Mixed-Integer Semidefinite Relaxation of Joint Admission Control and Beamforming: An SOC-Based Outer Approximation Approach with Provable Guarantees

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Abstract*—We consider the joint admission control and multicast downlink beamforming (JABF) problem, which is a fundamental problem in signal processing and admits a natural mixed-integer quadratically constrained quadratic program (MIQCQP) formulation. One popular approach to tackling such MIQCQP formulation is to develop convex relaxations of *both* the binary and continuous variables. However, most existing convex relaxations impose rather weak relationships between the binary and continuous variables and thus do not yield high-performance solutions. To overcome this weakness, we propose to keep the binary constraints intact and apply the semidefinite relaxation (SDR) technique to continuous variables. Although the resulting relaxation takes the form of a mixed-integer semidefinite program (MISDP) and is *theoretically* intractable in general, by exploiting the fact that such MISDP arises as a mixed-integer SDR of an MIQCQP and harnessing recent computational advances in solving large-scale mixed-integer second-order cone programming (MISOCP) problems, we develop a novel, *practically efficient* algorithm that *provably converges* to an optimal solution to the MISDP in a finite number of steps. The key idea of our algorithm is to construct successively tighter second-order cone (SOC) outer approximations of the constraints in the MISDP and solve a sequence of MISOCPs to obtain an optimal solution to the MISDP. Our work also provides, to the best of our knowledge, the first general framework for solving MISDPs that arise as mixed-integer SDRs of MIQCQPs. Next, we show that by applying a Gaussian randomization procedure to the optimal solution to the MISDP, we obtain a feasible solution to the JABF problem whose approximation accuracy is on the order of M, the number of users in the network. This improves upon the approximation accuracy guarantee of an existing convex relaxation method. Lastly, we present numerical results to demonstrate the viability of our proposed approach.

***Index Terms*—admission control, downlink beamforming, mixed-integer QCQP, mixed-integer SDP, outer approximation, approximation bound.**

I. Introduction

Downlink beamforming is among the fundamental problems in signal processing. Considering a cellular network in which multiple single-antenna users are served by a multi-antenna transmitter, one basic version of the problem is multicast beamforming, in which the goal is to design a beamformer that minimizes the transmit power while guaranteeing a certain level of quality of service (QoS) to each user [1]. However, when the user number is large, concurrently satisfying the QoS constraints for all users is generally impossible; hence admission control is indispensable in identifying a subset of users who can be served at the required QoS levels. As is well known, the user selection process can be modeled by discrete variables and incorporated into the optimization formulation of the beamformer design. This gives rise to a mixed-integer quadratically constrained quadratic program (MIQCQP) formulation of the joint admission control and beamforming (JABF) problem; see, e.g., [2]. In fact, many beamforming problems that integrate admission control or resource allocation, such as joint base-station activiation and multicast beamforming [3], [4], can also be formulated as MIQCQPs. Although MIQCQPs are non-convex and difficult to solve in general, their wide applicability in signal processing have generated much interest in developing practically efficient methods to tackle them. One popular approach is to apply convex relaxation techniques to the MIQCQP. This typically entails relaxing the discrete variables to lie in continuous intervals and applying the semidefinite relaxation (SDR) technique [5] to the quadratic constraints and objective. The resulting convex relaxation, which takes the form of a semidefinite program (SDP), can be solved by standard solvers. Subsequently, a post-processing procedure is employed to retrieve a feasible solution to the MIQCQP [6]–[8]. Nevertheless, the convex relaxations obtained from such approach are often too loose for the original MIQCQP, as the relationship between the integer and continuous variables is lost in the relaxation process. Moreover, there are very few theoretical results on the approximation quality of the solution obtained from such approach (the only one that we are aware of is in [7]). By contrast, the approximation quality of the SDR of many QCQPs that arise in signal processing is fairly well understood [5].

In view of the above, we propose a novel approach to tackling MIQCQPs. The idea of our approach is extremely simple: we apply the SDR technique to the quadratic constraints and objective and keep the integer constraints intact. The resulting formulation, which takes the form of a mixed-integer semidefinite program (MISDP), is a relaxation of the MIQCQP. Unfortunately, it is still challenging to solve the MISDP. One recent attempt is a branch-and-bound framework by Gally et al. [9] for solving a certain class of MISDPs. However, the framework does not cover the class of MISDPs that arise from the mixed-integer SDR of MIQCQPs. On another front, we note that both hardware and algorithmic developments over the past few decades have significantly advanced our capability in solving mixed-integer linear programming (MILP) and mixed-integer second-order cone programming (MISOCP) problems. Indeed, although these two classes of problems remain *theoretically* intractable, they have been shown to be *practically* tractable via commercial solvers such as CPLEX [10] and Gurobi [11]; see, e.g., [12]–[15]. Recently, Lubin et al. [16] have exploited such capability and introduced a polyhedral outer approximation (P-OA) approach for solving general mixed-integer convex programming (MICP) problems. The idea is to approximate the convex constraints in the MICP by successively tighter polyhedral sets and solve a sequence of MILPs to obtain an optimal solution to the MICP. However, when applied to MISDPs, the P-OA approach can be very slow, as the positive semidefinite (PSD) constraints cannot be well approximated by polyhedra.

To make our proposed approach viable, we exploit the fact that the MISDPs we are interested in arise as relaxations of MIQCQPs and hence the PSD constraints admit good second-order conic (SOC) outer approximations [17]. Consequently, we are able to extend the P-OA approach in [16] and develop a new, practically efficient, and provably convergent SOC-based outer approximation (OA) algorithm to solve such MISDPs. Our approach also allows us to establish theoretical guarantees on the approximation quality of the MISDP solution with respect to the optimal MIQCQP solution. To the best of our knowledge, this work is the first to give such guarantee for mixed-integer SDRs of MIQCQP. Although our approach applies to a general class of MIQCQPs, for the sake of concreteness and clarity, we shall present the technical developments of our algorithm and the numerical results in the context of JABF in the sequel.

Our notations are standard. Throughout the paper, we denote the set $\{1,\ldots,M\}$ by $[M]$; the Hermitian transpose by $(\cdot)^H$; the inner product of vectors or matrices by $\bullet$; the Frobenius norm by $\|\cdot\|$; the $n\times n$ identity matrix by $\boldsymbol{I}_n$; the sets of n-dimensional integer, complex, and nonnegative real vectors by $\mathbb{Z}^n$, $\mathbb{C}^n$, and $\mathbb{R}^n_+$, respectively; the set of $n\times n$ Hermitian matrices by $\mathbb{H}^n$; the set of $n\times n$ Hermitian PSD matrices by $\boldsymbol{X}\succeq \boldsymbol{0}$ or $\boldsymbol{X}\in\mathbb{H}^n_+$; the circularly symmetric complex Gaussian distribution with mean vector $\boldsymbol{0}$ and covariance matrix $\boldsymbol{W}$ by $\mathcal{CN}(\boldsymbol{0},\boldsymbol{W})$.

II. System Model and Problem Formulation

Consider a cellular network consisting of a single n-antenna transmiter and M single-antenna users. To fix ideas, we focus on the multicast setting, in which all users receive the same information from the transmitter. Let $\boldsymbol{h}_i\in\mathbb{C}^n$ be the channel vector between the transmitter and the i-th user and $\boldsymbol{w}\in\mathbb{C}^n$ be the beamforming vector at the transmitter. Assuming that the additive noise power at user i is σ_i^2, the signal-to-noise ratio (SNR) at user i is given by $\mathsf{SNR}_i = |\boldsymbol{w}^H\boldsymbol{h}_i|^2/\sigma_i^2$. If user i is served by the transmitter, then the QoS constraint $\mathsf{SNR}_i\geq\gamma_i$ should be satisfied, where $\gamma_i>0$ is a given threshold. Since it is generally not possible to satisfy the QoS constraints of all the users when the number of users is large, user admission control is needed to select a subset of users to serve. In this paper, we consider the strategy of selecting a subset of Q (where $1\leq Q\leq M$) users to serve. By redefining $\boldsymbol{h}_i$ by $\boldsymbol{h}_i/\sigma\sqrt{\gamma_i}$, we can formulate such joint admission control and multicast beamforming problem as the following MIQCQP [7]:

$$\begin{aligned} v^*_{\mathrm{JABF}} = \quad & \min_{\boldsymbol{w}\in\mathbb{C}^n,\ \boldsymbol{\beta}\in\mathbb{Z}^M} \quad \|\boldsymbol{w}\|^2 \qquad \text{(JABF)}\\ \text{s.t.} \quad & |\boldsymbol{w}^H\boldsymbol{h}_i|^2\geq\beta_i,\ \text{for } i\in[M],\\ & \sum_{i=1}^M \beta_i\geq Q,\ \beta_i\in\{0,1\},\ \text{for } i\in[M]. \end{aligned}$$

Here, we use the binary vector $\boldsymbol{\beta}$ to model the selection process. Note that (JABF) is easy when $M=1$ or $n=1$, as it reduces to a maximum eigenvalue problem. Hence, we shall assume that $M,n>1$ throughout the paper. Besides, we note that (JABF) is, in general, NP-hard, due to the fact that it is already NP-hard when $Q=M$ [18].

A popular approach to tackling (JABF) is to derive SDRs of *both* the non-convex quadratic constraints and the binary constraints [6]–[8]. This can be achieved by first expressing the binary constraints as quadratic constraints (e.g., by writing $\beta_i\in\{0,1\}$ as $\beta_i(\beta_i-1)=0$ or transforming $\beta_i\in\{0,1\}$ to $\hat{\beta}_i\in\{-1,1\}$ and writing the latter as $\hat{\beta}_i^2=1$) and then applying the SDR technique [5] to the resulting formulation. Although such an approach yields an SDP that can be solved in polynomial time, the solution quality is often poor, as the relaxations obtained from such an approach often impose rather weak relationships between the binary variable $\boldsymbol{\beta}$ and the continuous variable $\boldsymbol{w}$. To overcome this weakness, we propose to keep the binary constraints intact and apply SDR only to the quantities that involve the continuous variable $\boldsymbol{w}$. Specifically, by using the equivalence $\boldsymbol{W}=\boldsymbol{w}\boldsymbol{w}^H \iff \boldsymbol{W}\succeq\boldsymbol{0}$, $\operatorname{rank}(\boldsymbol{W})\leq 1$ and writing $\boldsymbol{H}_i=\boldsymbol{h}_i\boldsymbol{h}_i^H$, we obtain the following mixed-integer SDR of (JABF):

$$\begin{aligned} v^*_{\mathrm{MISDR}} = \quad & \min_{\boldsymbol{W}\in\mathbb{H}^n_+,\ \boldsymbol{\beta}\in\mathbb{Z}^M} \quad \boldsymbol{I}_n\bullet\boldsymbol{W} \qquad \text{(MISDR)}\\ \text{s.t.} \quad & \boldsymbol{H}_i\bullet\boldsymbol{W}\geq\beta_i,\ \text{for } i\in[M],\\ & \sum_{i=1}^M \beta_i\geq Q,\ \beta_i\in\{0,1\},\ \text{for } i\in[M]. \end{aligned}$$

Although (MISDR) yields a tighter relaxation of (JABF) than those in the literature, it involves binary constraints and is *in theory* intractable. Nevertheless, in the next section, we show that by exploiting the structure of (MISDR) and harnessing recent computational advances in MISOCP, it is possible to develop a *practically efficient* algorithm that *provably converges* to an optimal solution to (MISDR) in a finite number of steps. Once we obtain an optimal solution to (MISDR), we can apply the standard Gaussian randomization procedure (see [5]) to extract a feasible solution to (JABF).

III. Mixed-Integer SDR Framework for Solving (JABF)

A. An SOC-Based Outer Approximation Algorithm for Solving Mixed-Integer SDRs of MIQCQPs

As mentioned in the Introduction, existing algorithms for solving MISDPs are not well suited for solving (MISDR), as they do not exploit the fact that (MISDR) arises as a mixed-integer SDR of the MIQCQP (JABF). To exploit such structure in algorithm design, we first observe that by letting

$$\tilde{\boldsymbol{I}}_n = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0}^H & 0 \end{bmatrix} \in \mathbb{H}_+^{n+1},\ \tilde{\boldsymbol{H}}_i = \begin{bmatrix} \boldsymbol{H}_i & \boldsymbol{0} \\ \boldsymbol{0}^H & 0 \end{bmatrix} \in \mathbb{H}_+^{n+1},$$

we can reformulate (MISDR) as follows:

$$v^*_{\mathrm{MISDR}} = \min_{\tilde{\boldsymbol{W}}\in\mathbb{H}_+^{n+1},\ \boldsymbol{\beta}\in\mathbb{Z}^M} \quad \tilde{\boldsymbol{I}}_n \bullet \tilde{\boldsymbol{W}} \qquad \text{(MISDR-E)}$$

$$\text{s.t.} \quad \tilde{\boldsymbol{H}}_i \bullet \tilde{\boldsymbol{W}} \geq \beta_i,\ \text{ for } i \in [M], \qquad \text{(1a)}$$

$$\sum_{i=1}^{M} \beta_i \geq Q,\ \ \beta_i \in \{0,1\},\ \text{ for } i \in [M], \qquad \text{(1b)}$$

$$\tilde{\boldsymbol{W}} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{w} \\ \boldsymbol{w}^H & 1 \end{bmatrix} \in \mathbb{H}_+^{n+1}. \qquad \text{(1c)}$$

Now, observe that constraint (1c) is equivalent to $\boldsymbol{W} - \boldsymbol{w}\boldsymbol{w}^H \in \mathbb{H}_+^n$, which in turn is equivalent to $(\boldsymbol{W} - \boldsymbol{w}\boldsymbol{w}^H) \bullet \boldsymbol{Y} = \boldsymbol{W} \bullet \boldsymbol{Y} - \boldsymbol{w}^H\boldsymbol{Y}\boldsymbol{w} \geq 0$ for all $\boldsymbol{Y} \in \mathbb{H}_+^n$ by the self-duality of the PSD cone; see, e.g., [17]. In particular, for any $\mathcal{T} \subset \mathbb{H}_+^n$, the constraint

$$\boldsymbol{W} \bullet \boldsymbol{Y} - \boldsymbol{w}^H\boldsymbol{Y}\boldsymbol{w} \geq 0,\ \forall\ \boldsymbol{Y} \in \mathcal{T}$$

is an *outer approximation* of the constraint (1c). Hence, for any $\mathcal{T} \subset \mathbb{H}_+^n$, the following problem is an outer approximation for (MISDR-E):

$$v^*_{\mathrm{OA}}(\mathcal{T}) = \min_{\substack{\tilde{\boldsymbol{W}}\in\mathbb{H}^{n+1},\ \boldsymbol{W}\in\mathbb{H}^n,\\ \boldsymbol{w}\in\mathbb{C}^n,\ \boldsymbol{\beta}\in\mathbb{Z}^M}} \quad \tilde{\boldsymbol{I}}_n \bullet \tilde{\boldsymbol{W}} \qquad (\mathrm{OA}(\mathcal{T}))$$

$$\text{s.t.} \quad (1a) \text{ and } (1b) \text{ hold}, \qquad \text{(2a)}$$

$$\tilde{\boldsymbol{W}} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{w} \\ \boldsymbol{w}^H & 1 \end{bmatrix}, \qquad \text{(2b)}$$

$$\boldsymbol{W} \bullet \boldsymbol{Y} - \boldsymbol{w}^H\boldsymbol{Y}\boldsymbol{w} \geq 0,\ \forall\ \boldsymbol{Y} \in \mathcal{T}. \qquad \text{(2c)}$$

Next, observe that for each $\boldsymbol{Y} \in \mathbb{H}_+^n$ with $\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{U}^H$, $\boldsymbol{U} \in \mathbb{C}^{n\times n}$, the constraint $\boldsymbol{W} \bullet \boldsymbol{Y} - \boldsymbol{w}^H\boldsymbol{Y}\boldsymbol{w} \geq 0$ can be expressed as the SOC constraint

$$\left\| \begin{pmatrix} 1 - \boldsymbol{Y} \bullet \boldsymbol{W} \\ 2\boldsymbol{U}^H\boldsymbol{w} \end{pmatrix} \right\| \leq 1 + \boldsymbol{Y} \bullet \boldsymbol{W}.$$

Thus, if we choose $\mathcal{T}$ in (2c) to be a finite subset of $\mathbb{H}_+^n$, then $(\mathrm{OA}(\mathcal{T}))$ is an MISOCP, which can be well tackled by commercial solvers such as Gurobi [11]. Besides, it is clear that $v^*_{\mathrm{MISDR}} \geq v^*_{\mathrm{OA}}(\mathcal{T})$ for any $\mathcal{T} \subset \mathbb{H}_+^n$.

After obtaining an optimal solution $(\tilde{\boldsymbol{W}}^*_{\mathcal{T}}, \boldsymbol{\beta}^*_{\mathcal{T}})$ to $(\mathrm{OA}(\mathcal{T}))$, let us consider fixing the binary variable $\boldsymbol{\beta}$ in (MISDR-E) to $\boldsymbol{\beta}^*_{\mathcal{T}}$. Then, we obtain the following *inner approximation* of (MISDR-E):

$$v^*_{\mathrm{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}}) = \min_{\tilde{\boldsymbol{W}}\in\mathbb{H}_+^{n+1},\ \boldsymbol{W}\in\mathbb{H}_+^n,\ \boldsymbol{w}\in\mathbb{C}^n} \quad \tilde{\boldsymbol{I}}_n \bullet \tilde{\boldsymbol{W}} \qquad (\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$$

$$\text{s.t.} \quad \tilde{\boldsymbol{H}}_i \bullet \tilde{\boldsymbol{W}} \geq [\beta^*_{\mathcal{T}}]_i,\ \text{ for } i \in [M], \qquad \text{(3a)}$$

$$\tilde{\boldsymbol{W}} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{w} \\ \boldsymbol{w}^H & 1 \end{bmatrix}. \qquad \text{(3b)}$$

Note that $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ is an SDP and hence can be solved by standard solvers such as those in CVX [19]. Moreover, we have $v^*_{\mathrm{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}}) \geq v^*_{\mathrm{MISDR}}$. From the above discussion, we see that when the *integrality gap* $v^*_{\mathrm{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}}) - v^*_{\mathrm{OA}}(\mathcal{T})$ is zero, then we obtain an optimal solution to our original problem (MISDR). This motivates us to devise a strategy to update $\mathcal{T}$ iteratively so that the integrality gap can be decreased in each iteration. Towards that end, consider the dual of $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$:

$$\sup_{\tilde{\boldsymbol{Z}}\in\mathbb{H}_+^{n+1},\ \boldsymbol{\lambda}\in\mathbb{R}_+^M} \quad \boldsymbol{\lambda} \bullet \boldsymbol{\beta}^*_{\mathcal{T}} \qquad (\mathrm{DIA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$$

$$\text{s.t.} \qquad \tilde{\boldsymbol{I}}_n - \sum_{i=1}^{M} \lambda_i \tilde{\boldsymbol{H}}_i - \tilde{\boldsymbol{Z}} \succeq 0.$$

It is easy to verify that Slater's condition holds for $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$. This, together with the fact that $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ is bounded below, implies that strong duality between $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ and $(\mathrm{DIA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ holds. As it turns out, the dual variable $\tilde{\boldsymbol{Z}}$ can be used to update $\mathcal{T}$ to reduce the integrality gap.

Proposition 1. *Let $\mathcal{T} \subset \mathbb{H}_+^n$ be fixed. Consider the problem* $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ *and its dual* $(\mathrm{DIA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$.

(a) *(Optimality Cut) If* $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ *is feasible, then there exists an optimal solution $(\tilde{\boldsymbol{Z}}^*, \boldsymbol{\lambda}^*)$ to the dual* $(\mathrm{DIA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$. *Moreover, if we let*

$$\tilde{\boldsymbol{Z}}^* = \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{z} \\ \boldsymbol{z}^H & z \end{bmatrix}, \qquad (5)$$

then for any $\tilde{\boldsymbol{W}} \in \mathbb{H}^{n+1}$ satisfying (3a), (3b), *and $\boldsymbol{W} \bullet \boldsymbol{Z} - \boldsymbol{w}^H\boldsymbol{Z}\boldsymbol{w} \geq 0$, we have $\tilde{\boldsymbol{I}}_n \bullet \tilde{\boldsymbol{W}} \geq v^*_{\mathrm{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}})$.*

(b) *(Feasibility Cut) If* $(\mathrm{IA}(\boldsymbol{\beta}^*_{\mathcal{T}}))$ *is infeasible, then there exists a $\boldsymbol{\lambda}^*$ such that $\tilde{\boldsymbol{Z}}^* = \sum_{i=1}^{M} \lambda_i^* \tilde{\boldsymbol{H}}_i \succeq \boldsymbol{0}$ and $\boldsymbol{\lambda}^* \bullet (\boldsymbol{\beta}^*_{\mathcal{T}} + \boldsymbol{s}) < 0$ for any $\boldsymbol{s} \geq \boldsymbol{0}$. Moreover, if we let $\boldsymbol{Z}$ be as in* (5), *then for any $\tilde{\boldsymbol{W}} \in \mathbb{H}^{n+1}$ satisfying* (3a) *and* (3b), *we have $\boldsymbol{W} \bullet \boldsymbol{Z} - \boldsymbol{w}^H\boldsymbol{Z}\boldsymbol{w} < 0$.*

Now, if the integrality gap is non-zero (i.e., $v^*_{\mathrm{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}}) > v^*_{\mathrm{OA}}(\mathcal{T})$), then by setting $\mathcal{T}' \leftarrow \mathcal{T} \cup \{\boldsymbol{Z}\}$, Proposition 1 suggests that we can eliminate the binary solution $\boldsymbol{\beta}^*_{\mathcal{T}}$ from further consideration in $(\mathrm{OA}(\mathcal{T}'))$. This is because in (a) (resp. in (b)), such a solution can only increase the objective value (resp. is not feasible). Thus, in essence, the dual variable $\tilde{\boldsymbol{Z}}$ generates a *nonlinear* optimality or feasibility cut to the orignal outer approximation (OA($\mathcal{T}$)). The above process can then be repeated with $\mathcal{T}$ replaced by $\mathcal{T}'$. We summarize our proposed procedure for solving (MISDR) in Algorithm 1.

The proof of Proposition 1 is mainly based on Schur complement and strong duality between (IA($\boldsymbol{\beta}^*_{\mathcal{T}}$)) and (DIA($\boldsymbol{\beta}^*_{\mathcal{T}}$)). The existence of $\boldsymbol{\lambda}^*$ in Proposition 1(b) can be verified by separation theorem. Due to space limitation, we defer the full-length proof to the full version of this paper. Nevertheless, it is worth mentioning that Proposition 1 implies the binary solutions obtained during the course of Algorithm 1 are all distinct unless we have termination. Since there is only a finite number of binary solutions, we obtain the following result:

Theorem 1 (Finite Convergence of SOC-OA)**.** *The SOC-OA algorithm will converge to an optimal solution to* (MISDR) *(if one exists) in a finite number of steps.*

Our algorithm is primarily inspired by the P-OA approach proposed in [16], which uses *polyhedral* outer approximations to solve MICPs. We improve upon P-OA by providing much tighter SOC-based outer approximations of PSD constraints that arise from SDRs of quadratic constraints. It should be emphasized that although our development focuses on the mixed-integer SDR of (JABF), the techniques can be applied to general mixed-integer SDRs of MIQCQPs.

Algorithm 1 SOC-based Outer Approximation (SOC-OA)

Initialization: $\phi_U \leftarrow \infty$, $\phi_L \leftarrow -\infty$, Opt$\leftarrow \emptyset$, $\mathcal{T} \leftarrow \emptyset$, Tol = ϵ.
while $\phi_U - \phi_L > \text{Tol}$ **do**
  if (OA($\mathcal{T}$)) is infeasible **then**
    (MISDR) is also infeasible, terminate.
  end if
  Solve (OA($\mathcal{T}$)). Let $(\tilde{\boldsymbol{W}}^*_{\mathcal{T}}, \boldsymbol{\beta}^*_{\mathcal{T}})$ be an optimal solution with optimal value $v^*_{\text{OA}}(\mathcal{T})$. Update $\phi_L \leftarrow v^*_{\text{OA}}(\mathcal{T})$.
  Solve (IA($\boldsymbol{\beta}^*_{\mathcal{T}}$)).
  if (IA($\boldsymbol{\beta}^*_{\mathcal{T}}$)) is feasible **then**
    Let $\boldsymbol{W}^*$ and $(\tilde{\boldsymbol{Z}}^*, \boldsymbol{\lambda}^*)$ be the optimal solutions to (IA($\boldsymbol{\beta}^*_{\mathcal{T}}$)) and (DIA($\boldsymbol{\beta}^*_{\mathcal{T}}$)), respectively, with optimal value $v^*_{\text{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}})$. Construct $\boldsymbol{Z}$ according to Proposition 1(a).
    if $v^*_{\text{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}}) < \phi_U$ **then**
      Update $\phi_U \leftarrow v^*_{\text{IA}}(\boldsymbol{\beta}^*_{\mathcal{T}})$; Opt $\leftarrow (\tilde{\boldsymbol{W}}^*, \boldsymbol{\beta}^*_{\mathcal{T}})$.
    end if
  else
    Construct $\boldsymbol{Z}$ according to Proposition 1(b).
  end if
  Update $\mathcal{T} \leftarrow \mathcal{T} \cup \{\boldsymbol{Z}\}$.
end while

B. JABF Solution Extraction and Approximation Accuracy Analysis

After obtain an optimal solution $(\boldsymbol{W}^*, \boldsymbol{\beta}^*) \in \mathbb{H}^n_+ \times \mathbb{Z}^M$ to (MISDR) using Algorithm 1, we adopt the Gaussian randomization procedure shown in Algorithm 2 to obtain a feasible solution $(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*)$ to (JABF) with objective value $v(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*)$. In experiments, we repeat the random sampling and select the one that yields the best objective as our returned feasible solution. Naturally, we are interested in the approximation accuracy of the solution $(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*)$. This is given below:

Theorem 2 (Approximation Accuracy of Algorithm 2)**.** *Let* $(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*)$ *be the solution returned by the randomization procedure in Algorithm 2. Then, with probability at least* $1/6$*, we have* $v(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*) \leq 8M \cdot v^*_{\text{JABF}}$.

In other words, Theorem 2 shows that the transmit power used by the solution returned by Algorithm 2 is at most $\mathcal{O}(M)$ times the optimum. This improves upon the $\mathcal{O}(Q(M - Q + 1))$ approximation accuracy established in [7] for a convex relaxation approach. The proof of Theorem 2 follows the lines of [18, Lemma 3]; see also [5], [20]. Due to the page limit, we omit the proof here.

Algorithm 2 Randomization Procedure for JABF

Solve (MISDR) using Algorithm 1 to get $(\boldsymbol{W}^*, \boldsymbol{\beta}^*)$.
Define the index set $\mathcal{I} \triangleq \{i \in [M] : \beta^*_i = 1\}$.
Generate a random vector $\boldsymbol{\xi} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^*)$. Set $\hat{\boldsymbol{w}} = t\boldsymbol{\xi}$, where $t = \max_{i \in \mathcal{I}} \left\{(\boldsymbol{\xi}^H \boldsymbol{H}_i \boldsymbol{\xi})^{-1/2}\right\}$.
Return the feasible solution $(\hat{\boldsymbol{w}}, \boldsymbol{\beta}^*)$ for (JABF).

IV. Numerical Experiments

In this section we provide simulations to demonstrate the efficacy of the proposed scheme. The experimental setup is as follows: the dimension of beamformer is $n = 4, 8$; the number of users is $M = 8, 12, 16$; the number of users selected to serve is $Q = \frac{1}{4}M, \frac{1}{2}M, \frac{3}{4}M$. Channels are generated by $\boldsymbol{h}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_n)$ independently. In both figures, we averaged 300 channel realizations to obtain the plots. The number of Gaussian randomization is set to be 1000. All MISOCP subproblems are solved by Gurobi [11] and convex subproblems are solved by CVX [19].

In Figure 1, we show the minimal power required to satisfy QoS constraints of a fixed proportion of users as M varies but n fixed. Specifically, we compare against the semidefinite approximation (SDA) method proposed in [7]. While we directly solve (MISDR) for $(\boldsymbol{W}^*, \boldsymbol{\beta}^*)$, SDA approach consists of two steps. They first solve an approximate continuous relaxation by replacing (1b) with $\sum_{i=1}^{M} \beta_i = Q$ and $\beta_i \in [0, 1]$ for $i \in [M]$ to obtain $(\boldsymbol{W}^*_{\text{SDA}}, \boldsymbol{\beta}_{\text{SDA}}) \in \mathbb{H}^n_+ \times \mathbb{R}^M$; and then adopt a heuristic to retrieve the integer part $\boldsymbol{\beta}^*_{\text{SDA}}$. In simulations, we further refine the continuous part of their solution by solving another convex subproblem for $\boldsymbol{W}^R_{\text{SDA}}$ with the integer part fixed at $\boldsymbol{\beta}^*_{\text{SDA}}$. Subsequently, we do randomization for both methods with the beamformer generated by $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^*)$ and $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^R_{\text{SDA}})$, respectively. In the legend, "SOC-OA bound" and "SDA bound" denote the power budgets obtained by the two methods before rounding, respectively; "SOC-OA" and "SDA" refer to the actual powers required by the beamformer produced by the randomization procedure, respectively. Figure

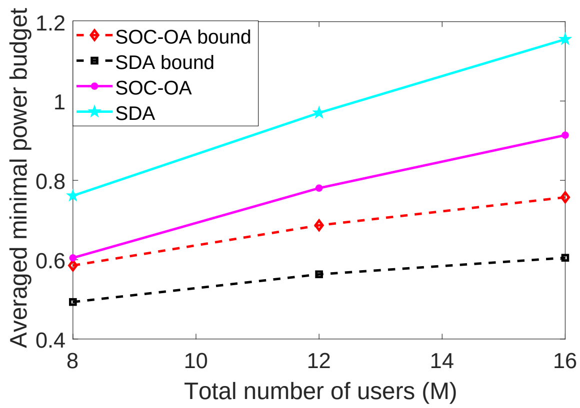

Fig. 1. Averaged minimal power budget with respect to the number of users M. Fix $n = 4$ and $Q = 3/4M$.

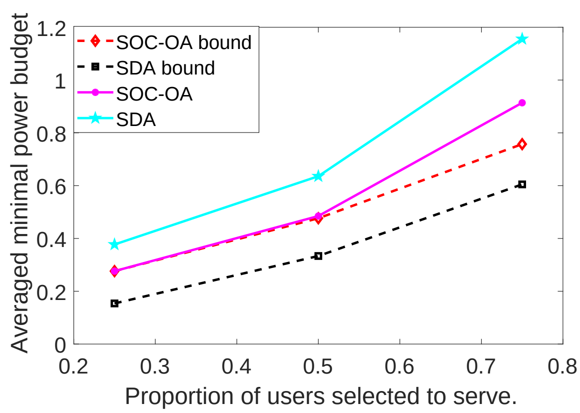

Fig. 2. Averaged minimal power budget with respect to the proportion of users selected to serve. Fix $n = 4$ and $M = 16$.

1 shows that the optimal values of SOC-OA serve as upper bounds of SDA prior to rounding, as we consider a tighter relaxation of (JABF) than SDA. On the other hand, SOC-OA scheme needs a lower power budget than the refined SDA approach after rounding, indicating that we acquire a better integer solution. From the plots, SOC-OA achieves tighter approximation ratios than SDA. The observation is consistent with the analytical results concerning approximation accuracy. To further investigate the performance of SOC-OA, we provide Figure 2 where n and M are fixed while the proportion of users selected changes. We see that when Q is less than or equal to half of the users, the approximation ratios achieved by SOC-OA are almost one, which reveals that SDR of (JABF) with respect to the beamformer variables is tight given β^*. In the future, it would be interesting to further reduce the time complexity of our proposed method.

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