ENGG 5501: Foundations of Optimization	**2023–24 First Term**
Homework Set 5	
Instructor: Anthony Man–Cho So	Due: December 13, 2023

SOLVE THE FOLLOWING PROBLEMS.

Problem 1 (25pts). Consider the following problem:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{n} \ln x_i + \ln(1 - e^T x) \\ \text{subject to} & e^T x \leq 1,\ x \geq \mathbf{0}. \end{array} \tag{1}$$

Here, as usual, $e = (1, \ldots, 1)$ is the vector of all ones. It can be shown that Problem (1) has an optimal solution (you do not need to show this).

(a) **(10pts).** Write down the first-order optimality conditions of Problem (1) and explain why they are necessary for optimality.

(b) **(15pts).** Using the result in (a), determine the optimal solution to Problem (1). Show all your work.

Problem 2 (25pts). Let $A, X_0 \in \mathcal{S}^n_+$ and $r > 0$ be given with $A \neq \mathbf{0}$. Consider the following problem:

$$\begin{array}{ll} \text{maximize} & \text{tr}(AX) \\ \text{subject to} & \|X - X_0\|_F^2 \leq r^2, \\ & X \in \mathcal{S}^n. \end{array} \tag{2}$$

Here, we have $\|M\|_F^2 = \text{tr}(M^2)$ for any $M \in \mathcal{S}^n$. Since the feasible region is compact and the objective function is continuous, Problem (2) has an optimal solution.

(a) **(10pts).** Write down the first-order optimality conditions of Problem (2) and explain why they are necessary for optimality.

(b) **(15pts).** Using the result in (a), determine the optimal solution X^* to Problem (2) and show that $X^* \in \mathcal{S}^n_+$. Show all your work.

Problem 3 (20pts). Consider the following problem:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 - 4 \geq 0, \quad x_1, x_2 \geq 0. \end{array} \tag{3}$$

(a) **(10pts).** Show that $(\bar{x}_1, \bar{x}_2) = (2, 2)$ is an optimal solution to Problem (3). You may use any method to answer this part.

(b) **(10pts).** By letting $\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\} = \mathbb{R}^2_+$ and

$$\theta(u) = \inf_{(x_1,x_2)\in\mathcal{X}} \left\{x_1^2 + x_2^2 + u(4 - x_1 - x_2)\right\},$$

we can formulate the following Lagrangian dual of Problem (3):

$$\sup_{u\geq 0} \theta(u). \tag{4}$$

Show that the duality gap between Problems (3) and (4) is zero.

Problem 4 (30pts). Let $A \in \mathcal{S}^n$ be given. Consider the following QCQP:

$$\begin{array}{ll} \text{minimize} & x^T A x \\ \text{subject to} & x_i^2 = 1 \quad \text{for } i = 1, \ldots, n. \end{array} \tag{5}$$

(a) **(5pts).** Derive the semidefinite relaxation of Problem (5) using the techniques introduced in class.

(b) **(10pts).** Write down the dual of the semidefinite relaxation you found in (a). Does the primal-dual pair of SDPs you obtained have zero duality gap? Justify your answer.

(c) **(15pts).** The Lagrangian dual of Problem (5) is given by

$$\sup_{w\in\mathbb{R}^n} \theta(w), \tag{6}$$

where

$$\theta(w) = \inf_{x\in\mathbb{R}^n} \left\{ x^T A x + \sum_{i=1}^n w_i(1 - x_i^2) \right\}.$$

Find an explicit expression for $\theta(w)$. Hence, or otherwise, show that Problem (6) is equivalent to the *dual* of the semidefinite relaxation you found in (b).