ENGG 5501: Foundations of Optimization **2023–24 First Term**

Homework Set 1

Instructor: Anthony Man–Cho So Due: September 28, 2023

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). A factory forecasts the demand for its product over the next n months to be $d_1, \ldots, d_n$. In any month, the factory has a regular production capacity of r units, at a cost of b dollars per unit. If it goes into overtime production, then it can produce additional units at a cost of c dollars per unit, where $c > b$. The factory can store units from month to month at a cost of s dollars per unit per month. The factory aims to determine the production schedule that minimizes cost. Formulate this task as a linear programming problem. Justify your answer.

Problem 2 (25pts).

(a) **(15pts).** Let $S = \left\{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\right\}$, where $A \in \mathcal{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given. Show that S is convex if $A \succeq \mathbf{0}$. Is the converse true? Explain.

(b) **(10pts).** Is the set of rational numbers discrete? Justify your answer.

Problem 3 (25pts). Let $B_\infty = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\} = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1 \text{ for } i = 1, \ldots, n\}$. For any $x \in B_\infty$, consider the set

$$N(x) = \{u \in \mathbb{R}^n : u^T(y - x) \leq 0 \text{ for all } y \in B_\infty\}.$$

(a) **(10pts).** Show that $N(x)$ is a convex cone for any $x \in B_\infty$.

(b) **(15pts).** Give an explicit description of $N(x)$. Simplify your answer as much as possible. Show all your work. (*Hint: Consider the index sets* $I_+ = \{i : x_i = 1\}$, $I_- = \{i : x_i = -1\}$, *and* $I_0 = \{i : -1 < x_i < 1\}$ *separately.*)

Problem 4 (30pts).

(a) **(10pts).** Let $P \in \mathbb{R}^{n \times n}$ be an orthogonal projection matrix (i.e., $P^2 = P$ and $P = P^T$; see Handout B, Section 1.6). Show that $\|P\| \leq 1$, where $\|P\|$ is the largest singular value of P.

(b) **(10pts).** Give an example of a projection matrix $P \in \mathbb{R}^{2 \times 2}$ with $\|P\| > 1$. Justify your answer.

(c) **(10pts).** Consider the hyperplane $H(s, c) = \{x \in \mathbb{R}^n : s^T x = c\}$, where $s \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $c \in \mathbb{R}$ are given. Let $x \in \mathbb{R}^n$ be arbitrary. Find a formula for $\Pi_{H(s,c)}(x)$ in terms of s, c and prove its correctness.