ENGG 5501: Foundations of Optimization **2025–26 First Term**

Homework Set 2

Instructor: Anthony Man–Cho So Due: November 3, 2025

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (10pts). Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a convex function and $C \subseteq \mathbb{R}^n$ be a non-empty convex set. Consider the function $g : \mathbb{R}^m \to \mathbb{R}$ defined by $g(x) = \inf_{y \in C} f(x, y)$. Suppose that $g(x) > -\infty$ for all $x \in \mathbb{R}^m$. Show that g is convex.

Problem 2 (20pts).

(a) **(10pts).** Let $C \subseteq \mathbb{R}^n$ be a non-empty convex set and $i_C : \mathbb{R}^n \to \{0, +\infty\}$ be the indicator function associated with C. Show that for each $x \in C$, we have

$$\partial i_C(x) = \{w \in \mathbb{R}^n : w^T(y - x) \leq 0 \text{ for all } y \in C\}.$$

(b) **(10pts).** Let $C_1, C_2 \subseteq \mathbb{R}^n$ be two non-empty convex sets. Is it true that $\partial(i_{C_1} + i_{C_2})(x) = \partial i_{C_1}(x) + \partial i_{C_2}(x)$ for every $x \in \mathrm{dom}(i_{C_1} + i_{C_2})$? If so, provide a proof. If not, provide a counter-example (in any dimension $n \geq 1$ will do) and justify your answer. (Recall that for any two sets $A, B \subseteq \mathbb{R}^n$, we have $A + B = \{x + y : x \in A,\, y \in B\}$.)

Problem 3 (20pts). We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is ρ-convex (where $\rho \in \mathbb{R}$) if for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y) + \frac{\rho \alpha (1 - \alpha)}{2} \|x - y\|_2^2.$$

Note that the usual notion of convexity is the same as 0-convexity.

(a) **(10pts).** Given $\rho \in \mathbb{R}$, show that the following statements are equivalent:

(1) The function f is ρ-convex.

(2) The function $x \mapsto f(x) + \frac{\rho}{2}\|x\|_2^2$ is convex.

(b) **(10pts).** Is the function $\mathbb{R}^n \ni x \mapsto -\|x\|_1$ ρ-convex for some $\rho \geq 0$? Justify your answer.

Problem 4 (15pts). Let $A \in \mathbb{R}^{m \times n}$ be given. Use the Farkas lemma to show that exactly one of the following systems has a solution:

$$\begin{aligned} &\text{(I)} \quad Ax \leq \mathbf{0},\ Ax \neq \mathbf{0},\ x \geq \mathbf{0}. \\ &\text{(II)} \quad A^T y \geq \mathbf{0},\ y > \mathbf{0}. \end{aligned}$$

Problem 5 (15pts). Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ be given. Consider a two-player game, in which player 1 chooses an $x \in \mathbb{R}^n_+$ and player 2 chooses a $y \in \mathbb{R}^m$, and then the payoff from player 1 to player 2 is given by

$$L(x, y) = c^T x + y^T(b - Ax).$$

Naturally, player 1 wants to minimize this payoff while player 2 wants to maximize it. We say that a pair $(x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}^m$ is a Nash equilibrium of this game if

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*) \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } y \in \mathbb{R}^m.$$

In other words, no player can improve her position by unilaterally changing her choice. Show that $(x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}^m$ is a Nash equilibrium of this game if and only if x^* is an optimal solution to the LP

$$\begin{array}{ll} \min & c^T x \\ \text{subject to} & Ax = b, \\ & x \in \mathbb{R}^n_+ \end{array}$$

and y^* is an optimal solution to the dual of the above LP.

Problem 6 (20pts). Let $\mathcal{U} = \{1, \ldots, n\}$ be a collection of n elements and $\mathscr{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$ be a collection of m subsets of $\mathcal{U}$ (i.e., $\mathcal{S}_j \subseteq \mathcal{U}$ for $j = 1, \ldots, m$) such that $\mathcal{U} = \cup_{j=1}^m \mathcal{S}_j$. The set $\mathcal{S}_j$ is associated with a non-negative cost c_j, for $j = 1, \ldots, m$. Consider the problem of finding a sub-collection $\mathscr{S}'$ of $\mathscr{S}$ so that (i) the union of the sets in $\mathscr{S}'$ equals $\mathcal{U}$ and (ii) the cost of $\mathscr{S}'$, which is defined as $\sum_{j:\mathcal{S}_j \in \mathscr{S}'} c_j$, is minimized among all sub-collections of $\mathscr{S}$ that satisfy (i).

(a) **(10pts).** Let $x_j \in \{0, 1\}$, where $j = 1, \ldots, m$, be the variable indicating whether the subset $\mathcal{S}_j$ is chosen in the sub-collection. Formulate the above problem as an integer program and derive its LP relaxation. Justify your answer. (*Hint: In the LP relaxation, it is natural to relax the binary constraint on x_j to the linear constraint $x_j \in [0, 1]$. Explain why you can simplify the latter to $x_j \geq 0$, just as in the vertex cover problem we discussed in class.*)

(b) **(10pts).** Write down the dual of the LP relaxation found in (a). Justify your answer.