* n *parallel additive white Gaussian noise (AWGN) channels. For* $i = 1, \ldots, n$*, the* i*–th channel is characterized by the channel power gain* $h_i \geq 0$ *and the additive Gaussian noise power* $\sigma_i > 0$*. Let* p_i *denote the transmit power allocated to the* i*–th channel, where* $i = 1, \ldots, n$*. The maximum information rate that can be reliably transmitted over the* i*–th channel is then given by*

$$
r_i = \log_2 \left(1 + \frac{h_i p_i}{\sigma_i} \right) = (\ln 2)^{-1} \ln \left(1 + \frac{h_i p_i}{\sigma_i} \right); \tag{29}
$$

see [4]. Given a budget P on the total transmit power over n channels, our goal is to allocate power $p_1, \ldots, p_n$ on each of the n channels such that the sum rate of all the channels is maximized. We are thus led to the following formulation:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{n} \ln\left(1 + \frac{h_i p_i}{\sigma_i}\right) & \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} p_i \leq P, & \\ & p_i \geq 0 & \text{for } i = 1, \ldots, n. \end{array} \tag{30}$$

It is easy to verify that the objective function of (30) is concave. Hence, problem (30) is a linearly constrained concave maximization problem. Now, by Theorems 5 and 6, every solution $(\bar{p}, \bar{v}) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$ to the following KKT system will yield an optimal solution $\bar{p} \in \mathbb{R}^n$ to problem (30):

$$\begin{array}{rcllr} v_0 - v_i & = & \dfrac{h_i}{h_i p_i + \sigma_i} & \text{for } i = 1, \ldots, n, & (a) \\ v_0 \left(\displaystyle\sum_{i=1}^{n} p_i - P \right) & = & 0, & & (b) \\ v_i p_i & = & 0 & \text{for } i = 1, \ldots, n, & (c) \\ v_i & \geq & 0 & \text{for } i = 0, 1, \ldots, n. & (d) \end{array} \tag{31}$$

To find a solution to the KKT system (31), we proceed as follows. Without loss of generality, we may assume that $h_i > 0$ for $i = 1, \ldots, n$. Then, we have $v_0 > v_i \geq 0$ by (31a) and (31d), which implies that

$$p_i = \frac{1}{v_0 - v_i} - \frac{\sigma_i}{h_i} \quad \text{for } i = 1, \ldots, n. \tag{32}$$

Now, if $p_i > 0$, then $v_i = 0$ by (31c). On the other hand, if $p_i = 0$, then in order to satisfy (32) with some $v_i \geq 0$, we must have

$$\frac{1}{v_0} - \frac{\sigma_i}{h_i} \leq 0.$$

Hence, we obtain

$$p_i = \left(\frac{1}{v_0} - \frac{\sigma_i}{h_i} \right)^+ \quad \text{for } i = 1, \ldots, n. \tag{33}$$

Moreover, since $v_0 > 0$, we have $\sum_{i=1}^{n} p_i = P$ by (31b). It follows that

$$\sum_{i=1}^{n} \left(\frac{1}{v_0} - \frac{\sigma_i}{h_i} \right)^+ = P.$$

In particular, we can solve for the unique positive root $\bar{v}_0$ of the above equation by a simple bisection search over the interval $0 < v_0 < \max_i(h_i/\sigma_i)$. Once we have $\bar{v}_0$, we can then extract the optimal power allocation $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ using (33).

4 Lagrangian Duality

In view of the development of the duality theories for LP and CLP, it is natural to ask whether one can construct a dual for a general optimization problem, and if so, whether there is a duality theory for the primal–dual pair of problems. To begin our investigation, let us focus on the following class of optimization problems:

$$\begin{array}{rccll} v_p^* & = & \inf & f(x) & \\ & & \text{subject to} & g_i(x) \le 0 & \text{for } i = 1, \ldots, m_1, \\ & & & h_j(x) = 0 & \text{for } j = 1, \ldots, m_2, \\ & & & x \in X. & \end{array} \tag{P}$$

Here, $f, g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2} : \mathbb{R}^n \to \mathbb{R}$ are *arbitrary* functions, and X is an *arbitrary* non–empty subset of $\mathbb{R}^n$. For the sake of brevity, we shall write the first two sets of constraints in (P) as $G(x) \le \mathbf{0}$ and $H(x) = \mathbf{0}$, where $G : \mathbb{R}^n \to \mathbb{R}^{m_1}$ is given by $G(x) = (g_1(x), \ldots, g_{m_1}(x))$ and $H : \mathbb{R}^n \to \mathbb{R}^{m_2}$ is given by $H(x) = (h_1(x), \ldots, h_{m_2}(x))$.

One way of constructing a dual of (P) is to reformulate it using a penalty function approach. Specifically, observe that (P) is equivalent to

$$\inf_{x \in X} \sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} L(x, v, w), \tag{34}$$

where $L : \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is the Lagrangian function associated with (P); i.e.,

$$L(x, v, w) = f(x) + v^T G(x) + w^T H(x).$$

This follows from the fact that for any $x \in X$,

$$\sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} \left\{ f(x) + v^T G(x) + w^T H(x) \right\} = \begin{cases} f(x) & \text{if } G(x) \le \mathbf{0} \text{ and } H(x) = \mathbf{0}, \\ +\infty & \text{otherwise.} \end{cases} \tag{35}$$

Now, it is clear that for any $\bar{x} \in X$ and $(\bar{v}, \bar{w}) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$,

$$\inf_{x \in X} L(x, \bar{v}, \bar{w}) \le L(\bar{x}, \bar{v}, \bar{w}) \le \sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} L(\bar{x}, v, w).$$

Hence, we have

$$\sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} \inf_{x \in X} L(x, v, w) \le \inf_{x \in X} \sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} L(x, v, w). \tag{36}$$

Observe that the right–hand side of (36) is precisely problem (P). This motivates us to define the following dual of (P):

$$v_d^* = \sup_{v \in \mathbb{R}_+^{m_1},\, w \in \mathbb{R}^{m_2}} \theta(v, w). \tag{D}$$

Here, $\theta : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is the value function given by

$$\theta(v, w) = \inf_{x \in X} L(x, v, w).$$

Problem (D) is known as a **Lagrangian dual** of problem (P). It is worth noting that since the set X is arbitrary, there can be many different Lagrangian duals for the same primal problem, depending on which constraints are handled as $G(x) \le \mathbf{0}$ and $H(x) = \mathbf{0}$, and which constraints are treated by X. However, different choices of the Lagrangian dual problem will in general lead to different outcomes, both in terms of the dual optimal value and the computational efforts required to solve the dual problem.

From the above construction of (D), the following is immediate:

Theorem 7 (Weak Duality Theorem) *Let $\bar{x} \in \mathbb{R}^n$ be feasible for (P) and $(\bar{v}, \bar{w}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ be feasible for (D). Then, we have $\theta(\bar{v}, \bar{w}) \le f(\bar{x})$.*

Note that the value function θ is the pointwise infimum of affine functions. As such, it is a concave function, *regardless of the convexity of* (P). In particular, the Lagrangian dual (D) is always a convex optimization problem. Such an observation suggests that strong duality between (P) and (D) may not hold in general. The following is an example of a primal–dual pair of problems with $v_p^* > v_d^*$.

Example 4 (A Primal–Dual Pair with Non–Zero Duality Gap) *Consider the following problem:*

$$\begin{array}{rcll} v_p^* & = & \text{minimize} & -x \\ & & \text{subject to} & x \le 1, \\ & & & x \in X = \{0, 2\}. \end{array} \tag{37}$$

It is clear that the optimal value of and optimal solution to (37) *are $v_p^* = 0$ and $x^* = 0$, respectively. By dualizing the inequality constraint, we obtain the following Lagrangian dual of* (37)*:*

$$v_d^* = \sup_{v \ge 0} \min_{x \in \{0,2\}} \{-x + v(x-1)\}. \tag{38}$$

Observe that for any $v \ge 0$, we have

$$\min_{x \in \{0,2\}} \{-x + v(x-1)\} = \min\{-v, v-2\}.$$

It follows that the optimal value of and optimal solution to (38) *are $v_d^* = -1$ and $v^* = 1$, respectively. In this case, we have $v_p^* > v_d^*$.*

The above example raises the important question of when strong duality holds. To address this question, let us introduce the following definition:

Definition 1 *We say that $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is a* **saddle point** *of the Lagrangian function L associated with (P) if the following conditions are satisfied:*

(a) $\bar{x} \in X$.

(b) $\bar{v} \ge \mathbf{0}$.

(c) For all $x \in X$ and $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, we have

$$L(\bar{x}, v, w) \le L(\bar{x}, \bar{v}, \bar{w}) \le L(x, \bar{v}, \bar{w}).$$

In particular, the point $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L if $\bar{x}$ *minimizes* L over all $x \in X$ when (v, w) is fixed at $(\bar{v}, \bar{w})$, and that $(\bar{v}, \bar{w})$ *maximizes* L over all $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ when x is fixed at $\bar{x}$.

The relevance of the notion of saddle point can be seen in the following theorem:

Theorem 8 *The point* $(\bar{x}, \bar{v}, \bar{w}) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ *is a saddle point of the Lagrangian function* L *associated with* (P) *iff the duality gap between* (P) *and* (D) *is zero and* $\bar{x}$ *and* $(\bar{v}, \bar{w})$ *are the optimal solutions to* (P) *and* (D)*, respectively.*

Proof Suppose that $(\bar{x}, \bar{v}, \bar{w})$ is a saddle point of L. From condition (c), we have $L(\bar{x}, v, w) \leq L(\bar{x}, \bar{v}, \bar{w})$ for all $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$. It follows from condition (a) and the identity (35) that $\bar{x}$ is feasible for (P). It is also clear from condition (b) that $(\bar{v}, \bar{w})$ is feasible for (D). Hence, by condition (c), we have

$$\theta(\bar{v}, \bar{w}) = \min_{x \in X} L(x, \bar{v}, \bar{w}) = L(\bar{x}, \bar{v}, \bar{w}) = \max_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(\bar{x}, v, w) = f(\bar{x}); \tag{39}$$

i.e., the duality gap between (P) and (D) is zero, and the common optimal value $v_p^* = v_d^*$ is attained by the primal solution $\bar{x}$ and dual solution $(\bar{v}, \bar{w})$.

Conversely, suppose that $\bar{x}$ and $(\bar{v}, \bar{w})$ are optimal for (P) and (D), respectively, with $f(\bar{x}) = \theta(\bar{v}, \bar{w})$. Then, we have $\bar{x} \in X$, $G(\bar{x}) \leq \mathbf{0}$, $H(\bar{x}) = \mathbf{0}$, and $\bar{v} \geq \mathbf{0}$; i.e., conditions (a) and (b) are satisfied. Moreover, by the primal feasibility of $\bar{x}$ and dual feasibility of $(\bar{v}, \bar{w})$, we have

$$\theta(\bar{v}, \bar{w}) = \inf_{x \in X} L(x, \bar{v}, \bar{w}) \leq L(\bar{x}, \bar{v}, \bar{w}) \leq \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(\bar{x}, v, w) = f(\bar{x}).$$

Since we have $f(\bar{x}) = \theta(\bar{v}, \bar{w})$ by assumption, equality must hold throughout the above chain of inequalities. In particular, for any $x \in X$ and $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$, we have

$$L(\bar{x}, v, w) \leq \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(\bar{x}, v, w) = L(\bar{x}, \bar{v}, \bar{w}) = \inf_{x \in X} L(x, \bar{v}, \bar{w}) \leq L(x, \bar{v}, \bar{w});$$

i.e., condition (c) is satisfied. This completes the proof. □

From the proof of Theorem 8, particularly the chain of equalities in (39), we see that the existence of a saddle point $(\bar{x}, \bar{v}, \bar{w})$ of L implies

$$\sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} \inf_{x \in X} L(x, v, w) \geq \theta(\bar{v}, \bar{w}) = f(\bar{x}) \geq \inf_{x \in X} \max_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(x, v, w).$$

This, together with (36), yields the following minimax relationship:

$$\sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} \inf_{x \in X} L(x, v, w) = \inf_{x \in X} \sup_{v \in \mathbb{R}_+^{m_1}, w \in \mathbb{R}^{m_2}} L(x, v, w).$$

Moreover, the common value is attained by the saddle point $(\bar{x}, \bar{v}, \bar{w})$. It is natural to ask whether the above relationship holds under other conditions. The following result, which is a special case of Sion's minimax theorem [7], shows one possibility:

Theorem 9 *Let* L *be the Lagrangian function associated with* (P)*. Suppose that*

(a) X *is a compact convex subset of* $\mathbb{R}^n$*,*

(b) $(v, w) \mapsto L(x, v, w)$ *is continuous and concave on* $\mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ *for each* $x \in X$*, and*

(c) $x \mapsto L(x, v, w)$ *is continuous and convex on* X *for each* $(v, w) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$*.*

Then, we have

$$\sup_{v\in\mathbb{R}^{m_1}_+,\, w\in\mathbb{R}^{m_2}} \min_{x\in X} L(x,v,w) = \min_{x\in X} \sup_{v\in\mathbb{R}^{m_1}_+,\, w\in\mathbb{R}^{m_2}} L(x,v,w).$$

For a proof of Theorem 9, we refer the reader to [5].

Since saddle points of L are primal–dual pairs of optimal solutions to (P) and (D), one should be able to characterize them using certain optimality conditions. This is achieved in the following theorem:

Theorem 10 (Saddle Point Optimality Conditions) *The point $(\bar{x},\bar{v},\bar{w}) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is a saddle point of the Lagrangian function L associated with (P) iff the following hold:*

(a) (Primal Feasibility) $\bar{x} \in X$, $G(\bar{x}) \le \mathbf{0}$, and $H(\bar{x}) = \mathbf{0}$.

(b) (Lagrangian Optimality) $\bar{v} \ge \mathbf{0}$ and $\bar{x} = \arg\min_{x\in X} L(x,\bar{v},\bar{w})$.

(c) (Complementarity) $\bar{v}^T G(\bar{x}) = 0$.

Proof Suppose that $(\bar{x},\bar{v},\bar{w})$ is a saddle point of L. Then, conditions (a) and (b) follow from Definition 1 and Theorem 8. Now, Definition 1 implies that

$$f(\bar{x}) = L(\bar{x},\mathbf{0},\mathbf{0}) \le L(\bar{x},\bar{v},\bar{w}) = f(\bar{x}) + \bar{v}^T G(\bar{x}),$$

or equivalently, $\bar{v}^T G(\bar{x}) \ge 0$. On the other hand, since $\bar{v} \ge \mathbf{0}$ and $G(\bar{x}) \le \mathbf{0}$, we have $\bar{v}^T G(\bar{x}) \le 0$. This gives condition (c).

Conversely, suppose that $(\bar{x},\bar{v},\bar{w}) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ satisfies conditions (a)–(c) above. Then, we have $L(\bar{x},\bar{v},\bar{w}) \le L(x,\bar{v},\bar{w})$ for all $x \in X$. Moreover, we have

$$L(\bar{x},\bar{v},\bar{w}) = f(\bar{x}) + \bar{v}^T G(\bar{x}) + \bar{w}^T H(\bar{x}) \ge f(\bar{x}) + v^T G(\bar{x}) + w^T H(\bar{x}) = L(\bar{x},v,w)$$

for all $(v,w) \in \mathbb{R}^{m_1}_+ \times \mathbb{R}^{m_2}$, since $\bar{v}^T G(\bar{x}) = 0$, $G(\bar{x}) \le \mathbf{0}$, and $H(\bar{x}) = \mathbf{0}$. By Definition 1, we conclude that $(\bar{x},\bar{v},\bar{w})$ is a saddle point of L. □

It is instructive to consider conditions (a)–(c) in Theorem 10 in the context of a convex optimization problem. Specifically, suppose that X is an open convex set, $f, g_1, \ldots, g_{m_1}$ are convex and continuously differentiable on X, and $h_1, \ldots, h_{m_2}$ are affine. Then, the Lagrangian function L associated with (P) is convex on X. By Proposition 2, the condition $\bar{x} = \arg\min_{x\in X} L(x,\bar{v},\bar{w})$ is equivalent to

$$\nabla f(\bar{x}) + \sum_{i=1}^{m_1} \bar{v}_i \nabla g_i(\bar{x}) + \sum_{j=1}^{m_2} \bar{w}_j \nabla h_j(\bar{x}) = \mathbf{0}.$$

Thus, conditions (a)–(c) are simply the KKT conditions of (P). This, together with the machinery we developed earlier, leads to the following strong duality results for convex optimization problems:

Corollary 2 *Consider problem (P), where X is open and convex, $f, g_1, \ldots, g_{m_1}$ are convex and continuously differentiable on X, and $h_1, \ldots, h_{m_2}$ are affine. Suppose that (P) has an optimal solution and satisfies the Slater condition. Then, the dual (D) also has an optimal solution. Moreover, we have $v_p^* = v_d^*$.*

Proof Let $\bar{x}$ be an optimal solution to (P). By Theorem 4, there exist $\bar{v} \in \mathbb{R}^{m_1}$ and $\bar{w} \in \mathbb{R}^{m_2}$ such that $(\bar{x},\bar{v},\bar{w})$ satisfies the KKT conditions of (P). From the discussion in the preceding paragraph, we see that $(\bar{x},\bar{v},\bar{w})$ is a saddle point of the Lagrangian function L associated with (P). It follows from Theorems 8 and 10 that $(\bar{v},\bar{w})$ is an optimal solution to (D) and $v_p^* = v_d^*$. □

Corollary 3 *Consider problem (P), where X is open and convex, f is convex and continuously differentiable on X, and $g_1, \ldots, g_{m_1}, h_1, \ldots, h_{m_2}$ are affine. Suppose that (P) has an optimal solution. Then, the dual (D) also has an optimal solution. Moreover, we have $v_p^* = v_d^*$.*

The proof of the above corollary is essentially the same as that of Corollary 2, except that we invoke Theorem 5 instead of Theorem 4.

Let us now illustrate the above theory with some examples.

Example 5 (Lagrangian Duals of Some Optimization Problems)

1. **Semidefinite Programming.** *Consider the following standard form SDP:*

$$\begin{array}{ll} \inf & C \bullet Z \\ \text{subject to} & A_j \bullet Z = b_j \quad \text{for } j = 1, \ldots, m, \\ & Z \in X = \mathcal{S}_+^n, \end{array} \tag{40}$$

where $C, A_1, \ldots, A_m \in \mathcal{S}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. The Lagrangian dual of (40) is

$$\sup_{w \in \mathbb{R}^m} \theta(w), \tag{41}$$

where

$$\theta(w) = \inf_{Z \in \mathcal{S}_+^n} \left\{ C \bullet Z + \sum_{j=1}^m w_j (b_j - A_j \bullet Z) \right\}.$$

Now, for any fixed $w \in \mathbb{R}^m$, we have

$$\theta(w) = \begin{cases} b^T w & \text{if } C - \sum_{j=1}^m w_j A_j \in \mathcal{S}_+^n, \\ -\infty & \text{otherwise.} \end{cases} \tag{42}$$

To see this, let $U \Lambda U^T$ be a spectral decomposition of $C - \sum_{j=1}^m w_j A_j$, and suppose that $\Lambda_{ii} < 0$ for some $i = 1, \ldots, n$. Consider the matrix $Z(\alpha) = \alpha U e_i e_i^T U$. It is clear that $Z(\alpha) \in \mathcal{S}_+^n$ for all $\alpha > 0$. Moreover, as $\alpha \nearrow \infty$, we have

$$\left(C - \sum_{j=1}^m w_j A_j \right) \bullet Z(\alpha) = \alpha (U \Lambda U^T) \bullet (U e_i e_i^T U^T) = \alpha \Lambda \bullet e_i e_i^T = \alpha \Lambda_{ii} \searrow -\infty,$$

which implies that

$$\theta(w) = b^T w + \inf_{Z \in \mathcal{S}_+^n} \left(C - \sum_{j=1}^m w_j A_j \right) \bullet Z = -\infty.$$

On the other hand, if $C - \sum_{j=1}^m w_j A_j \in \mathcal{S}_+^n$, then we have $(C - \sum_{j=1}^m w_j A_j) \bullet Z \geq 0$ for any $Z \in \mathcal{S}_+^n$. It follows that $\theta(w) = b^T w$ in this case (by taking, say, $Z = \mathbf{0}$).

Now, using (42), we see that (41) is equivalent to

$$\begin{array}{ll} \sup & b^T w \\ \text{subject to} & C - \sum_{j=1}^{m} w_j A_j \in \mathcal{S}_+^n, \end{array}$$

which is precisely the dual SDP we defined before.

2. **Quadratic Programming.** *Consider the optimization problem*

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} x^T Q x + c^T x \\ \text{subject to} & Ax \leq b, \end{array} \tag{43}$$

where $Q \in \mathcal{S}_{++}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, *and* $c \in \mathbb{R}^n$ *are given. The Lagrangian dual of* (43) *is given by*

$$\sup_{v \in \mathbb{R}_+^m} \theta(v),$$

where

$$\theta(v) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^T Q x + c^T x + v^T (Ax - b) \right\}. \tag{44}$$

By considering the first–order optimality condition of (44), *we see that the infimum is attained at*

$$x^*(v) = -Q^{-1}(c + A^T v).$$

Upon substituting the above expression into (44), *we obtain*

$$\begin{aligned} \theta(v) &= \frac{1}{2}(c + A^T v)^T Q^{-1} (c + A^T v) - c^T Q^{-1} (c + A^T v) - v^T (A Q^{-1} (c + A^T v) + b) \\ &= -\frac{1}{2} v^T A Q^{-1} A^T v - (A Q^{-1} c + b)^T v - \frac{1}{2} c^T Q^{-1} c. \end{aligned}$$

Note that $(1/2)c^T Q^{-1} c$ *is a constant. Hence, the Lagrangian dual of* (43) *is equivalent to*

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} v^T A Q^{-1} A^T v + (A Q^{-1} c + b)^T v \\ \text{subject to} & v \geq \mathbf{0}. \end{array}$$

3. **Fenchel Dual.** *Consider the optimization problem*

$$\inf_{x \in X_1 \cap X_2} \{ f_1(x) - f_2(x) \},$$

where $X_1, X_2 \subseteq \mathbb{R}^n$ *and* $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ *are given with* $X_1 \cap X_2 \neq \emptyset$. *Observe that the above problem is equivalent to*

$$\begin{array}{ll} \inf & f_1(y) - f_2(z) \\ \text{subject to} & y = z, \\ & (y, z) \in X_1 \times X_2. \end{array} \tag{45}$$

The Lagrangian dual of (45) *is given by*

$$\sup_{w \in \mathbb{R}^n} \theta(w),$$

where

$$\theta(w) = \inf_{(y,z) \in X_1 \times X_2} \left\{ f_1(y) - f_2(z) + w^T(z - y) \right\} = g_2(w) - g_1(w)$$

and

$$g_1(w) = \sup_{y \in X_1} \left\{ w^T y - f_1(y) \right\}, \quad g_2(w) = \inf_{z \in X_2} \left\{ w^T z - f_2(z) \right\}.$$

The above construction can be advantageous when g_1 and g_2 admit explicit expressions.

References

[1] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear Programming: Theory and Algorithms*. Wiley–Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, second edition, 1993.

[2] D. P. Bertsekas. *Nonlinear Programming*. Athena Scientific, Belmont, Massachusetts, second edition, 1999.

[3] M. Brookes. The Matrix Reference Manual. Available online at `http://www.ee.ic.ac.uk/hp/staff/dmb/matrix/intro.html`, 2011.

[4] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., Hoboken, New Jersey, second edition, 2006.

[5] H. Komiya. Elementary Proof for Sion's Minimax Theorem. *Kodai Mathematical Journal*, 11(1):5–7, 1988.

[6] W. Rudin. *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics. McGraw–Hill Book Co., Singapore, third edition, 1976.

[7] M. Sion. On General Minimax Theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 1958.