ENGG 5501: Foundations of Optimization **2023–24 First Term**

Homework Set 1 Solution

Instructor: Anthony Man–Cho So September 28, 2023

Problem 1 (20pts). Let us assume that the factory has to meet the forecasted demand every month. Let x_i, y_i be the number of regular production units and overtime production units, respectively, where $i = 1, \ldots, n$. Furthermore, let s_i be the number of stored units in month i, where $i = 0, 1, \ldots, n-1$, with $s_0 = 0$. Then, the minimum-cost production schedule can be found via the following LP:

$$
\begin{array}{lll}
\min & b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} y_i + s\sum_{i=1}^{n-1} s_i & \\
\text{subject to} & s_i = x_i + y_i + s_{i-1} - d_i & \text{for } i = 1, \ldots, n-1, \\
& x_n + y_n + s_{n-1} - d_n \geq 0, & \\
& 0 \leq x_i \leq r, & \text{for } i = 1, \ldots, n, \\
& y_i \geq 0, & \text{for } i = 1, \ldots, n, \\
& s_i \geq 0, & \text{for } i = 1, \ldots, n-1.
\end{array}
$$

Problem 2 (25pts).

(a) **(15pts).** Let $x_1, x_2 \in S$ and $\alpha \in (0, 1)$. Then, we have

$$
\begin{aligned}
x_1^T A x_1 + b^T x_1 + c &\leq 0, && (1) \\
x_2^T A x_2 + b^T x_2 + c &\leq 0. && (2)
\end{aligned}
$$

Now, we compute

$$
\begin{aligned}
& (\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + b^T (\alpha x_1 + (1-\alpha)x_2) + c \\
= \; & (\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) + \alpha \left(b^T x_1 + c\right) + (1-\alpha)\left(b^T x_2 + c\right) \\
\leq \; & (\alpha x_1 + (1-\alpha)x_2)^T A (\alpha x_1 + (1-\alpha)x_2) - \alpha x_1^T A x_1 - (1-\alpha) x_2^T A x_2 && (3) \\
= \; & -\alpha(1-\alpha) x_1^T A x_1 - (1-\alpha)(1-(1-\alpha)) x_2^T A x_2 + 2\alpha(1-\alpha) x_1^T A x_2 \\
= \; & -\alpha(1-\alpha)\left(x_1^T A x_1 - 2x_1^T A x_2 + x_2^T A x_2\right) \\
= \; & -\alpha(1-\alpha)(x_1 - x_2)^T A (x_1 - x_2) \\
\leq \; & 0, && (4)
\end{aligned}
$$

where (3) follows from the fact that $b^T x_i + c \leq -x_i^T A x_i$ for $i = 1, 2$ (by (1) and (2)), and (4) follows from the assumption that $A \succeq \mathbf{0}$. This proves that S is convex if $A \succeq \mathbf{0}$.

Note that the converse of the claim need not be true. Indeed, let $n = 1$, and let $A = -1$, $b = c = 0$. Then, we have $S = \{x \in \mathbb{R} : -x^2 \leq 0\} = \mathbb{R}$, which is trivially convex.

(b) **(10pts).** The set $\mathbb{Q}$ of rational numbers is not discrete. Indeed, let $x \in \mathbb{Q}$ and $\epsilon > 0$ be arbitrary. Consider the point $\bar{x} = x + 2^{-k}$, where $k \geq \lceil \log_2 1/\epsilon \rceil$. Clearly, we have $\bar{x} \in \mathbb{Q}$ and $\bar{x} \neq x$. Moreover, we have $\bar{x} \in B(x, \epsilon)$. It follows that $\{x\} \subsetneq \mathbb{Q} \cap B(x, \epsilon)$, as desired.

Problem 3 (25pts).

(a) **(10pts).** Let $x \in B_\infty$ be fixed. For any $\alpha > 0$ and $u \in N(x)$, we have $\alpha u^T(y - x) \leq 0$ for all $y \in B_\infty$. Hence, $N(x)$ is a cone. Moreover, for any $u, v \in N(x)$ and $\alpha \in (0, 1)$, we have

$$(\alpha u + (1 - \alpha)v)^T(y - x) = \alpha u^T(y - x) + (1 - \alpha)v^T(y - x) \leq 0 \text{ for all } y \in B_\infty.$$

It follows that $N(x)$ is convex.

(b) **(15pts).** Recall that $I_0 = \{i : -1 < x_i < 1\}$, $I_+ = \{i : x_i = 1\}$, and $I_- = \{i : x_i = -1\}$. Define

$$S(x) = \left\{ u \in \mathbb{R}^n : \begin{array}{ll} u_i = 0 & \text{if } i \in I_0, \\ u_i \geq 0 & \text{if } i \in I_+, \\ u_i \leq 0 & \text{if } i \in I_-, \end{array} \quad \text{for } i = 1, \ldots, n \right\}.$$

We claim that $N(x) = S(x)$. Suppose that $u \in S(x)$. Then, for each $y \in B_\infty$, we have $|y_i| \leq 1$ for $i = 1, \ldots, n$, which implies that

$$u^T(y - x) = \sum_{i \in I_0} u_i(y_i - x_i) + \sum_{i \in I_+} u_i(y_i - 1) + \sum_{i \in I_-} u_i(y_i + 1) \leq 0.$$

It follows that $u \in N(x)$.

Conversely, suppose that $u \in N(x)$. Let $i \in \{1, \ldots, n\}$ be fixed and $\alpha \in [-1, 1]$ be arbitrary. Define the vector $y(i, \alpha) \in \mathbb{R}^n$ by

$$[y(i, \alpha)]_j = \begin{cases} x_j & \text{if } j \neq i, \\ \alpha & \text{otherwise.} \end{cases}$$

Since $x \in B_\infty$, we have $y(i, \alpha) \in B_\infty$. This, together with the definition of u, yields $u^T(y(i, \alpha) - x) = u_i(\alpha - x_i) \leq 0$. Since the preceding inequality holds for any $\alpha \in [-1, 1]$, we must have $u_i = 0$ if $x_i \in I_0$, $u_i \geq 0$ if $x_i \in I_+$, and $u_i \leq 0$ if $x_i \in I_-$. It follows that $u \in S(x)$. This completes the proof.

Problem 4 (30pts).

(a) **(10pts).** If $P = \mathbf{0}$, then we clearly have $\|P\| = 0 \leq 1$. Hence, suppose that $P \neq \mathbf{0}$. Note that if (λ, v) is an eigenpair of P, then by the property $P^2 = P$, we have

$$\lambda v = Pv = P^2 v = P(Pv) = \lambda P v = \lambda^2 v.$$

Since $v \neq \mathbf{0}$, the above yields $\lambda \in \{0, 1\}$. To complete the proof, we use the fact that

$$\|P\|^2 = \lambda_{\max}(P^T P) = \lambda_{\max}(P^2) = 1.$$

(b) **(10pts).** Consider the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that $P^2 = P$, which implies that P is a projection matrix. Now, we compute

$$P^T P = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

which shows that $\|P\| = \sqrt{\lambda_{\max}(P^T P)} = \sqrt{2} > 1$.

(c) **(10pts).** Intuitively, the vector $x - \Pi_{H(s,c)}(x)$ should be normal to the hyperplane $H(s,c)$. Hence, we should have $x - \Pi_{H(s,c)}(x) = \alpha s$ for some $\alpha \in \mathbb{R}$. Since $\Pi_{H(s,c)}(x) \in H(s,c)$, this requires that $s^T(x - \alpha s) = c$, which implies that $\alpha = (s^T x - c)/s^T s$. This yields the following candidate for $\Pi_{H(s,c)}(x)$:

$$\Pi_{H(s,c)}(x) = x - \frac{s^T x - c}{s^T s} s. \tag{5}$$

To prove the correctness of the above formula, we use Theorem 3 of Handout 2. Let $y \in H(s,c)$ be arbitrary. Since $s^T y = c$, we obtain

$$\begin{aligned} \left(y - \Pi_{H(s,c)}(x)\right)^T \left(x - \Pi_{H(s,c)}(x)\right) &= \left(y - x + \frac{s^T x - c}{s^T s} s\right)^T \left(\frac{s^T x - c}{s^T s} s\right) \\ &= \frac{s^T x - c}{s^T s} s^T y - \frac{s^T x - c}{s^T s} s^T x + \frac{(s^T x - c)^2}{s^T s} \\ &= 0. \end{aligned}$$

This establishes the correctness of the formula in (5).