ENGG 5501: Foundations of Optimization	2025–26 First Term
Homework Set 1	
Instructor: Anthony Man–Cho So	Due: September 22, 2025

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Recall the knapsack problem in Section 2.2 of Handout 1. Suppose that we have a knapsack with capacity $C = 5$, and there are three items with sizes $s_1 = 1$, $s_2 = 2$, $s_3 = 4$ and values $v_1 = 2$, $v_2 = 3$, $v_3 = 5$.

(a) **(10pts).** List out all the feasible solutions to the above instance of the knapsack problem. Hence, determine the optimal solution to and optimal value of the instance.

(b) **(10pts).** Show that the LP relaxation of the above instance has a fractional optimal solution (i.e., an optimal solution where one or more variables take non-integer values).

Problem 2 (10pts). Let $S \subset \mathbb{R}^n$ be a discrete set. Show that each $x \in S$ can be associated with a radius $\epsilon_x > 0$ such that (i) $S \cap B(x, \epsilon_x) = \{x\}$ and (ii) $B(x, \epsilon_x) \cap B(y, \epsilon_y) = \emptyset$ for distinct $x, y \in S$.

Problem 3 (20pts). Let $P = \left\{x \in \mathbb{R}^n : a_i^T x \le b_i \text{ for } i = 1, \ldots, m\right\}$, where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. Recall that a ball with center $\bar{x} \in \mathbb{R}^n$ and radius $r > 0$ is defined as the set $B(\bar{x}, r) = \{x \in \mathbb{R}^n : \|x - \bar{x}\|_2 \le r\}$. We are interested in finding a ball with the largest possible radius, subject to the condition that it is entirely contained within the set P. Give a linear programming formulation of this problem.

Problem 4 (30pts).

(a) **(10pts).** Given an integer $k \in \{1, \ldots, \min\{m, n\}\}$, consider the set $S = \{X \in \mathbb{R}^{m \times n} : \text{rank}(X) \le k\}$. Is S convex? Justify your answer.

(b) **(10pts).** Let $P \in \mathbb{R}^{n \times n}$ be an orthogonal projection matrix (i.e., $P^2 = P$ and $P = P^T$; see Handout B, Section 1.6). Show that $\|P\| \le 1$, where $\|P\|$ is the largest singular value of P.

(c) **(10pts).** Give an example of a projection matrix $P \in \mathbb{R}^{2 \times 2}$ with $\|P\| > 1$. Justify your answer.

Problem 5 (20pts). Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Define

$$K^\circ = \{w \in \mathbb{R}^n : w^T x \le 0 \text{ for all } x \in K\}$$

to be the *polar cone* of K.

(a) **(5pts).** Show that K° is a closed convex cone.

(b) **(15pts).** Show that for any $x \in \mathbb{R}^n$, we have $z^* = \Pi_K(x)$ if and only if

$$z^* \in K, \quad x - z^* \in K^\circ, \quad (x - z^*)^T z^* = 0.$$