**ENGG 5501: Foundations of Optimization** **2025–26 First Term**

# Homework Set 1 Solution

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**Problem 1 (20pts).**

(a) **(10pts).** Let the three items be numbered $1, 2, 3$. Then, the feasible solutions to the given knapsack instance are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, and $\{1, 3\}$. The optimal solution is $\{1, 3\}$ with an optimal value of $v_1 + v_3 = 7$.

(b) **(10pts).** The LP relaxation of the given knapsack instance is given by

$$\begin{aligned} \text{maximize} \quad & 2x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 4x_3 \leq 5, \\ & x_1, x_2, x_3 \in [0, 1]. \end{aligned}$$

Observe that the solution $\bar{x}_1 = 1$, $\bar{x}_2 = 1$, $\bar{x}_3 = 1/2$ is feasible with an objective value of 7.5. Since every integer feasible solution to the given knapsack instance is also feasible for the above LP relaxation, we conclude that every optimal solution to the LP relaxation must be fractional.

**Problem 2 (10pts).** Since $S \subset \mathbb{R}^n$ is a discrete set, for each $x \in S$, there exists an $\epsilon'_x > 0$ such that $S \cap B(x, \epsilon'_x) = \{x\}$. Now, set $\epsilon_x = \epsilon'_x/2 > 0$ for $x \in S$. Clearly, we have $S \cap B(x, \epsilon_x) = \{x\}$. Moreover, for distinct $x, y \in S$, if $z \in B(x, \epsilon_x) \cap B(y, \epsilon_y)$, then

$$\|x - y\|_2 \leq \|x - z\|_2 + \|z - y\|_2 \leq 2 \max\{\epsilon_x, \epsilon_y\} = \max\{\epsilon'_x, \epsilon'_y\}.$$

This implies that either $y \in S \cap B(x, \epsilon'_x)$ or $x \in S \cap B(y, \epsilon'_y)$, both of which lead to a contradiction.

**Problem 3 (20pts).** We begin by observing that the ball $B(y, r)$ with center $y$ and radius $r$ can be represented as

$$B(y, r) = \{x \in \mathbb{R}^n : \|x - y\|_2 \leq r\} = \{y + u \in \mathbb{R}^n : \|u\|_2 \leq r\}.$$

Now, consider a fixed $i \in \{1, \ldots, m\}$ and assume without loss of generality that $a_i \neq \mathbf{0}$. Observe that $B(y, r) \subset H_i = \left\{x \in \mathbb{R}^n : a_i^T x \leq b_i\right\}$ iff $a_i^T(y+u) \leq b_i$ for all $\|u\|_2 \leq r$. By the Cauchy–Schwarz inequality, we have

$$\sup_{u \in \mathbb{R}^n : \|u\|_2 \leq r} a_i^T u = a_i^T \left( r \cdot \frac{a_i}{\|a_i\|_2} \right) = r\|a_i\|_2.$$

It follows that $B(y, r) \subset H_i$ iff

$$a_i^T y + r\|a_i\|_2 \leq b_i, \tag{1}$$

which is a linear inequality in $y \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$. In particular, we have $B(y, r) \subset P = \left\{x \in \mathbb{R}^n : a_i^T x \leq b_i \text{ for } i = 1, \ldots, m\right\}$ iff (1) holds for $i = 1, \ldots, m$. Hence, the problem of finding the largest inscribed ball in $P$ can be formulated as the following LP:

$$\begin{aligned} \text{maximize} \quad & r \\ \text{subject to} \quad & a_i^T y + r\|a_i\|_2 \leq b_i \quad \text{for } i = 1, \ldots, m, \\ & y \in \mathbb{R}^n,\ r \geq 0. \end{aligned}$$

**Problem 4 (30pts).**

(a) **(10pts).** The set $S$ is not necessarily convex. Indeed, consider the set $S = \{X \in \mathbb{R}^{2\times 2} : \text{rank}(X) \leq 1\}$ and the matrices

$$X_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Clearly, we have $X_1, X_2 \in S$, but $(X_1 + X_2)/2 = I \notin S$.

(b) **(10pts).** If $P = \mathbf{0}$, then we clearly have $\|P\| = 0 \leq 1$. Hence, suppose that $P \neq \mathbf{0}$. Note that if $(\lambda, v)$ is an eigenpair of $P$, then by the property $P^2 = P$, we have

$$\lambda v = Pv = P^2 v = P(Pv) = \lambda P v = \lambda^2 v.$$

Since $v \neq \mathbf{0}$, the above yields $\lambda \in \{0, 1\}$. To complete the proof, we use the fact that

$$\|P\|^2 = \lambda_{\max}(P^T P) = \lambda_{\max}(P^2) = 1.$$

(c) **(10pts).** Consider the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that $P^2 = P$, which implies that $P$ is a projection matrix. Now, we compute

$$P^T P = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

which shows that $\|P\| = \sqrt{\lambda_{\max}(P^T P)} = \sqrt{2} > 1$.

**Problem 5 (20pts).**

(a) **(5pts).** First, let us establish the convexity of $K^\circ$. Let $w, w' \in K^\circ$ and $\alpha \in [0, 1]$ be arbitrary. Then, by definition of $K^\circ$, for all $x \in K$, we have

$$(\alpha w + (1-\alpha)w')^T x = \alpha w^T x + (1-\alpha)(w')^T x \leq 0,$$

which shows that $K^\circ$ is convex. Note that the above argument does not depend on the convexity of $K$. In other words, $K^\circ$ is convex regardless of whether $K$ is convex.

Next, let us show that $K^\circ$ is a cone. Let $w \in K^\circ$ and $\alpha > 0$. Then, by definition of $K^\circ$, for all $x \in K$, we have $(\alpha w)^T x = \alpha w^T x \leq 0$. This completes the proof.

Lastly, since $K^\circ$ is the (possibly infinite) intersection of halfspaces, its closedness follows from the closedness of halfspaces.

(b) **(15pts).** By Theorem 3 of Handout 2, we have $z^* = \Pi_K(x)$ if and only if $z^* \in K$ and

$$(x - z^*)^T (z - z^*) \leq 0 \quad \text{for all } z \in K. \tag{2}$$

Suppose that $z^* = \Pi_K(x)$. Since $z^* \in K$ and $K$ is a cone, we have $\alpha z^* \in K$ for all $\alpha > 0$. In particular, by taking $z = \alpha z^*$ in (2), we have

$$(\alpha - 1)(x - z^*)^T z^* \leq 0 \quad \text{for all } \alpha > 0.$$

This implies that $(x - z^*)^T z^* = 0$. Upon substituting this into (2), we have $(x - z^*)^T z \leq 0$ for all $z \in K$, which, by definition of $K^\circ$, means that $x - z^* \in K^\circ$.

Conversely, if $z^* \in K$, $x - z^* \in K^\circ$, and $(x - z^*)^T z^* = 0$, then (2) holds, which implies that $z^* = \Pi_K(x)$.