ENGG 5501: Foundations of Optimization **2023–24 First Term**

Homework Set 3

Instructor: Anthony Man–Cho So Due: November 7, 2023

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (15pts). Construct a primal-dual pair of linear programs such that both the primal and the dual have a unique optimal solution. Justify your answer.

Problem 2 (20pts). Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ be given. Consider a two-player game, in which player 1 chooses an $x \in \mathbb{R}^n_+$ and player 2 chooses a $y \in \mathbb{R}^m$, and then the payoff from player 1 to player 2 is given by

$$L(x, y) = c^T x + y^T (b - Ax).$$

Naturally, player 1 wants to minimize this payoff while player 2 wants to maximize it. We say that a pair $(x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}^m$ is a Nash equilibrium of this game if

$$L(x^*, y) \le L(x^*, y^*) \le L(x, y^*) \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } y \in \mathbb{R}^m.$$

In other words, no player can improve her position by unilaterally changing her choice. Show that $(x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}^m$ is a Nash equilibrium of this game if and only if x^* is an optimal solution to the LP

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathbb{R}^n_+ \end{array}$$

and y^* is an optimal solution to the dual of the above LP.

Problem 3 (20pts). Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$ be given. Let $v : \mathbb{R}^m \to \mathbb{R}$ be the function defined by

$$\begin{array}{rcll} v(b) & = & \text{minimize} & c^T x \\ & & \text{subject to} & Ax \ge b, \\ & & & x \ge \mathbf{0}. \end{array} \tag{1}$$

In other words, $v(b)$ is the optimal value of the LP (1) when the right-hand side of the first inequality constraint is b.

(a) **(10pts).** Let $b \in \mathbb{R}^m$ be fixed. Find the dual of Problem (1).

(b) **(10pts).** Using the result in (a), or otherwise, show that the function $v(\cdot)$ is convex on the set $\{b \in \mathbb{R}^m : v(b) \text{ is finite}\}$.

Problem 4 (25pts). Let $\mathcal{U} = \{1, \dots, n\}$ be a collection of n elements and $\mathscr{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_m\}$ be a collection of m subsets of $\mathcal{U}$ (i.e., $\mathcal{S}_j \subseteq \mathcal{U}$ for $j = 1, \dots, m$) such that $\mathcal{U} = \cup_{j=1}^m \mathcal{S}_j$. The set $\mathcal{S}_j$ is associated with a non-negative cost c_j, for $j = 1, \dots, m$. Consider the problem of finding a sub-collection $\mathscr{S}'$ of $\mathscr{S}$ so that (i) the union of the sets in $\mathscr{S}'$ equals $\mathcal{U}$ and (ii) the cost of $\mathscr{S}'$, which is defined as $\sum_{j:\mathcal{S}_j\in\mathscr{S}'} c_j$, is minimized among all sub-collections of $\mathscr{S}$ that satisfies (i).

(a) **(10pts).** Let $x_j \in \{0,1\}$, where $j = 1,\ldots,m$, be the variable indicating whether the subset $\mathcal{S}_j$ is chosen in the sub-collection. Formulate the above problem as an integer program and derive its LP relaxation. Justify your answer. (*Hint: In the LP relaxation, it is natural to relax the binary constraint on x_j to the linear constraint $x_j \in [0,1]$. Explain why you can simplify the latter to $x_j \geq 0$, just as in the vertex cover problem we discussed in class.*)

(b) **(15pts).** Write down the dual of the LP relaxation found in (a). Justify your answer.

Problem 5 (20pts). Consider a *production game* defined as follows. Let $\mathcal{N} = \{1,\ldots,n\}$ be the set of players, each of whom is given a vector $b^i = (b_1^i,\ldots,b_m^i)$ $(i = 1,\ldots,n)$ of resources. These resources can be used to produce goods, which in turn can be sold at a given market price. Specifically, we assume the following production model: For player $i \in \mathcal{N}$, a unit of the j-th good $(j = 1,\ldots,p)$ requires a_{kj}^i units of the k-th resource $(k = 1,\ldots,m)$ to produce. Furthermore, the j-th good can be sold at a price c_j, where $j = 1,\ldots,p$.

Now, let $S \subseteq \mathcal{N}$ be a coalition of players. Such a coalition will possess

$$b_k(S) = \sum_{i \in S} b_k^i$$

units of the k-th resource, where $k = 1,\ldots,m$. Using all of their resources, the coalition S can produce any vector $x = (x_1,\ldots,x_p) \in \mathbb{R}_+^p$ of goods that satisfies $A(S)x \leq b(S)$, where

$$A(S)_{kj} = \min_{i \in S} \left\{ a_{kj}^i \right\} \quad \text{for } k = 1,\ldots,m;\ j = 1,\ldots,p \quad \text{and} \quad b(S) = (b_1(S),\ldots,b_m(S)).$$

Naturally, a coalition S would like to maximize its revenue. The optimization problem it faces can be formulated as the following LP:

$$\begin{aligned} v(S) \quad = \quad & \text{maximize} && c^T x \\ & \text{subject to} && A(S)x \leq b(S), \\ & && x \geq \mathbf{0}. \end{aligned} \tag{2}$$

The function v will be the value function of this game. Recall that an allocation vector $z = (z_1,\ldots,z_n) \in \mathbb{R}^n$ is in the core if and only if $\sum_{i \in \mathcal{N}} z_i = v(\mathcal{N})$ and $\sum_{i \in S} z_i \geq v(S)$ for all $S \subset \mathcal{N}$.

(a) **(10pts).** Consider the LP faced by the grand coalition $\mathcal{N}$. Write down its dual.

(b) **(10pts).** Suppose that the dual LP given in (a) is feasible. Let y^* be one of its optimal solutions. Show that the allocation vector

$$z^* = \left(\left(b^1\right)^T y^*, \left(b^2\right)^T y^*, \ldots, \left(b^n\right)^T y^* \right) \in \mathbb{R}^n$$

belongs to the core.