ENGG 5501: Foundations of Optimization	2023–24 First Term
Homework Set 4	
Instructor: Anthony Man–Cho So	Due: November 28, 2023

SOLVE THE FOLLOWING PROBLEMS.

Problem 1 (25pts).

(a) **(10pts).** Show that $\mathcal{S}_+^n$ is self-dual; i.e., $\mathcal{S}_+^n = (\mathcal{S}_+^n)^*$.

(b) **(15pts).** Let $Q \in \mathcal{S}_+^n$ be given. Show that there exist a matrix P, vectors q, u, and scalars α, β such that $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ satisfies $x^T Q x \le t$ if and only if

$$\left\| P \begin{bmatrix} t \\ x \end{bmatrix} + q \right\|_2 \le \alpha t + u^T x + \beta.$$

(*Remark: The above result shows that the constraint $x^T Q x \le t$ can be formulated as an SOC constraint.*)

Problem 2 (15pts). Let $A \in \mathbb{R}^{m \times n}$; $P \in \mathbb{R}^{\ell \times n}$; $c, u \in \mathbb{R}^n$; $b \in \mathbb{R}^m$; $q \in \mathbb{R}^\ell$; and $v \in \mathbb{R}$ be given. Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & \|Px + q\|_2 \le u^T x + v, \\ & x \ge \mathbf{0}. \end{array} \tag{1}$$

By formulating Problem (1) as a conic LP in standard dual form, derive the dual of Problem (1). Show all your work. (*Hint: You may find Propositions 2(b) and 3 in Handout 5 useful.*)

For Problems 3 and 4, we consider the following setting. Let $\mathcal{E}$ be a finite-dimensional Euclidean space equipped with an inner product $\bullet$. We define a norm on $\mathcal{E}$ by $\|x\| = (x \bullet x)^{1/2}$ and the interior of a set $S \subseteq \mathcal{E}$ by

$$\text{int}(S) = \{x \in \mathcal{E} : \text{there exists an } \epsilon > 0 \text{ such that } x + \epsilon d \in S \text{ for any } d \in \mathcal{E} \text{ with } \|d\| \le 1\}.$$

Furthermore, let $K \subseteq \mathcal{E}$ be a non-empty closed cone and consider the dual cone associated with K; i.e., $K^* = \{w \in \mathcal{E} : x \bullet w \ge 0 \text{ for all } x \in K\}$.

Problem 3 (35pts). The goal of this problem is to show that

$$\text{int}(K^*) = \{w \in \mathcal{E} : x \bullet w > 0 \text{ for all } x \in K \backslash \{\mathbf{0}\}\}.$$

Note that if $K = \{\mathbf{0}\}$, then $K^* = \mathcal{E} = \text{int}(K^*)$, and the result holds automatically. Thus, let us assume that $K \neq \{\mathbf{0}\}$.

(a) **(10pts).** Suppose that $w \in \text{int}(K^*)$. Show that $x \bullet w > 0$ for all $x \in K \backslash \{\mathbf{0}\}$.

(b) **(10pts).** Show that the inner product $\bullet$ satisfies the generalized Cauchy–Schwarz inequality; i.e., for any $u, v \in \mathcal{E}$, we have

$$|u \bullet v| \leq \|u\| \cdot \|v\|.$$

(*Hint: Observe that* $(u - v) \bullet (u - v) = \|u - v\|^2 \geq 0$.)

(c) **(15pts).** Suppose that $w \in \mathcal{E}$ satisfies $x \bullet w > 0$ for all $x \in K \backslash \{\mathbf{0}\}$. Set

$$\epsilon = \inf_{x \in K : \|x\| = 1} x \bullet w.$$

Show that $\epsilon > 0$. Hence, conclude that $w \in \text{int}(K^*)$.

Problem 4 (25pts). Let $a_1, \ldots, a_m \in \mathcal{E}$ be given. The goal of this problem is to show that the Slater condition

$$\text{there exists a } \bar{y} \in \mathbb{R}^m \text{ such that } \sum_{i=1}^{m} \bar{y}_i a_i \in \text{int}(K^*) \tag{$\mathscr{S}$}$$

implies the closedness of the set

$$C = \{(a_1 \bullet x, \ldots, a_m \bullet x) \in \mathbb{R}^m : x \in K\}.$$

Towards that end, let $\{b_j\}_{j \geq 1}$ be a sequence in C such that $b_j \to b$. We need to show that $b \in C$. To begin, we note that since $b_j \in C$, there exists an $x_j \in K$ such that $b_j = (a_1 \bullet x_j, \ldots, a_m \bullet x_j)$. Our goal then is to show that there exists an $x \in K$ satisfying $b = (a_1 \bullet x, \ldots, a_m \bullet x)$.

(a) **(10pts).** Show that there exists a constant $M_1 > 0$ satisfying $\|b_j\| \leq M_1$ for all $j \geq 1$. Hence, show that there exists a constant $M_2 > 0$ satisfying $|\bar{y}^T b_j| \leq M_2$ for all $j \geq 1$.

(b) **(15pts).** Observe that

$$\bar{y}^T b_j = \left(\sum_{i=1}^{m} \bar{y}_i a_i \right) \bullet x_j.$$

Using the Slater condition ($\mathscr{S}$) and the results in Problem 3 and Problem 4(a), show that there exists a constant $M_3 > 0$ satisfying $\|x_j\| \leq M_3$ for all $j \geq 1$.

Now, the proof can be completed as follows. The result in (b) implies that the sequence $\{x_j\}_{j \geq 1}$ is bounded. Hence, there exists a subsequence $x_{j_1}, x_{j_2}, \ldots$ that converges to some x (see p.3 of Handout C; this result is known as the *Bolzano–Weierstrass theorem*). Since $x_{j_\ell} \in K$ for all $\ell \geq 1$ and K is closed, we have $x \in K$. Moreover, observe that

$$b = \lim_{\ell \to \infty} b_{j_\ell} = \lim_{\ell \to \infty} (a_1 \bullet x_{j_\ell}, \ldots, a_m \bullet x_{j_\ell}) = (a_1 \bullet x, \ldots, a_m \bullet x).$$

It follows that $b \in C$, as desired.