ENGG 5501: Foundations of Optimization	**2023–24 First Term**
Homework Set 2	
Instructor: Anthony Man–Cho So	Due: October 19, 2023

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Recall that every closed convex set S is the intersection of all the halfspaces containing S. The goal of this problem is to verify this result for $\mathcal{S}_+^n$, the set of $n \times n$ symmetric positive semidefinite matrices. Given any $A, B \in \mathcal{S}^n$, we define $A \bullet B = \mathrm{tr}(AB)$ to be the inner product between A and B.

(a) **(10pts).** Show that for any $A, B \in \mathcal{S}_+^n$, we have $A \bullet B \geq 0$.

(b) **(10pts).** The result in (a) implies that $\mathcal{S}_+^n \subseteq \{X \in \mathcal{S}^n : A \bullet X \geq 0\}$ for any $A \in \mathcal{S}_+^n$. Show that in fact

$$\mathcal{S}_+^n = \bigcap_{A \in \mathcal{S}_+^n} \{X \in \mathcal{S}^n : A \bullet X \geq 0\}.$$

Problem 2 (30pts).

(a) **(10pts).** Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be given by

$$f(x) = \begin{cases} x \ln x & \text{if } x \geq 0, \\ +\infty & \text{otherwise} \end{cases}$$

(note that $0 \ln 0 = 0$). Give an explicit expression of f^*, the conjugate of f. Show your calculations.

(b) **(10pts).** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$. Give an explicit expression of f^*, the conjugate of f. Show your calculations.

(c) **(10pts).** Let $C = \{x \in \mathbb{R}_+^n : \|x\|_2 \leq 1\}$ and $i_C : \mathbb{R}^n \to \{0, +\infty\}$ be the indicator function associated with C. Give an explicit expression of i_C^*, the conjugate of i_C. Show your calculations.

Problem 3 (25pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $L > 0$ be a constant such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2 \quad \text{for all } x, y \in \mathbb{R}^n.$$

(a) **(15pts).** Show that

$$\left| f(y) - f(x) - \nabla f(x)^T (y - x) \right| \leq \frac{L}{2} \|x - y\|_2^2 \quad \text{for all } x, y \in \mathbb{R}^n.$$

(*Hint: Fix $x, y \in \mathbb{R}^n$ and apply the Fundamental Theorem of Calculus to the function $t \mapsto f(x + t(y - x))$.*)

(b) **(10pts).** Given a point $x \in \mathbb{R}^n$ and a scalar $\alpha > 0$, let $y = x - \alpha \nabla f(x)$. Show that

$$f(y) - f(x) \leq \left(-\frac{1}{\alpha} + \frac{L}{2} \right) \|x - y\|_2^2.$$

(The above result shows that if $\alpha \in (0, 2/L)$ and $\nabla f(x) \neq \mathbf{0}$, then $f(y) < f(x)$; i.e., starting at x, the function value decreases as one moves in the direction $-\nabla f(x)$ with a step size $\alpha \in (0, 2/L)$.)

Problem 4 (25pts).

(a) **(10pts).** Let $A \in \mathbb{R}^{m \times n}$ be given. Use the Farkas lemma to show that exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(I)} & Ax \leq \mathbf{0},\ Ax \neq \mathbf{0},\ x \geq \mathbf{0}. \\ \text{(II)} & A^T y \geq \mathbf{0},\ y > \mathbf{0}. \end{array}$$

(*Hint: Follow the idea in the proof of Corollary 2 in Handout 3.*)

(b) **(15pts).** Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$ be given. Suppose that there exists an $\bar{x} \in \mathbb{R}^n$ satisfying $A\bar{x} \leq b$. Show that exactly one of the following systems has a solution:

$$\begin{array}{ll} \text{(I)} & Ax \leq b,\ \ c^T x > d. \\ \text{(II)} & A^T y = c,\ \ b^T y \leq d,\ \ y \geq \mathbf{0}. \end{array}$$