ENGG 5501: Foundations of Optimization	2023–24 First Term
Homework Set 3 Solution	
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Problem 1 (15pts). There are many possible constructions. For instance, consider the following primal-dual pair of standard-form LP:

$$
\text{(P)} \quad \begin{array}{rl} \text{minimize} & x_1 \\ \text{subject to} & x_1 + x_2 = 1, \\ & x_1 - x_2 = 0, \\ & x_1, x_2 \geq 0. \end{array}
\qquad
\text{(D)} \quad \begin{array}{rl} \text{maximize} & y_1 \\ \text{subject to} & y_1 + y_2 \leq 1, \\ & y_1 - y_2 \leq 0. \end{array}
$$

Clearly, $x^* = (1/2, 1/2)$ is the only feasible solution to Problem (P) and hence is also optimal. Now, let y^* be an optimal solution to Problem (D), whose existence is guaranteed by the LP strong duality theorem. By complementary slackness, we see that y^* must satisfy

$$
y_1^* + y_2^* = 1, \qquad y_1^* - y_2^* = 0.
$$

This implies that $y^* = (1/2, 1/2)$ is the unique optimal solution to Problem (D).

Problem 2 (20pts). Suppose that $(x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}^m$ is a Nash equilibrium of the game. We first show that $Ax^* = b$. Suppose there exists an index i such that $(b - Ax^*)_i \neq 0$. For any $\alpha > 0$, let $y_\alpha = \alpha \cdot \text{sgn}((b - Ax^*)_i) \cdot e_i \in \mathbb{R}^m$. Since $L(x^*, y_\alpha) \leq L(x^*, y^*)$, we have

$$
L(x^*, y_\alpha) = c^T x^* + y_\alpha^T (b - Ax) = c^T x^* + \alpha |(b - Ax^*)_i| \leq L(x^*, y^*) = c^T x^* + (y^*)^T (b - Ax^*).
$$

Since the above inequality holds for all $\alpha > 0$ and the rightmost quantity is fixed, we obtain a contradiction by letting $\alpha \nearrow +\infty$. It follows that $Ax^* = b$. This, together with the assumption that $x^* \in \mathbb{R}^n_+$, implies that x^* is feasible for the LP

$$
\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathbb{R}^n_+. \end{array} \tag{1}
$$

Now, observe that for any feasible solution $\bar{x}$ to Problem (1), we have $L(x^*, y^*) \leq L(\bar{x}, y^*)$, which implies that $c^T x^* \leq c^T \bar{x}$. Hence, we conclude that x^* is optimal for Problem (1).

Next, we show that $c - A^T y^* \in \mathbb{R}^n_+$. Suppose there exists an index i such that $(c - A^T y^*)_i < 0$. For any $\alpha > 0$, let $x_\alpha = \alpha \cdot e_i \in \mathbb{R}^n_+$. Since $L(x^*, y^*) \leq L(x_\alpha, y^*)$, we have

$$
L(x^*, y^*) = c^T x^* + (y^*)^T (b - Ax^*) \leq L(x_\alpha, y^*) = c^T x_\alpha + (y^*)^T (b - Ax_\alpha) = \alpha (c - A^T y^*)_i + b^T y^*.
$$

Since the above inequality holds for all $\alpha > 0$ and the leftmost quantity is fixed, we obtain a contradiction by letting $\alpha \nearrow +\infty$. It follows that $c - A^T y^* \in \mathbb{R}^n_+$; i.e., y^* is feasible for the dual LP

$$
\begin{array}{rl} \text{maximize} & b^T y \\ \text{subject to} & c - A^T y \in \mathbb{R}^n_+, \\ & y \in \mathbb{R}^m. \end{array} \tag{2}
$$

Now, since $L(x^*, y^*) \leq L(\mathbf{0}, y^*)$ and $Ax^* = b$, we have $c^T x^* \leq b^T y^*$. Moreover, since x^* is feasible for (1) and y^* is feasible for (2), by the LP weak duality theorem, we have $c^T x^* = b^T y^*$. It follows that y^* is optimal for Problem (2).

Conversely, suppose that x^* is an optimal solution to (1) and y^* is an optimal solution to (2). Then, since $Ax^* = b$, for any $y \in \mathbb{R}^m$, we have $L(x^*, y) = L(x^*, y^*)$. Moreover, since $Ax^* = b$ and $c - A^T y^* \in \mathbb{R}^n_+$, for any $x \in \mathbb{R}^n_+$, we have

$$L(x, y^*) = (c - A^T y^*)^T x + b^T y^* \geq b^T y^* = c^T x^* = L(x^*, y^*),$$

where the second-to-last equality follows from the LP strong duality theorem. This completes the proof.

Problem 3 (20pts).

(a) **(10pts).** The given LP can be written in the standard dual form

$$\begin{array}{rcll} v(b) & = & -\text{maximize} & (-c)^T x \\ & & \text{subject to} & \begin{bmatrix} -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} -b \\ \mathbf{0} \end{bmatrix}. \end{array}$$

It follows that the dual is given by

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z + w = c, \\ & z, w \geq \mathbf{0}, \end{array} \quad \text{or equivalently,} \quad \begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z \leq c, \\ & z \geq \mathbf{0}. \end{array}$$

(b) **(10pts).** For any $b \in \mathbb{R}^m$ such that $v(b)$ is finite, we have

$$\begin{array}{rcll} v(b) & = & \text{maximize} & b^T z \\ & & \text{subject to} & A^T z \leq c, \\ & & & z \geq \mathbf{0} \end{array}$$

by the LP strong duality theorem. Since $v(\cdot)$ is a pointwise supremum of the collection $\{b \mapsto b^T z : A^T z \leq c,\ z \geq \mathbf{0}\}$ of linear functions, we conclude that $v(\cdot)$ is convex.

Problem 4 (25pts).

(a) **(10pts).** The integer programming formulation of the problem is given by

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{m} c_j x_j \\ \text{subject to} & \displaystyle\sum_{j : i \in \mathcal{S}_j} x_j \geq 1 \quad \text{for } i = 1, \ldots, n, \\ & x_j \in \{0, 1\} \quad \text{for } j = 1, \ldots, m. \end{array} \tag{3}$$

Indeed, given a feasible solution $\bar{x} \in \{0,1\}^m$ to Problem (3), let $\bar{\mathscr{S}} = \{\mathcal{S}_j : \bar{x}_j = 1,\ j \in \{1, \ldots, m\}\}$ be the corresponding sub-collection. Then, the objective function gives the total cost of the subsets in $\bar{\mathscr{S}}$. Moreover, observe that the union of the subsets in $\bar{\mathscr{S}}$ equals $\mathcal{U}$ if and only if for each element $i \in \mathcal{U}$, there exists a subset $\mathcal{S}_j \in \bar{\mathscr{S}}$ such that $i \in \mathcal{S}_j$. This yields the first set of constraints in Problem (3).

The LP relaxation of Problem (3) is given by

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{j=1}^{m} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j:i\in\mathcal{S}_j} x_j \geq 1 & \text{for } i = 1,\dots,n, \\ & x_j \geq 0 & \text{for } j = 1,\dots,m. \end{array} \tag{4}$$

Note that if x^* is an optimal solution to Problem (4), then we necessarily have $x^* \leq e$. This follows from the non-negativity of the cost c and the observation that the vector $\bar{x}' \in [0,1]$ defined by $\bar{x}'_j = \min\{\bar{x}_j, 1\}$ for $j = 1,\dots,m$ is feasible for Problem (4) whenever the vector $\bar{x}$ is feasible for Problem (4).

(b) **(15pts).** Let y_i be the dual variable corresponding to the constraint $\sum_{j:i\in\mathcal{S}_j} x_j \geq 1$, where $i = 1,\dots,n$. Note that the first set of constraints in Problem (4) takes the form $Ax \geq e$, where $A \in \{0,1\}^{n\times m}$ and the (i,j)-th entry of A equals 1 if $i \in \mathcal{S}_j$. It follows that the dual of Problem (4) is given by

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{n} y_i & \\ \text{subject to} & \displaystyle\sum_{i\in\mathcal{S}_j} y_i \leq c_j & \text{for } j = 1,\dots,m, \\ & y_i \geq 0 & \text{for } i = 1,\dots,n. \end{array}$$

Problem 5 (20pts).

(a) **(10pts).** The dual of the LP faced by the grand coalition is given by

$$\begin{array}{ll} \text{minimize} & b(\mathcal{N})^T y \\ \text{subject to} & A(\mathcal{N})^T y \geq c, \\ & y \geq \mathbf{0}. \end{array}$$

(b) **(10pts).** By the LP strong duality theorem, we have

$$\sum_{i\in\mathcal{N}} z_i^* = \left(\sum_{i\in\mathcal{N}} b^i\right)^T y^* = b(\mathcal{N})^T y^* = v(\mathcal{N}).$$

Now, let $S \subseteq \mathcal{N}$ be an arbitrary coalition. By definition, $A(S)$ satisfies the following component-wise inequality:

$$A(S) \geq A(\mathcal{N}).$$

Since y^* satisfies $y \geq \mathbf{0}$ and $A(\mathcal{N})^T y^* \geq c$, we conclude that

$$A(S)^T y^* \geq A(\mathcal{N})^T y^* \geq c.$$

In other words, y^* is feasible for the dual of the LP that is faced by S. Hence, by the LP weak duality theorem, we have

$$\sum_{i \in S} z_i^* = b(S)^T y^* \geq v(S),$$

as desired.