Online Nonlinear AUC Maximization for Imbalanced Datasets

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***Abstract*—Classifying binary imbalanced streaming data is a significant task in both machine learning and data mining. Previously, online AUC (area under the ROC curve) maximization has been proposed to seek a linear classifier. However, it is not well suited for handling nonlinearity and heterogeneity of the data. In this work, we propose the kernelized online imbalanced learning (KOIL) algorithm, which produces a nonlinear classifier for the data by maximizing the AUC score while minimizing a functional regularizer. We address four major challenges that arise from our approach. First, to control the number of support vectors without sacrificing the model performance, we introduce two buffers with fixed budgets to capture the global information on the decision boundary by storing the corresponding learned support vectors. Second, to restrict the fluctuation of the learned decision function and achieve smooth updating, we confine the influence on a new support vector to its k-nearest opposite support vectors. Third, to avoid information loss, we propose an effective compensation scheme after the replacement is conducted when either buffer is full. With such a compensation scheme, the performance of the learned model is comparable to the one learned with infinite budgets. Fourth, to determine good kernels for data similarity representation, we exploit the multiple kernel learning (MKL) framework to automatically learn a set of kernels. Extensive experiments on both synthetic and real-world benchmark datasets demonstrate the efficacy of our proposed approach.**

***Index Terms*—Imbalanced data, AUC maximization, Kernel, Budget**

I. Introduction

IMBALANCED streaming data are prevalent in various real-world applications, such as network intrusion detection [41], purchasing or clicking analysis for customer relationship [12], [16], etc. These data exhibit the following prominent characteristics:

1) Huge volume: The volume of the data increases tremendously, from Petabyte to Exabyte, or even Zettabyte.
2) High velocity: They are streaming data, generated in seconds or microseconds, from various online applications. The data may change dynamically.
3) Extreme imbalance: The imbalanced ratio can be $100:1$, or even $10,000:1$ for a standard binary classification task, where the important class is very rare due to the nature of human attention.
4) Nonlinearity and heterogeneity: Only nonlinear classifiers can produce a more accurate decision boundary; see Fig. 1(a) for an example. The heterogeneity poses difficulty in defining data similarity.

Learning binary classification models from imbalanced data has become an important research topic in both machine learning and data mining [3], [30], [45], [49]. In the literature, researchers aim at maximizing the area under the ROC curve (AUC) instead of accuracy because the AUC score is effective in measuring the performance of classifiers for imbalanced data [1], [2], [17], [19], [23]. To deal with imbalanced streaming data, researchers have proposed the online AUC maximization approach [15], [55]. However, the resulting algorithms only produce a linear classifier and are not well suited for handling the nonlinearity and heterogeneity of the data.

In this work, we focus on seeking an online nonlinear classifier with kernels—a less explored but important research topic in the literature. There are three major obstacles to this approach. First, the learned kernel-based estimator becomes more complex as the number of samples increases [27], [52]. Without a suitable stream oblivious strategy, the number of learned support vectors may grow to infinity, which is obviously undesirable for large-scale applications. In the literature, various refinement techniques have been proposed. They include projection-based methods [9], [13], [32], fixed-budget strategies [4], [10], and sparse kernel learning via weighted sampling [53]. However, extending the above methods to tackle imbalanced data seems to be a non-trivial task. Second, fluctuation due to outliers is unavoidable in online learning [6], [25], [35]. Thus, additional effort is required to achieve smooth updating. Third, the kernel representation is effective in capturing nonlinearity and heterogeneity of the data [27], [21]. However, it is not clear how to effectively determine a good kernel representation.

To overcome the above obstacles, we propose the Kernelized Online Imbalanced Learning (KOIL) algorithm with fixed budgets to achieve online nonlinear AUC maximization. We highlight our contributions as follows:

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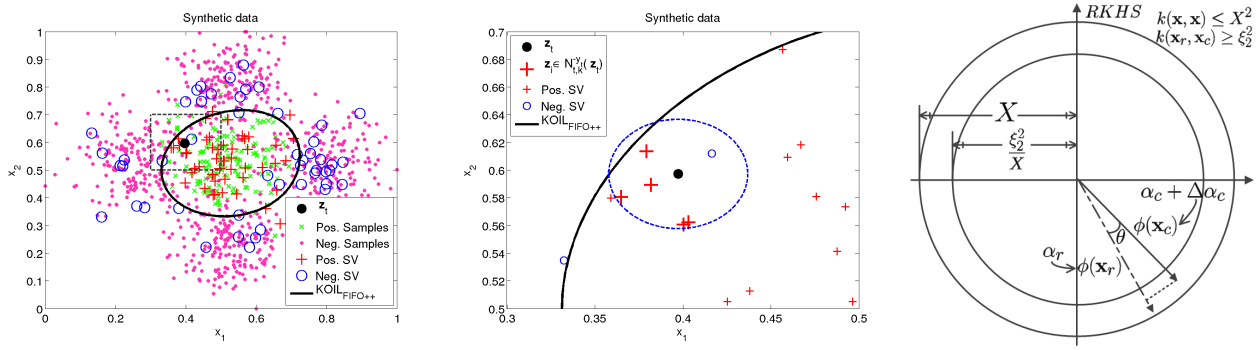


(a) (b) (c)

Fig. 1. Illustration of the KOIL algorithm with k-nearest neighbor confinement and the extended updating policy on a synthetic data in 2-D space. Figure 1(a) shows the decision function in black solid curve, the new instance in big $\bullet$, the positive samples in small $\times$'s, the negative samples in small $\bullet$'s, the positive support vectors in big $+$'s, and the negative support vectors in big $\circ$'s. It is shown that the decision function learned by our proposed KOIL algorithm with the extended FIFO updating policy can classify the data well. Figure 1(b) zooms into the local region of a new instance $\mathbf{z}_t$ and shows how its influence is being controlled. Here, it can only affect its k-nearest opposite support vectors (big $+$'s), where $k = 5$. Obviously, restricting the influence of the new instance to a local region is safe since it will not affect those positive support vectors that are far away from it. Figure 1(c) shows the removed support vector $\mathbf{x}_r$ in the dotted arrow, the compensated support vector $\mathbf{x}_c$ in the solid arrow, and the angle θ between them. By the two assumptions $k(\mathbf{x},\mathbf{x}) \le X^2$ and $k(\mathbf{x}_r,\mathbf{x}_c) \ge \xi_2^2$, we have $\|\phi(\mathbf{x}_c)\|_{\mathcal{H}} \cos\theta \ge \frac{\xi_2^2}{X}$, where $\phi(\mathbf{x}_c) = k(\mathbf{x}_c,\cdot)$.

1) To better control the computational cost, we fix the budget (buffer size) of the buffer for each data class in the KOIL algorithm to store the learned support vectors.
2) We propose a smooth update rule by confining the influence on a new instance to its k-nearest opposite support vectors; see Fig. 1(b) for an example. Our KOIL algorithm can thus limit the effect of outliers.
3) We design an effective scheme to compensate for the loss when a support vector is removed. The idea is to transfer the weight of the removed support vector to its closest support vector in the buffer; see Fig. 1(c) for an illustration. As a result, the learned model typically approaches the one learned with infinite budgets.
4) We exploit the online multiple kernel learning (MKL) framework to automatically determine a good kernel representation. Specifically, we try to learn multiple kernel classifiers and the corresponding linear combination coefficients from a pool of predefined kernels in an online mode. Different from existing online MKL algorithms [22], our KOIL algorithm focuses on the pairwise loss function and discounts the weights of multiple kernel classifiers when there are errors. Empirical results show that online MKL is effective in determining the kernel representation.

II. Related Work

We review some prior work in closely related areas: machine learning from imbalanced data, online learning, and multiple kernel learning.

Learning from imbalanced data is an important task in machine learning and data mining [3], [30], [45]. Some algorithms have been developed to train classifiers by maximizing the AUC metric, such as Wilcoxon-Mann-Whitney statistic optimization [48] and RankOpt [19]. Some investigations extend SVM to optimize the AUC metric [2]. A general framework for optimizing multivariate nonlinear performance measures, such as AUC and F1, is proposed in [23]. Cost-sensitive multilayer Perceptron is also proposed to improve the discrimination ability of MLPs [3]. One major weakness of these methods is that they train the model in the batch-mode, which is inefficient when new training samples appear sequentially.

Online learning algorithms are important as they can adaptively update the models based on new training samples. The oldest and most well-known online learning algorithm is the Perceptron [34]. Many variants have been proposed in the literature [5], [14]. Some are inspired by the maximum margin principle [8], [29], [54]. To learn from imbalanced data, algorithms for online AUC maximization are proposed in [11], [15], [55]. Several works have established generalization error bounds for online learning algorithms with pairwise loss functions [24], [44]. However, these algorithms only focus on linear classifiers, which are not sufficient to capture the heterogeneity and nonlinearity embedded in the data [50], [51]. In the literature, various kernel-based online learning algorithms have been proposed. They include online learning algorithms in a reproducing kernel Hilbert space (RKHS) [10], [27], [32], [39], online Gaussian Process [9], [18], [26], [38], and kernelized recursive least-square algorithms [13], [42], etc. A key challenge in online learning with kernels is that the computational complexity scales with the number of training samples. To tackle this challenge, various strategies have been proposed, including projection-based methods [9], [13], [26], [32], fixed-budget strategies [4], [10], and sparse kernel learning via weighted sampling [53]. However, these strategies aim at directly maximizing the accuracy and cannot handle the task of imbalanced learning properly. Some other methods adopt the strategy of projecting or deleting support vectors to maintain the buffer size [9], [32], [42]. However, such strategy could lead to unbounded support vectors or information loss.

The multiple kernel learning (MKL) framework is a well-known and effective tool for kernel learning. It aims to combine multiple kernels by optimizing the performance of kernel-based learning methods (e.g., support vector machine) [33], [40]. To attain good model performance, MKL with different

norm regularizers have been proposed [28], [47], [51]. Recently, online MKL (OMKL) has been proposed to simultaneously learn multiple kernel classifiers and the corresponding coefficients from a pool of predefined kernels in an online mode [20], [22]. Similar ideas have been applied to solve problems in image search and regression [36], [46]. However, existing algorithms do not consider the task of imbalanced learning.

In summary, all the aforementioned algorithms cannot handle well the nonlinearity and heterogeneity in imbalanced streaming data. This motivates us to seek for a nonlinear classifier for imbalanced data classification in the online training mode.

III. KOIL for AUC Maximization

A. Notations and Problem Definition

Throughout the paper, bold small letter, e.g., $\mathbf{x}$, denotes a vector. Letter in calligraphic font, e.g., $\mathcal{X}$, indicates a set. We use $\mathbb{R}^d$ to denote a d-dimensional Euclidean space and $\mathcal{H}$ to denote a Hilbert space. The inner product of $\mathbf{x}$ and $\mathbf{y}$ on $\mathcal{H}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}}$.

We are interested in the imbalanced binary classification problem, where our goal is to learn a nonlinear decision function $f : \mathbb{R}^d \to \mathbb{R}$ from a sequence of feature-labeled pair instances $\{\mathbf{z}_t = (\mathbf{x}_t, y_t) \in \mathcal{Z}, t \in [T]\}$, where $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, $\mathbf{x}_t \in \mathcal{X} \subseteq \mathbb{R}^d$, $y_t \in \mathcal{Y} = \{-1, +1\}$, and $[T] = \{1, \ldots, T\}$. Without loss of generality, we assume that the positive class is the minority class while the negative class is the majority class. We denote by $N^{\tilde{y}}_{t,k}(\mathbf{z})$ the set of feature-labeled pair instances that are the k-nearest neighbors of $\mathbf{z}$ and have the label of $\tilde{y}$ at the t-th trial. Here, the neighborhood is defined by the distance or the similarity between two instances; i.e., the smaller the distance between or the more similar the instances, the closer the neighbors. We define the index sets I^+_t and I^-_t to record the indices of positive and negative support vectors at the t-th trial, respectively. Moreover, for simplicity, we define two buffers $\mathcal{K}^+_t$ and $\mathcal{K}^-_t$ to store the learned information, namely the weight and support vector, from the two classes at the t-th trial, respectively:

$$\mathcal{K}^+_t.\mathcal{A}=\{\alpha^+_{i,t} \,|\, \alpha^+_{i,t}\neq 0, i\in I^+_t\}, \quad \mathcal{K}^+_t.\mathcal{B}=\{\mathbf{z}_i \,|\, y_i=+1, i\in I^+_t\},$$
$$\mathcal{K}^-_t.\mathcal{A}=\{\alpha^-_{i,t} \,|\, \alpha^-_{i,t}\neq 0, i\in I^-_t\}, \quad \mathcal{K}^-_t.\mathcal{B}=\{\mathbf{z}_i \,|\, y_i=-1, i\in I^-_t\}.$$

Here, $\alpha_{i,t}$ denotes the weight of the support vector that first occurred at the i-th trial and updated at the t-th trial. We fix the budgets (the buffer sizes) to be the same; i.e., $|I^+_t| = |I^-_t| = N$ for all t.

At the t-th trial, our proposed KOIL algorithm computes a decision function f_t of the form

$$f_t(\mathbf{x}) = \sum_{i \in I^+_t} \alpha^+_{i,t} k(\mathbf{x}_i, \mathbf{x}) + \sum_{j \in I^-_t} \alpha^-_{j,t} k(\mathbf{x}_j, \mathbf{x}), \qquad (1)$$

where $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a predefined kernel [27]. The corresponding weights and support vectors are stored in $\mathcal{K}^+_t$ and $\mathcal{K}^-_t$, respectively. Then, given a new instance $\mathbf{x}$, we can predict its class by $\text{sgn}(f_t(\mathbf{x}))$, where f_t encodes the nonlinearity and heterogeneity of the data and is generally an element of a RKHS $\mathcal{H}$; i.e., $f_t(\mathbf{x}) = \langle f_t(\cdot), k(\mathbf{x}, \cdot) \rangle_{\mathcal{H}}$ for all $\mathbf{x} \in \mathcal{X}$, where $k(\mathbf{x}, \cdot) \in \mathcal{H}$ [37]. In the following, we will motivate and describe our strategy for updating f_t.

Algorithm 1 Kernelized Online Imbalanced Learning (KOIL) with Fixed Budgets

Input:
- penalty parameter C and learning rate η
- maximum positive budget N^+ and negative budget N^-
- number of nearest neighbors k

Initialize $\mathcal{K}^+.\mathcal{A} = \mathcal{K}^-.\mathcal{A} = \emptyset$, $\mathcal{K}^+.\mathcal{B} = \mathcal{K}^-.\mathcal{B} = \emptyset$, $N_p = N_n = 0$
for $t = 1$ **to** T **do**
  receive a training sample $\mathbf{z}_t = (\mathbf{x}_t, y_t)$
  if $y_t == +1$ **then**
    $N_p = N_p + 1$
    $[\mathcal{K}^-, \mathcal{K}^+, \alpha] = \text{UpdateKernel}(\mathbf{z}_t, \mathcal{K}^-, \mathcal{K}^+, C, \eta, k)$
    $\mathcal{K}^+ = \text{UpdateBuffer}(\alpha, \mathbf{z}_t, \mathcal{K}^+, k, N^+, N_p)$
  else
    $N_n = N_n + 1$
    $[\mathcal{K}^+, \mathcal{K}^-, \alpha] = \text{UpdateKernel}(\mathbf{z}_t, \mathcal{K}^+, \mathcal{K}^-, C, \eta, k)$
    $\mathcal{K}^- = \text{UpdateBuffer}(\alpha, \mathbf{z}_t, \mathcal{K}^-, k, N^-, N_n)$
  end if
end for

B. Learning with Kernels for AUC Maximization

Given the positive dataset $\mathcal{D}^+ = \{\mathbf{z}_i | y_i = +1, i \in I^+\}$ and the negative dataset $\mathcal{D}^- = \{\mathbf{z}_j | y_j = -1, j \in I^-\}$, the AUC metric for a kernel representation function f is calculated by

$$AUC(f) = \frac{\sum_{i\in I^+}\sum_{j\in I^-} \mathbb{I}[f(\mathbf{x}_i) - f(\mathbf{x}_j) > 0]}{|I^+||I^-|} = 1 - \frac{\sum_{i\in I^+}\sum_{j\in I^-} \mathbb{I}[f(\mathbf{x}_i) - f(\mathbf{x}_j) \leq 0]}{|I^+||I^-|},$$

where $\mathbb{I}[\pi]$ is the indicator function; i.e., $\mathbb{I}[\pi] = 1$ when π is true and $\mathbb{I}[\pi] = 0$ otherwise. It is clear that maximizing $AUC(f)$ is equivalent to minimizing $\sum_{i\in I^+}\sum_{j\in I^-} \mathbb{I}[f(\mathbf{x}_i) - f(\mathbf{x}_j) \leq 0]$. Since the direct maximization of the AUC score is an NP-hard problem [7], the indicator function is usually replaced by a convex surrogate, which may yield sub-optimal performance. A widely used surrogate is the pairwise hinge loss function [15], [55]:

$$\ell_h(f, \mathbf{z}, \mathbf{z}') = \frac{|y - y'|}{2}\left[1 - \frac{1}{2}(y - y')(f(\mathbf{x}) - f(\mathbf{x}'))\right]_+, \qquad (2)$$

where $[v]_+ = \max\{0, v\}$. This gives rise to the problem of regularized minimization as follows:

$$\mathcal{L}(f) = \frac{1}{2}\|f\|^2_{\mathcal{H}} + C \sum_{i\in I^+}\sum_{j\in I^-} \ell_h(f, \mathbf{z}_i, \mathbf{z}_j). \qquad (3)$$

Here, $\frac{1}{2}\|f\|^2_{\mathcal{H}}$ is a regularization term that controls the functional complexity and $C > 0$ is a penalty parameter associated with the training errors.

C. Online AUC Maximization by KOIL

Following the derivation in [21], we update the kernel decision function by minimizing the following *localized instantaneous regularized risk of AUC* associated with the arrival of a new instance $\mathbf{z}_t$:

$$\hat{\mathcal{L}}_t(f) := \hat{\mathcal{L}}(f, \mathbf{z}_t) = \frac{1}{2}\|f\|_{\mathcal{H}}^2 + C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \ell_h(f, \mathbf{z}_t, \mathbf{z}_i), \tag{4}$$

where k is a predefined constant. Two remarks are in order:

- The risk defined in Eq. (4) measures the pairwise losses between $\mathbf{z}_t$ and its k-nearest opposite support vectors in the buffer. This can resolve the scalability issue and is different from NORMA [27], whose risk only measures the predictive error of the new instance.
- The advantage of our approach is twofold. First, two buffers with relatively large budgets can keep track of the global information on the decision boundary. Second, by considering the k-nearest opposite support vectors of the new instance, we can utilize the local information around the new instance and avoid the fluctuation of the decision function.

Algorithm 1 shows the KOIL framework, which consists of two key components: UpdateKernel (Algorithm 2) and UpdateBuffer (Algorithm 3).

Algorithm 2 UpdateKernel

Input:
- newly received sample with label $\mathbf{z}_t$
- $\mathcal{K}^{-y_t}$ and $\mathcal{K}^{y_t}$ for support vectors with the opposite label to $\mathbf{z}_t$ and the same label as $\mathbf{z}_t$, respectively
- penalty parameter C, learning rate η, and number of the nearest neighbors k

Output: updated $\mathcal{K}^{-y_t}$, $\mathcal{K}^{y_t}$ and weight $\alpha_{t,t}$ for $\mathbf{z}_t$
Initialize: $V_t = \emptyset$, compute f_{t-1} by Eq. (1)
for $i \in I_t^{-y_t}$ **do**
  if $1 > y_t(f_{t-1}(\mathbf{x}_t) - f_{t-1}(\mathbf{x}_i))$ **then**
    $V_t = V_t \cup \{i\}$
  end if
end for
if $|V_t| > k$ **then**
  $Sim(i) = k(\mathbf{x}_t, \mathbf{x}_i), \quad \forall\, i \in V_t$
  $[Sim', idx] = \text{Sort}(Sim, \text{'descend'})$
  $idx_k = idx(1:k)$
  $V_t = V_t(idx_k)$
end if
update $\alpha_{i,t}$ by Eq. (8)
return $\mathcal{K}^{-y_t}, \mathcal{K}^{y_t}, \alpha_{t,t}$

1) UpdateKernel: We apply the gradient descent method to update the decision function at each trial; i.e.,

$$f_t := f_{t-1} - \eta \partial_f \hat{\mathcal{L}}_t(f_{t-1}), \tag{5}$$

where ∂_f is shorthand for $\partial/\partial f$ (the gradient with respect to f) and $\eta \in (0, 1)$ is the learning rate, which can be a constant or decreases with the number of trials, as long as it guarantees descent; i.e., $\hat{\mathcal{L}}_t(f_t) \leq \hat{\mathcal{L}}_t(f_{t-1})$. We initialize $f_0 = 0$. To compute $\partial_f \hat{\mathcal{L}}_t(f)$, we first calculate $\partial_f \ell_h(f, \mathbf{z}_t, \mathbf{z}_i)$ by

$$\partial_f \ell_h(\cdot) = \begin{cases} 0, & \ell_h(f, \mathbf{z}_t, \mathbf{z}_i) = 0, \\ -\varphi(\mathbf{z}_t, \mathbf{z}_i), & \ell_h(f, \mathbf{z}_t, \mathbf{z}_i) > 0, \end{cases} \tag{6}$$

where $\varphi(\mathbf{z}_t, \mathbf{z}_i) = y_t(k(\mathbf{x}_t, \cdot) - k(\mathbf{x}_i, \cdot))$. Using Eq. (4) and Eq. (6), we then obtain

$$\begin{aligned} &\partial_f \hat{\mathcal{L}}_t(f_{t-1}) \\ =& f_{t-1} - C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1}, \mathbf{z}_t, \mathbf{z}_i) > 0] \varphi(\mathbf{z}_t, \mathbf{z}_i). \end{aligned} \tag{7}$$

Now, define V_t to be the set of indices for which the indicator function in Eq. (7) evaluates to 1 (the valid set) and $\overline{V}_t$ to be its complement; i.e.

$$\begin{aligned} V_t &:= \{i \in I_t^{-y_t} \mid \mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t) \wedge \ell_h(f_{t-1}, \mathbf{z}_t, \mathbf{z}_i) > 0\}, \\ \overline{V}_t &:= I_t^{-y_t} \setminus V_t. \end{aligned}$$

It then follows from Eq. (5) and Eq. (7) that

$$f_t = (1-\eta) f_{t-1} + \eta C y_t |V_t| k(\mathbf{x}_t, \cdot) - \eta C y_t \sum_{i \in V_t} k(\mathbf{x}_i, \cdot).$$

In particular, since $I_t^{y_t} = I_{t-1}^{y_t} \cup \{t\}$ and $I_t^{-y_t} = I_{t-1}^{-y_t} = V_t \cup \overline{V}_t$, we see that if f_{t-1} is of the form in Eq. (1), then so is f_t as long as we make the following correspondence between the kernel weights at the $(t-1)$-st trial and the t-th trial:

$$\alpha_{i,t} = \begin{cases} \eta C y_t |V_t|, & i = t, \\ (1-\eta)\alpha_{i,t-1} - \eta C y_t, & i \in V_t, \\ (1-\eta)\alpha_{i,t-1}, & i \in I_{t-1}^{y_t} \cup \overline{V}_t. \end{cases} \tag{8}$$

It is instructive to take a closer look at the update rule in Eq. (8). It divides the data into three classes. The first involves the new instance $\mathbf{z}_t$. In this case, at most k of the opposite support vectors are used in the pairwise loss calculation. This prevents the fluctuation of the decision function. The second involves the k-nearest opposite support vectors of the new instance $\mathbf{z}_t$; i.e., the support vectors in $N_{t,k}^{-y_t}(\mathbf{z}_t)$. In this case, their corresponding weights change by a magnitude of $|\eta C y_t|$, which favors a more balanced updating. The third covers the case where the new instance does not incur errors or the labels of the previously learned support vectors are the same as that of the new instance. The corresponding weights of those previously learned support vectors are then reduced by a factor of $1-\eta$, which is the same as NORMA [27].

2) UpdateBuffer: Since the buffers have a fixed budget, they have to be updated when they are full. Traditional stream oblivious policies such as First-In-First-Out (FIFO) and Reservoir Sampling (RS) [43] have been adopted in online linear AUC maximization [55] and shown to be effective in that setting. However, these policies will discard support vectors, which could lead to a degradation in the performance of kernel-based online learning algorithms [21].

To avoid information loss, we need to design a more sophisticated compensation scheme. Towards that end, let $\mathbf{z}_r = (\mathbf{x}_r, y_r)$ be the removed support vector. We find the support vector $\mathbf{z}_c = (\mathbf{x}_c, y_c)$ with $y_c = y_r$ in $\mathcal{K}_t^{y_r}$ that is most similar to $\mathbf{z}_r$ and update its corresponding weight to compensate for the loss of information due to the removal

Algorithm 3 UpdateBuffer–RS++

1: **Input:**
   - received sample $\mathbf{z}_t$ and its weight α_t
   - buffer $\mathcal{K}$ to be updated
   - buffer size N
   - number of instances received until trial t, N_t
2: **Output:** updated buffer $\mathcal{K}$
3: **if** $|\mathcal{K}.\mathcal{B}| < N$ **then**
4: $\mathcal{K}.\mathcal{A} = \mathcal{K}.\mathcal{A} \cup \{\alpha_t\}$, $\mathcal{K}.\mathcal{B} = \mathcal{K}.\mathcal{B} \cup \{\mathbf{z}_t\}$
5: **else**
6: sample Z from a Bernoulli distribution with $\Pr(Z = 1) = N/N_t$
7: **if** $Z = 1$ **then**
8: uniformly select an instance $\mathbf{z}_r$
9: update $\mathcal{K}.\mathcal{A}$: $\mathcal{K}.\mathcal{A} = \mathcal{K}.\mathcal{A} \setminus \{\alpha_{r,t}\} \cup \{\alpha_{t,t}\}$
10: update $\mathcal{K}.\mathcal{B}$: $\mathcal{K}.\mathcal{B} = \mathcal{K}.\mathcal{B} \setminus \{\mathbf{z}_r\} \cup \{\mathbf{z}_t\}$
11: **else**
12: $\mathbf{z}_r = \mathbf{z}_t$, $\alpha_{r,t} = \alpha_{t,t}$
13: **end if**
14: find $\mathbf{z}_c = \arg\max_{\mathbf{z}_i \in \mathcal{K}.\mathcal{B}} \{k(\mathbf{x}_r, \mathbf{x}_i)\}$
15: set $\alpha_{c,t} = \alpha_{c,t} + \alpha_{r,t}$ and update $\alpha_{c,t}$ in $\mathcal{K}.\mathcal{A}$
16: **end if**
17: **return** $\mathcal{K}$

of $\mathbf{z}_r$. Specifically, let $\Delta\alpha_{c,t}$ be the updated weight of the compensated support vector $\mathbf{z}_c$. By keeping track of the change in the value of the decision function, we would like to find $\Delta\alpha_{c,t}$ such that

$$f_t(\mathbf{x}) \approx f_t(\mathbf{x}) - \alpha_{r,t} k(\mathbf{x}_r, \mathbf{x}) + \Delta\alpha_{c,t} \cdot k(\mathbf{x}_c, \mathbf{x}).$$

This suggests that we should set $\Delta\alpha_{c,t} = \alpha_{r,t} \frac{k(\mathbf{x}_r, \mathbf{x})}{k(\mathbf{x}_c, \mathbf{x})} \approx \alpha_{r,t}$. Consequently, we propose the following update rule for the compensated version of f_t, which we denote by f_t^{++}:

$$\begin{aligned} f_t^{++} := (1-\eta) f_{t-1}^{++} - \eta \partial_f \hat{\mathcal{L}}_t(f_{t-1}^{++}) \\ + \alpha_{r,t} \left(k(\mathbf{x}_c, \cdot) - k(\mathbf{x}_r, \cdot) \right). \end{aligned} \tag{9}$$

Here, f_{t-1}^{++} is the compensated decision function from the previous trial. When neither buffer is full, we have $f_t^{++} = f_t$ and the update is done by Eq. (5). Ideally, if $k(\mathbf{x}_c, \mathbf{x})$ equals $k(\mathbf{x}_r, \mathbf{x})$, then f_t^{++} incorporates all the learned support vectors and is equivalent to the one learned with infinite budgets.

Algorithm 3 shows the procedure of the extended Reservoir Sampling (RS++). Some elaborations are in order:

1) In lines 3-4, if the buffer is not full (i.e., $|\mathcal{K}.\mathcal{B}| < N$), then the new instance becomes a support vector and is directly added into the buffer $\mathcal{K}$.
2) In lines 6-10, if the buffer is full, then reservoir sampling is performed. Specifically, with probability $\frac{N}{N_t}$, we update the buffer by randomly replacing one support vector $\mathbf{z}_r$ in $\mathcal{K}.\mathcal{B}$ with $\mathbf{z}_t$.
3) In line 12, if replacement is not conducted, then the removed support vector $\mathbf{z}_r$ is set to be the new instance $\mathbf{z}_t$.
4) In lines 14-15, we extend the classic RS strategy by finding the support vector $\mathbf{z}_c$ that is most similar to the removed support vector $\mathbf{z}_r$, updating its weight, and putting its weight back to the buffer $\mathcal{K}.\mathcal{A}$.

In a similar manner, we can define the extended FIFO strategy, namely **FIFO++**. For FIFO++, we modify lines 6-13 in Algorithm 3 so that the first support vector in the buffer is removed and the new instance is added to the end of the buffer as a new support vector.

IV. Regret Analysis

In this section, we derive a regret bound for the KOIL algorithm with update rule in Eq. (5) under the non-smooth pairwise hinge loss in Eq. (2). Recall that the regret at time T is defined as the difference between the objective value up to the T-th trial and the smallest objective value from hindsight; i.e.,

$$R_T = \sum_{t=1}^{T} \left(\hat{\mathcal{L}}_t(f_t) - \hat{\mathcal{L}}_t(f^*) \right), \tag{10}$$

where f^* is the optimal decision function obtained in hindsight by minimizing Eq. (3) and $\{f_t\}_{t=1}^T$ are obtained by Eq. (5).

In the following, unless otherwise specified, we assume that $\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)$; i.e., $\mathbf{z}_i$ is one of the k-nearest opposite support vectors of $\mathbf{z}_t$. We first establish some auxiliary results that will be useful for our derivation of the regret bound.

Lemma 1. *Suppose that for all $\mathbf{x} \in \mathbb{R}^d$, $k(\mathbf{x}, \mathbf{x}) \le X^2$, where $X > 0$. Let $0 < \xi_1 \le X$ be such that $k(\mathbf{x}_t, \mathbf{x}_i) \ge \xi_1^2$ for all $\mathbf{z}_i = (\mathbf{x}_i, y_i) \in N_{t,k}^{-y_t}(\mathbf{z}_t)$. With $f_0 = 0$ and the update rule in Eq. (5), we have*

$$\|f_t\|_{\mathcal{H}} \le Ckc_p$$

for $t \in [T]$, where

$$c_p := \sqrt{2X^2 - 2\xi_1^2}. \tag{11}$$

Lemma 2. *Suppose that the assumptions of Lemma 1 hold. With $f_0 = 0$ and the update rule in Eq. (5), the pairwise hinge loss function $\ell_h : \mathcal{H} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ defined in Eq. (2) satisfies*

$$\ell_h(f_{t-1}, \mathbf{z}_t, \mathbf{z}_i) \le U := 1 + Ckc_p^2$$

for $t \in [T]$, where c_p is defined in Eq. (11).

The proofs of the above lemmas can be found in Appendices A and B. Now, we are ready to present the advertised regret bound.

Theorem 1. *Suppose that the assumptions of Lemma 1 hold. With $f_0 = 0$ and the update rule in Eq. (5), where $\eta \in (0, 1)$ at each trial is chosen to guarantee descent, we have*

$$R_T \le \frac{\|f^*\|_{\mathcal{H}}^2}{2\eta} + \eta Ck \left((U-1) + \frac{1}{2}(k+1)Cc_p^2 \right) T, \tag{12}$$

where c_p is defined in Eq. (11).

The proof of Theorem 1 is given in Appendix C. Before we proceed, let us make several remarks.

- The regret bound R_T can be further bounded by $O(\sqrt{T})$ if we set η to $O(1/\sqrt{T})$. This bound is the same as that for standard online learning algorithms, but it is different from the mistake bounds derived in [4], [10], [32], which aim at maximizing classification accuracy.

- The expression in Eq. (12) seems to suggest that the smallest regret bound is attained at $k = 1$. However, when $k = 1$, the learned decision function cannot utilize the localized information in the buffers and will yield sub-optimal performance. Our empirical evaluation shows that the best choice of k is around 10% of the budget; see detailed results in Section VII. We conjecture that a more accurate surrogate of the AUC metric can provide a better indication on the regret-minimizing value of k. We leave this as a future direction.
- By exploiting the convexity of the localized instantaneous regularized risk of AUC defined in Eq. (4) and confining the range of $|\alpha_t|$ to $[0, \gamma\eta]$, we can derive the corresponding regret bound for the update rule in (9); cf. [21]. However, the regret bound we obtained via this approach is proportional to T. We leave the derivation of a tighter bound as a future work.

V. Extension to a Smooth Pairwise Hinge Loss

In this section, we extend the results developed earlier to the case of a smooth pairwise hinge loss function. Specifically, consider the square of the pairwise hinge loss function; i.e.,

$$\ell_{sh}(f, \mathbf{z}, \mathbf{z}') = \left(\frac{|y - y'|}{2} \left[1 - \frac{1}{2}(y - y')(f(\mathbf{x}) - f(\mathbf{x}')) \right]_+ \right)^2. \tag{13}$$

We substitute Eq. (13) into Eq. (4) and compute the decision function by minimizing the following *smooth localized instantaneous regularized risk of AUC* associated with $\mathbf{z}_t$:

$$\tilde{\mathcal{L}}_t(f) := \tilde{\mathcal{L}}_t(f) = \frac{1}{2}\|f\|_{\mathcal{H}}^2 + C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \ell_{sh}(f, \mathbf{z}_t, \mathbf{z}_i). \tag{14}$$

As before, we initialize $\tilde{f}_0 = 0$ and apply the standard gradient descent method to update the decision function at each trial; i.e.,

$$\tilde{f}_t := \tilde{f}_{t-1} - \eta \partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}), \tag{15}$$

where $\eta \in (0, 1)$ is the learning rate and

$$\partial \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) = \tilde{f}_{t-1} - 2C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \Big[\mathbb{I}[\ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i) > 0] \times \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i) \cdot \varphi(\mathbf{z}_t, \mathbf{z}_i) \Big].$$

In addition, we define the valid set V_t and its complement $\overline{V}_t$ at the t-th trial as follows:

$$V_t := \{ i \in I_t^{-y_t} \mid \mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t) \wedge \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i) > 0 \},$$
$$\overline{V}_t := I_t^{-y_t} \setminus V_t.$$

Then, the corresponding update rule for the kernel weights at the t-th trial is given by

$$\alpha_{i,t} = \begin{cases} 2\eta C y_t \sum_{i \in V_t} \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i), & i = t, \\ (1 - \eta)\alpha_{i,t-1} & \\ \quad -2\eta C y_t \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i), & i \in V_t, \\ (1 - \eta)\alpha_{i,t-1}, & i \in I_{t-1}^{y_t} \cup \overline{V}_t. \end{cases}$$

Lastly, we have the following update rule for the compensated version $\tilde{f}_t^{++}$ of $\tilde{f}_t$:

$$\tilde{f}_t^{++} := (1 - \eta)\tilde{f}_{t-1}^{++} - \eta \partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}^{++}) + \alpha_{r,t} \left(k(\mathbf{x}_c, \cdot) - k(\mathbf{x}_r, \cdot) \right),$$

where $\tilde{f}_{t-1}^{++}$ is the compensated decision function from the previous trial. When neither buffer is full, we have $\tilde{f}_t^{++} = \tilde{f}_t$ and the update is done by Eq. (15).

By defining the regret at time T as

$$\tilde{R}_T = \sum_{t=1}^{T} \left(\tilde{\mathcal{L}}_t(\tilde{f}_t) - \tilde{\mathcal{L}}_t(\tilde{f}^*) \right), \tag{16}$$

where $\tilde{f}^*$ is the optimal decision function obtained in hindsight by minimizing Eq. (3) with ℓ_h replaced by ℓ_{sh} defined in Eq. (13), and $\{\tilde{f}_t\}_{t=1}^T$ are obtained by the update rule in Eq. (15), we have the following regret bound:

Theorem 2. *Suppose that the assumptions of Lemma 1 hold. Suppose further that* $\frac{1}{T}\sum_{t=1}^{T} \tilde{\mathcal{L}}_t(\tilde{f}^*) \leq L^*$ *for some* $L^* > 0$. *With* $\tilde{f}_0 = 0$ *and the update rule in Eq. (15), where* $\eta \in (0, 1)$ *at each trial is chosen to guarantee descent, we have*

$$\tilde{R}_T \leq \frac{1}{1 - (1 + \zeta)\eta} \left(\frac{1}{2\eta}\|\tilde{f}^*\|_{\mathcal{H}}^2 + (1 + \zeta)\eta L^* T \right),$$

where $\zeta = 2Ck^2 c_p^2$ *and* c_p *is defined in Eq.* (11).

The proof of Theorem 2 is provided in Appendix D. The result shows that in general, our KOIL algorithm can attain an $O(\sqrt{T})$ regret bound under the smooth pairwise loss function in Eq. (13). Again, we leave the derivation of a tighter regret bound as future work.

VI. KOIL with Multiple Kernel Learning

In this section, we exploit the MKL framework to obtain an accurate data representation for good performance. Given a set of kernel functions $K = \{k_l : \mathcal{X} \times \mathcal{X} \to \mathbb{R}, l \in [m]\}$, we aim to learn a linear combination of these functions to obtain the decision function

$$F_t(\mathbf{x}) = \sum_{l=1}^{m} q_l^t \cdot \text{sgn}(f_{l,t}(\mathbf{x})),$$

where $\mathbf{q}^t = [q_1^t, \ldots, q_m^t]$ is the normalized (i.e., $\sum_{l=1}^m q_l^t = 1$) weight for multiple kernel classifiers learned up to the t-th trial and $f_{l,t}$ is an element of the RKHS $\mathcal{H}_{k_l}$ endowed with the inner product k_l. The l-th kernel classifier at the t-th trial is defined to have the same form as in Eq. (1):

$$f_{l,t}(\mathbf{x}) = \sum_{i \in I_t^+} \alpha_{l,i,t}^+ k_l(\mathbf{x}_i, \mathbf{x}) + \sum_{j \in I_t^-} \alpha_{l,j,t}^- k_l(\mathbf{x}_j, \mathbf{x}).$$

As before, we define two buffers $\mathcal{K}_{l,t}^+$ and $\mathcal{K}_{l,t}^-$ to store the corresponding information (i.e., weights and support vectors) for the l-th kernel classifier at the t-th trial.

Algorithm 4 shows the KOIL algorithm with multiple kernels.

- In line 6, we select the classifier based on the Bernoulli distribution that is proportional to the weight of the classifier. Since the weight is divided by the maximum

weight of all classifiers, at least one classifier will be selected at each trial.

- In lines 7-15, we update the predictor of the sampled classifier. To avoid excessive update fluctuation, we define the loss function $\check{\mathcal{L}}_t$ as in Eq. (4) and Eq. (14), but without the regularization term. This necessitates a change in the update rule for $\alpha_{i,t}$ in the function UpdateKernel. Specifically, in UpdateKernel2, we update $\alpha_{i,t}$ by

$$\alpha_{i,t} = \begin{cases} \eta C y_t |V_t|, & i = t, \\ \alpha_{i,t-1} - \eta C y_t, & i \in V_t, \\ \alpha_{i,t-1}, & i \in I_{t-1}^{y_t} \cup \overline{V}_t. \end{cases}$$

if the pairwise hinge loss function in Eq. (2) is used, and by

$$\alpha_{i,t} = \begin{cases} 2\eta C y_t \sum_{i \in V_t} \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i), & i = t, \\ \alpha_{i,t-1} - 2\eta C y_t \ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i), & i \in V_t, \\ \alpha_{i,t-1}, & i \in I_{t-1}^{y_t} \cup \overline{V}_t. \end{cases}$$

if the smooth pairwise hinge loss function in Eq. (13) is used.

- In line 16, the weight of the sampled kernel is updated by the exponential weighted average algorithm, where the weight is discounted by a large factor when the loss is large.

It should be noted that in order to avoid fluctuation, we do not add a smoothing term to update the probability of selecting classifiers as in [20], [22], [36], [46], [52].

Similar to Eq. (10) and Eq. (16), we can define the corresponding regret for $\{f_{l,t}\}$ and obtain an expected regret bound for Algorithm 4.

Theorem 3. *Suppose that the loss function is non-negative,* $\max_{t=1}^{T} \check{\mathcal{L}}_t(f_{l,t-1}) \leq L$, *and* $\|\partial_f \check{\mathcal{L}}_t(f_{l,t-1})\|_{\mathcal{H}_{k_l}} \leq G$ *for some* $L, G > 0$. *With* $f_{l,0} = 0$ *and suitable choices of* $\eta \in (0,1)$, $\lambda > 0$, *we have*

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{l=1}^{m} q_l^t \check{\mathcal{L}}_t(f_{l,t})\right] \leq \min_{l\in[m]} \min_{f \in \mathcal{H}_{k_l}} \left(\sum_{t=1}^{T} \check{\mathcal{L}}_t(f) + \frac{\|f\|_{\mathcal{H}_{k_l}}^2}{2\lambda}\right) + \frac{T}{2}(\eta L^2 + \lambda G^2),$$

where $q_l^t = w_l^t / [\sum_{l=1}^{m} w_l^t]$.

Note that by assuming the boundedness of the optimal kernel predictor and setting $\eta, \lambda = O(1/\sqrt{T})$, we can obtain a regret bound of $O(\sqrt{T})$ by following the proof in [52]. Alternatively, we can utilize the AM-GM inequality to remove the term $\ln m/\eta$ in [52]. Due to space limitation, we omit the proof here.

VII. Experiments

In this section, we conduct extensive experiments on both synthetic and benchmark datasets to evaluate the performance of our proposed KOIL algorithm.[1]

[1]Demo codes written in both C++ and Matlab can be downloaded at https://github.com/JunjieHu/koil.

Algorithm 4 KOIL with MKL

1: **Input:**
   - penalty parameter C and learning rates η, λ
   - maximum positive and negative budgets N^+ and N^-, respectively
   - number of nearest neighbors k
2: **Initialize** $\mathbf{w}^1 = \mathbf{1}$, $\mathcal{K}_l^+.\mathcal{A} = \mathcal{K}_l^-.\mathcal{A} = \emptyset$, $\mathcal{K}_l^+.\mathcal{B} = \mathcal{K}_l^-.\mathcal{B} = \emptyset$, $N_{p,l} = N_{n,l} = 0$, for $l \in [m]$
3: **for** $t = 1$ **to** T **do**
4: receive a training sample $\mathbf{z}_t = (\mathbf{x}_t, y_t)$
5: **for** $l = 1$ **to** m **do**
6: **if** BernSample($\mathbf{w}_l^t / [\max_j \mathbf{w}_j^t]$) $== 1$ **then**
7: **if** $y_t == +1$ **then**
8: $N_{p,l} = N_{p,l} + 1$
9: $[\mathcal{K}_l^-, \mathcal{K}_l^+, \alpha_l]$ =UpdateKernel2($\mathbf{z}_t, \mathcal{K}_l^-, \mathcal{K}_l^+, C, \eta, k$)
10: $\mathcal{K}_l^+$ =UpdateBuffer($\alpha_l, \mathbf{z}_t, \mathcal{K}_l^+, k, N^+, N_{p,l}$)
11: **else**
12: $N_{n,l} = N_{n,l} + 1$
13: $[\mathcal{K}_l^+, \mathcal{K}_l^-, \alpha_l]$=UpdateKernel2($\mathbf{z}_t, \mathcal{K}_l^+, \mathcal{K}_l^-, C, \eta, k$)
14: $\mathcal{K}_l^-$ =UpdateBuffer($\alpha_l, \mathbf{z}_t, \mathcal{K}_l^-, k, N^-, N_{n,l}$)
15: **end if**
16: $w_l^{t+1} = w_l^t \exp(-\lambda \check{\mathcal{L}}_t(f_{l,t}))$
17: **end if**
18: **end for**
19: $\mathbf{q}^{t+1} = \mathbf{w}^{t+1} / |\mathbf{w}^{t+1}|$
20: **end for**

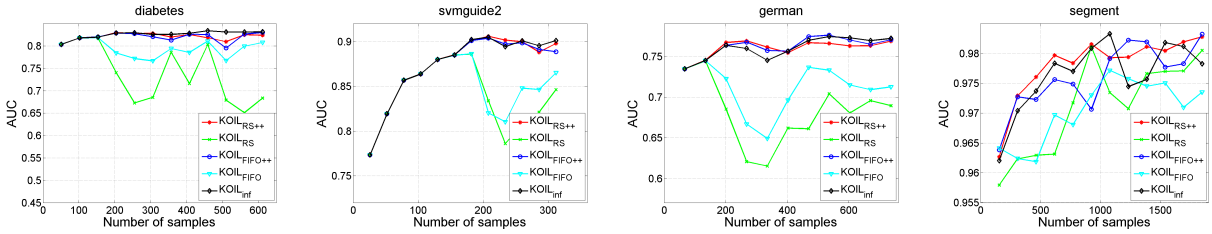
A. Compared Algorithms

We compare our proposed KOIL algorithm with state-of-the-art online learning algorithms. Since our focus is on online imbalanced learning, for fairness' sake, we do not consider batch-trained imbalanced learning algorithms in our comparison. Rather, we consider online linear algorithms and kernel-based online learning algorithms with a finite or infinite buffer size.

- "Perceptron": the classical perceptron algorithm [34]
- "OAM_{seq}": an online linear AUC maximization algorithm [55]
- "OPAUC": One-pass AUC maximization [15]
- "NORMA": online learning with kernels [27]
- "RBP": Randomized budget perceptron [4]
- "Forgetron": a kernel-based perceptron on a fixed budget [10]
- "Projectron/**Projectron++**": a bounded kernel-based perceptron [32]
- "$\text{KOIL}_{\text{RS++}}$/$\text{KOIL}_{\text{FIFO++}}$: our proposed algorithm with the pairwise hinge loss function in Eq. (2) and fixed budgets updated by RS++ and FIFO++, respectively
- "$\text{KOIL}^2_{\text{RS++}}$/$\text{KOIL}^2_{\text{FIFO++}}$": our proposed algorithm with the smooth pairwise hinge loss function in Eq. (13) and fixed budgets updated by RS++ and FIFO++, respectively

B. Experimental Setup

To ensure a fair comparison, we adopt the same setup for all algorithms. For KOIL, we set the learning rate $\eta = 0.01$ and apply a 5-fold cross validation to find the penalty cost



(a) diabetes (b) svmguide2 (c) german (d) segment

Fig. 2. Average AUC performance on four datasets obtained by different updating policies of the KOIL algorithm.

$C \in 2^{[-10:10]}$. For kernel-based methods, we use the Gaussian kernel and tune its parameter $\sigma \in 2^{[-10:10]}$ by a 5-fold cross validation. For NORMA, we apply a 5-fold cross validation to select λ and $\nu \in 2^{[-10:10]}$. For Projectron, we apply a similar 5-fold cross validation to select the parameter of projection difference $\eta \in 2^{[-10:10]}$.

TABLE I
SUMMARY OF ALL DATASETS (C^* AND γ^* ARE THE CORRESPONDING OPTIMAL HYPERPARAMETERS TUNED BY 5-FOLD CROSS VALIDATION)

Dataset	T	d	T^-/T^+	C^*	γ^*
Syn1	1,000	2	4	2^4	2^7
Syn2	1,100	2	10	2^{-9}	2^5
Syn3	5,100	2	50	2^{-10}	2^4
Syn4	10,100	2	100	2^{-6}	2^5
sonar	208	60	1.144	2^4	1
australian	690	14	1.248	2^2	1
heart	270	13	1.250	2^7	2^{-8}
ionosphere	351	34	1.786	2^5	2
diabetes	768	8	1.866	2^5	2
glass	214	9	2.057	2^5	2^5
german	1,000	24	2.333	2^7	2^{-4}
svmguide2	391	20	2.342	2^8	2^{-5}
segment	2,310	19	6.000	2^3	2^3
satimage	4,435	36	9.687	2^6	2
vowel	528	10	10.000	2^4	2^3
letter	15,000	16	26.881	2^5	2^5
poker	25,010	10	47.752	2^5	2^{-4}
shuttle	43,500	9	328.546	2^7	2^{10}

C. Experiments on Synthetic Datasets

To illustrate the KOIL algorithm and show the power of the kernel method, we generate a synthetic dataset in 2D space; see the example in Fig 1(a). The positive samples are generated from the 2-dimensional Gaussian distribution with mean $(\frac{1}{2}, \frac{1}{2})$ and standard deviation 0.1. The negative samples are generated from a mixture of four Gaussians with the same standard deviation as the positive samples and means at $(\frac{1}{6}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{6})$, $(\frac{1}{2}, \frac{5}{6})$, $(\frac{5}{6}, \frac{1}{2})$, respectively.

Following the above setup, we generate different synthetic datasets with different imbalanced ratios to explore the performance of KOIL in different scenarios. The datasets consist of

- Syn1: a set of data with imbalanced ratio 1:4 consisting of 200 positive samples and 800 negative samples;
- Syn2: a set of data with imbalanced ratio 1:10 consisting of 100 positive samples and 1,000 negative samples;
- Syn3: a set of data with imbalanced ratio 1:50 consisting of 100 positive samples and 5,000 negative samples;
- Syn4: a set of data with imbalanced ratio 1:100 consisting of 100 positive samples and 10,000 negative samples.

Obviously, these four datasets are linearly non-separable in the original space. Kernel-based online learning algorithms significantly outperform the online linear algorithms. For example, in the Syn1 dataset, Perceptron and the OAM_{seq} with buffer size 100 for each class only attain AUC scores of 0.495 ± 0.031 and 0.467 ± 0.027, respectively. These are even poorer than random guesses. For NORMA with an infinite buffer size, it achieves an AUC score of 0.940 ± 0.013. Our proposed $\text{KOIL}_{\text{RS++}}$ and $\text{KOIL}_{\text{FIFO++}}$ with a buffer size of only 50 for each class and $k = 5$ can improve the AUC scores to 0.961 ± 0.016 and 0.960 ± 0.014, respectively. Our KOIL algorithm with the smooth pairwise loss function can attain comparable or even better performance than that with the non-smooth loss function.

D. Experiments on Benchmark Real-world Datasets

14 well-known benchmark datasets, whose imbalanced ratios range from 1.144 to 328.546, are obtained from the UCI and LIBSVM websites for evaluation. Table I summarizes the detailed statistics of the datasets.

For each dataset, we conduct 5-fold cross validation on all the algorithms, where four folds of the data are used for training while the rest for testing. The 5-fold cross validation is independently repeated four times. We set the buffer size to 100 for each class for all related algorithms, including OAM_{seq}, RBP, and Forgetron. We then average the AUC performance of 20 runs. The results are reported in Table III. From the table, we have the following observations:

- Our KOIL algorithm with the RS++ and FIFO++ updating policies performs better than online linear AUC maximization algorithms in most datasets. By examining the results of OAM_{seq} on australian, heart, diabetes, german, and shuttle, as well as the results of OPAUC on australian and german, we speculate that a linear classifier is enough to achieve good performance on these datasets while a nonlinear classifier can be adversely affected by outliers.
- Under the pairwise hinge loss function, the KOIL algorithm significantly outperforms all competing kernel-based algorithms in nearly all datasets. The results demonstrate the effectiveness of our proposed approach.

TABLE II
AVERAGE AUC PERFORMANCE (MEAN±STD) ON THE SYNTHETICS DATASETS. ●/○ (-) INDICATES THAT BOTH/ONE OF $\text{KOIL}_{\text{RS++}}$ AND $\text{KOIL}_{\text{FIFO++}}$ ARE/IS SIGNIFICANTLY BETTER (WORSE) THAN THE CORRESPONDING METHOD (PAIRWISE t-TESTS AT 95% SIGNIFICANCE LEVEL)

Data	$\text{KOIL}_{\text{RS++}}$	$\text{KOIL}_{\text{FIFO++}}$	$\text{KOIL}^2_{\text{RS++}}$	$\text{KOIL}^2_{\text{FIFO++}}$	Perceptron	OAM_{seq}	OPAUC	NORMA	RBP	Forgetron	Projectron	Projectron++
Syn1	.961±.016	.960±.014	.967±.011	**.968**±.011	.495±.031●	.501±.021●	.503±.032●	.940±.013●	.948±.021●	.878±.147●	.954±.019	.953±.017
Syn2	.959±.022	.958±.018	.961±.017	**.962**±.018	.484±.037●	.502±.032●	.508±.032●	.937±.041●	.887±.062●	.954±.023	.941±.032●	.944±.023●
Syn3	.939±.029	.941±.025	**.943**±.022	.942±.023	.495±.025●	.499±.022●	.492±.020●	.769±.087●	.872±.081●	.807±.130●	.901±.064●	.922±.039●
Syn4	.965±.014	.966±.013	**.968**±.013	.966±.015	.510±.023●	.495±.026●	.499±.022●	.834±.205●	.892±.069●	.844±.097●	.962±.015	.948±.024●
win/tie/loss			0/4/0	0/4/0	4/0/0	4/0/0	4/0/0	4/0/0	4/0/0	3/1/0	2/2/0	3/1/0

TABLE III
AVERAGE AUC PERFORMANCE (MEAN±STD) ON THE BENCHMARK DATASETS. ●/○ (-) INDICATES THAT BOTH/ONE OF $\text{KOIL}_{\text{RS++}}$ AND $\text{KOIL}_{\text{FIFO++}}$ ARE/IS SIGNIFICANTLY BETTER (WORSE) THAN THE CORRESPONDING METHOD (PAIRWISE t-TESTS AT 95% SIGNIFICANCE LEVEL)

Data	$\text{KOIL}_{\text{RS++}}$	$\text{KOIL}_{\text{FIFO++}}$	$\text{KOIL}^2_{\text{RS++}}$	$\text{KOIL}^2_{\text{FIFO++}}$	Perceptron	OAM_{seq}	OPAUC	NORMA	RBP	Forgetron	Projectron	Projectron++
sonar	.955±.028	.955±.028	**.957**±.031	**.957**±.031	.803±.083●	.843±.056●	.844±.077●	.925±.044●	.913±.032●	.896±.054●	.896±.049●	.896±.049●
australian	.923±.023	.922±.026	.919±.024	.920±.026	.869±.035●	**.925**±.024	.923±.025	.919±.023	.911±.017●	.912±.026●	.923±.024	.923±.024
heart	.908±.040	.910±.040	.911±.038	.908±.037	.876±.066●	**.912**±.040	.901±.043○	.890±.051●	.865±.043●	.900±.053○	.902±.038	.905±.042
ionosphere	**.985**±.015	**.985**±.015	.959±.026●	.952±.031●	.851±.056●	.905±.041●	.888±.046●	.961±.016●	.960±.030●	.945±.031●	.964±.025●	.963±.027●
diabetes	.826±.036	.830±.030	.817±.037○	.825±.028	.726±.059●	.827±.033	.805±.035●	.792±.032●	.828±.034	.820±.027○	.832±.033	**.833**±.033
glass	**.887**±.053	.884±.054	.885±.048	.885±.048	.810±.065●	.827±.064●	.800±.074●	.811±.077●	.811±.071●	.813±.075●	.811±.070●	.781±.076●
german	.769±.032	.778±.031	.774±.030	.769±.037○	.748±.033●	.777±.027	**.787**±.026 -	.766±.032○	.699±.038●	.712±.054●	.769±.028○	.770±.024
svmguide2	**.897**±.040	.885±.043	.891±.042	.882±.040○	.860±.037●	.886±.045○	.859±.050●	.865±.046●	.890±.038	.864±.045●	.886±.044○	.886±.045○
segment	.983±.008	**.985**±.012	.970±.012●	.959±.015●	.875±.020●	.919±.020●	.882±.019●	.910±.042●	.969±.017●	.943±.038●	.979±.013●	.978±.016●
satimage	**.924**±.012	.923±.015	.922±.012	.922±.013	.700±.015●	.755±.018●	.724±.016●	.914±.025●	.899±.018●	.892±.032●	.910±.015●	.904±.011●
vowel	**1.000**±.000	**1.000**±.001	.998±.007●	.993±.014●	.848±.070●	.905±.024●	.885±.034●	.996±.005●	.968±.017●	.987±.027●	.982±.013●	.994±.019●
letter	.933±.021	**.942**±.017	.926±.022●	.932±.017○	.767±.029●	.827±.021●	.823±.018●	.910±.027●	.928±.011○	.815±.102●	.926±.016●	.926±.015●
poker	.681±.031	**.693**±.032	.654±.023●	.676±.031●	.514±.030●	.503±.024●	.509±.031●	.577±.040●	.501±.031●	.572±.029●	.675±.027●	.675±.027●
shuttle	.950±.040	.956±.021	.946±.039	.953±.020	.520±.134●	**.999**±.000 -	.754±.043●	.725±.053●	.844±.041●	.839±.060●	.873±.063●	.795±.063●
win/tie/loss			6/8/0	7/7/0	14/0/0	9/4/1	12/1/1	13/1/0	12/2/0	14/0/0	11/3/0	10/4/0

- We observe that the KOIL algorithm with non-smooth loss beats the one with smooth loss in 5 datasets while being comparable in the remaining 9 datasets.
- In most of the datasets, kernel-based algorithms show better AUC performance than the linear algorithms. This again demonstrates the power of kernel methods in classifying real-world datasets.
- We observe that the performance of OAM_{seq} on satimage is not as good as that in [55] and [21]. This can be attributed to the different partition of the training and test data.

E. Evaluation of Updating Policies

We compare the compensation schemes RS++ and FIFO++ with the original updating policies RS and FIFO and show the average AUC performance of 20 runs on four typical datasets in Fig. 2. Here, KOIL_{inf} denotes KOIL learned with infinite budgets and is used as a reference. From Fig. 2, we have the following observations:

- $\text{KOIL}_{\text{RS++}}$ and $\text{KOIL}_{\text{FIFO++}}$ have nearly the same performance as KOIL_{inf}. This confirms that the extended policies indeed compensate for the lost information when a support vector is replaced.
- The KOIL algorithm with extended updating policies significant outperforms the one with original stream oblivious policy when either buffer is full. Without compensation, the performance fluctuates and decays when support vectors are removed. With compensation, the performance is rather stable.

F. Sensitivity Analysis

In this subsection, we study the sensitivity of the KOIL algorithm to the input parameters. First, we test the performance of the KOIL algorithm as the buffer size varies. From Fig. 3, we observe that the performance follows similar trend in [21], [55]; i.e., improving gradually with the increase of the buffer size and becoming stable when the size is relatively large.

Next, we test the performance of the KOIL algorithm as the number of localized support vectors k varies. From Fig. 4, we have the following observations:

- When $k = 1$, the smallest possible value of k, the performance of the KOIL algorithm is usually poor because it only considers the pairwise loss incurred by the nearest opposite support vector of the new instance and cannot fully utilize the localized information.
- The KOIL algorithm usually attains the best performance when k is approximately 10% of the buffer size. As k further increases, the performance starts to deteriorate. Our results consistently demonstrate that the effect of outliers can be alleviated by utilizing localized information of the new instance.
- For some datasets, such as svmguide2 and german, the performance of the KOIL algorithm is not too sensitive to k. The reason could be that the learned support vectors in these datasets are well-separated when the buffers are full. As a result, new instances have little influence on the updating of the decision function.

In sum, a key step in maintaining the model performance is the compensation scheme; see the results in Fig. 2. The setting of localized AUC is also crucial to good performance as it can mitigate the effect of noise; see results in Fig. 4. Fig. 3 suggests that the budget just needs to be sufficiently large, say several hundred.

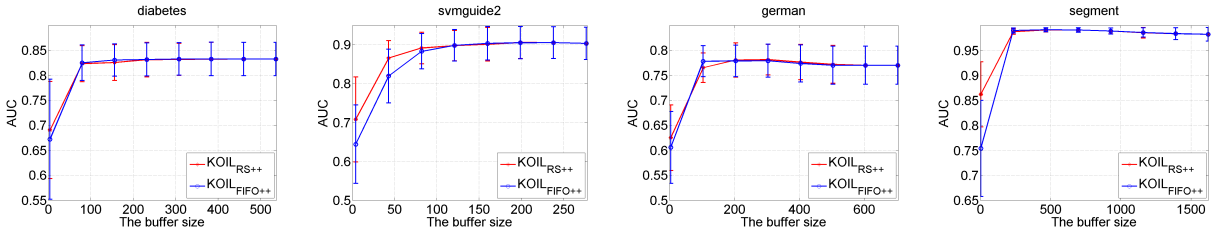


Fig. 3. Average AUC of the KOIL algorithm with different buffer sizes.

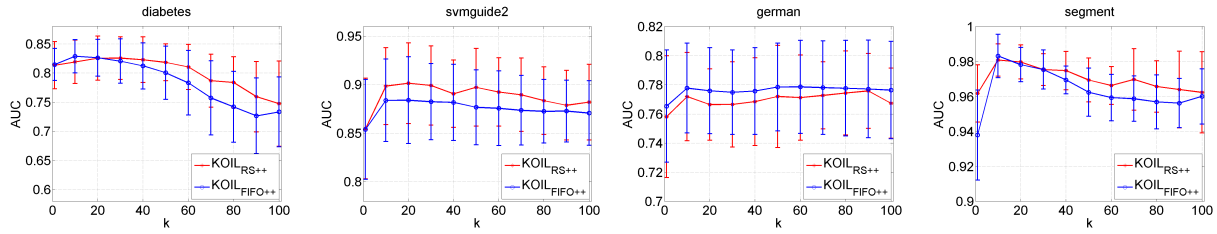


Fig. 4. Average AUC of the KOIL algorithm with different k. Here $k = [1, 10{:}10{:}100]$ and the budget is 100 for each buffer.

G. Evaluation of the KOIL Algorithm with MKL

We evaluate the performance of the KOIL algorithm with MKL using the setting in [22]. Specifically, we use 16 kernel functions in our experiment, including 3 polynomial kernels (i.e., $k(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i^T \mathbf{x}_j)^p$ with degree parameters $p = 1$, 2, and 3) and 13 Gaussian kernels (i.e., $k(\mathbf{x}_i, \mathbf{x}_j) = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|^2/2\sigma^2)$ with kernel width parameter $\sigma \in 2^{[-6:1:6]}$). For simplicity, the learning rates η and λ are both set to 0.01. A 5-fold cross validation is applied to find the best penalty cost C from $2^{[-10:1:10]}$. Table IV summarizes the results and reveals the following:

- The KOIL algorithm with MKL attains better or comparable performance than the one with tuned optimal kernel. Indeed, under the smooth loss function, the former has a better performance in at least 6 out of the 14 datasets. On the other hand, under the non-smooth loss function, both versions of the KOIL algorithm have comparable performance on most datasets. We conjecture that this may be due to the non-smoothness of the loss function.
- For some datasets, such as sonar and ionosphere, the KOIL algorithm with MKL cannot beat the one with the tuned optimal kernel. We conjecture that this may be due to the limitation of the training data in these datasets. Training with multi-epoches [52] could be a promising approach to improving the model performance.

TABLE IV
AVERAGE AUC PERFORMANCE (MEAN±STD) ON THE BENCHMARK DATASETS. ● (-) INDICATES THAT THE PERFORMANCE BY KOIL WITH MKL IS SIGNIFICANTLY BETTER THAN (COMPARABLE TO) THAT BY KOIL WITH THE TUNED OPTIMAL KERNEL (PAIRWISE t-TESTS AT 95% SIGNIFICANCE LEVEL)

Data	$\text{KOIL}^{\text{MKL}}_{\text{RS++}}$	$\text{KOIL}^{\text{MKL}}_{\text{FIFO++}}$	$\text{KOIL}^{\text{MKL 2}}_{\text{RS++}}$	$\text{KOIL}^{\text{MKL 2}}_{\text{FIFO++}}$
sonar	0.893±0.053	0.899±0.047	0.946±0.040 -	0.949±0.031 -
australian	0.922±0.027 -	0.919±0.028 -	0.918±0.026 -	0.911±0.024
heart	0.906±0.044 -	0.907±0.042 -	0.906±0.040 -	0.904±0.038 -
ionosphere	0.953±0.062	0.957±0.073	0.972±0.039 -	0.972±0.042●
diabetes	0.826±0.035 -	0.831±0.032 -	0.827±0.036●	0.822±0.033 -
glass	0.890±0.056 -	0.891±0.051 -	0.890±0.053 -	0.893±0.052 -
german	0.771±0.042 -	0.769±0.033 -	0.774±0.033 -	0.768±0.039 -
svmguide2	0.906±0.040●	0.896±0.049●	0.905±0.041●	0.903±0.043●
segment	0.993±0.004●	0.994±0.004●	0.991±0.005●	0.990±0.009●
satimage	0.937±0.015●	0.939±0.015●	0.938±0.012●	0.937±0.014●
vowel	0.999±0.002	0.999±0.002 -	0.999±0.002 -	0.998±0.003 -
letter	0.954±0.013●	0.959±0.014●	0.962±0.011●	0.968±0.008●
poker	0.690±0.035●	0.707±0.027●	0.709±0.023●	0.705±0.020●
shuttle	0.948±0.028 -	0.926±0.032	0.888±0.029	0.886±0.032
win/tie/loss	5/6/3	5/3/4	6/7/1	6/6/2

VIII. CONCLUSION

We focused on the imbalanced streaming binary classification problem and proposed a kernel-based online learning algorithm to seek a nonlinear classifier. Our algorithm is based on three crucial ideas. First, we adopt two fixed-budget buffers to control the number of support vectors and maintain the global information on the decision boundary. Second, we update the weight of a new arriving support vector by confining its influence on only its k-nearest opposite support vectors. Third, we transfer the weight of the removed support vector to its most similar one when either buffer is full, so as to avoid information loss. We also exploited the MKL framework to determine the kernel our KOIL algorithm. Lastly, we conducted extensive experiments to demonstrate the efficacy and superiority of our proposed approach.

Several challenging but promising directions can be considered in the future. First, the current KOIL algorithm only explores a localized surrogate of the AUC metric. Investigating more accurate surrogate functions for the AUC metric is significant in both theory and practice. Second, the current regret bound only applies to the case where there is no compensation. A natural direction is to derive a regret bound for the case where the compensation scheme is used. Third, it would be interesting to investigate and evaluate more efficient update rules for the KOIL algorithm with MKL.

APPENDIX A
PROOF OF LEMMA 1

Proof. First, the assumptions $k(\mathbf{x},\mathbf{x}) \leq X^2$ for all $\mathbf{x} \in \mathbb{R}^d$ and $k(\mathbf{x}_t,\mathbf{x}_i) \geq \xi_1^2 > 0$ yield

$$\|\varphi(\mathbf{z}_t,\mathbf{z}_i)\|_{\mathcal{H}} = \sqrt{k(\mathbf{x}_t,\mathbf{x}_t) - 2k(\mathbf{x}_t,\mathbf{x}_i) + k(\mathbf{x}_i,\mathbf{x}_i)} \leq c_p, \tag{17}$$

where c_p is defined in Eq. (11). Now, using Eq. (5), Eq. (7), and the triangle inequality, we compute

$$\begin{aligned}
&\|f_t\|_{\mathcal{H}} \\
\leq&(1-\eta)\|f_{t-1}\|_{\mathcal{H}} \\
&+ \eta C \sum_{\mathbf{z}_i} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \cdot \|\varphi(\mathbf{z}_t,\mathbf{z}_i)\|_{\mathcal{H}} \\
\leq&(1-\eta)\|f_{t-1}\|_{\mathcal{H}} + \eta C k c_p,
\end{aligned}$$

where the last inequality is due to Eq. (17) and the fact that the number of elements in $N_{t,k}^{-y_t}(\mathbf{z}_t)$ is at most k.

By expanding $\|f_t\|_{\mathcal{H}}$ iteratively and using $f_0 = 0$, we have

$$\|f_t\|_{\mathcal{H}} \leq (1-\eta)^t \|f_0\|_{\mathcal{H}} + \frac{1-(1-\eta)^t}{\eta} \eta C k c_p \leq C k c_p,$$

where the second inequality is due to the fact that when $\eta \in (0,1)$, we have $1-(1-\eta)^t \leq 1$ for $t \in [T]$. This completes the proof. □

APPENDIX B
PROOF OF LEMMA 2

Proof. Based on the pairwise hinge loss defined in Eq. (2), we have

$$\begin{aligned}
\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) &\leq 1 + |f_{t-1}(\mathbf{x}_t) - f_{t-1}(\mathbf{x}_i)| \\
&= 1 + |\langle f_{t-1}, k(\mathbf{x}_t,\cdot) - k(\mathbf{x}_i,\cdot)\rangle_{\mathcal{H}}| \\
&\leq 1 + \|f_{t-1}\|_{\mathcal{H}} \cdot \|k(\mathbf{x}_t,\cdot) - k(\mathbf{x}_i,\cdot)\|_{\mathcal{H}} \\
&\leq 1 + C k c_p^2 \quad (:= U),
\end{aligned}$$

where the first inequality is due to the triangle inequality and $\frac{1}{2}|y_t - y_i| \leq 1$; the second inequality is due to the Cauchy-Schwarz inequality; the third inequality is due to Lemma 1 and Eq. (17). □

APPENDIX C
PROOF OF THEOREM 1

Proof. Since the learning rate $\eta \in (0,1)$ at each trial is chosen to guarantee descent and $\hat{\mathcal{L}}_t$ is convex, we have

$$\begin{aligned}
R_T &= \sum_{t=1}^{T} \left(\hat{\mathcal{L}}_t(f_t) - \hat{\mathcal{L}}_t(f^*)\right) \leq \sum_{t=1}^{T} \left(\hat{\mathcal{L}}_t(f_{t-1}) - \hat{\mathcal{L}}_t(f^*)\right) \\
&\leq \sum_{t=1}^{T} \langle \partial \hat{\mathcal{L}}_t(f_{t-1}), f_{t-1} - f^* \rangle_{\mathcal{H}}.
\end{aligned} \tag{18}$$

Now, observe that

$$\begin{aligned}
&\|f_t - f^*\|_{\mathcal{H}}^2 - \|f_{t-1} - f^*\|_{\mathcal{H}}^2 \\
=&\|f_{t-1} - \eta \partial_f \hat{\mathcal{L}}_t(f_{t-1}) - f^*\|_{\mathcal{H}}^2 - \|f_{t-1} - f^*\|_{\mathcal{H}}^2 \\
=&\eta^2 \|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2 - 2\eta \langle \partial_f \hat{\mathcal{L}}_t(f_{t-1}), f_{t-1} - f^* \rangle_{\mathcal{H}}.
\end{aligned}$$

By summing the above identity over $t \in [T]$, we have

$$\begin{aligned}
&\|f_T - f^*\|_{\mathcal{H}}^2 - \|f_0 - f^*\|_{\mathcal{H}}^2 \\
=&-2\eta \sum_{t=1}^{T} \langle \partial_f \hat{\mathcal{L}}_t(f_{t-1}), f_{t-1} - f^* \rangle_{\mathcal{H}} \\
&+ \eta^2 \sum_{t=1}^{T} \|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2.
\end{aligned} \tag{19}$$

Upon combining Eq. (18), Eq. (19) and using the fact that $f_0 = 0$, $\|f_T - f^*\|_{\mathcal{H}}^2 \geq 0$, we obtain

$$R_T \leq \frac{\|f^*\|_{\mathcal{H}}^2}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2.$$

We now proceed to bound $\|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2$. Observe that

$$\begin{aligned}
&\|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2 \\
=&\left\| f_{t-1} - C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \cdot \varphi(\mathbf{z}_t,\mathbf{z}_i) \right\|_{\mathcal{H}}^2 \\
=&\|f_{t-1}\|_{\mathcal{H}}^2 \\
&+ \left\| C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \cdot \varphi(\mathbf{z}_t,\mathbf{z}_i) \right\|_{\mathcal{H}}^2 \\
&- 2C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \langle f_{t-1}, \varphi(\mathbf{z}_t,\mathbf{z}_i) \rangle_{\mathcal{H}}.
\end{aligned}$$

From Lemma 1, we know that the first term above is bounded by

$$\|f_{t-1}\|_{\mathcal{H}}^2 \leq C^2 k^2 c_p^2. \tag{20}$$

Now, by Eq. (17) and the fact that the number of elements in $N_{t,k}^{-y_t}(\mathbf{z}_t)$ is at most k, we can bound the second term by

$$\begin{aligned}
&\left\| C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \cdot \varphi(\mathbf{z}_t,\mathbf{z}_i) \right\|_{\mathcal{H}}^2 \\
\leq&C^2 \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \cdot \|\varphi(\mathbf{z}_t,\mathbf{z}_i)\|_{\mathcal{H}}^2 \\
\leq&C^2 k c_p^2.
\end{aligned} \tag{21}$$

Lastly, since f_{t-1} is an element of a RKHS and $\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) \leq U$ by Lemma 2, we have

$$\begin{aligned}
\langle f_{t-1}, \varphi(\mathbf{z}_t,\mathbf{z}_i) \rangle_{\mathcal{H}} &= \frac{1}{2}(y_t - y_i)\left(f_{t-1}(\mathbf{x}_t) - f_{t-1}(\mathbf{x}_i)\right) \\
&\geq 1 - U.
\end{aligned}$$

It follows that the third term can be bounded by

$$\begin{aligned}
&-2C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0] \langle f_{t-1}, \varphi(\mathbf{z}_t,\mathbf{z}_i) \rangle_{\mathcal{H}} \\
\leq&2C \sum_{\mathbf{z}_i \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}[\ell_h(f_{t-1},\mathbf{z}_t,\mathbf{z}_i) > 0](U-1) \\
\leq&2Ck(U-1),
\end{aligned} \tag{22}$$

where the second inequality is again due to the fact that the number of elements in $N_{t,k}^{-y_t}(\mathbf{z}_t)$ is at most k.

By combining Eq. (20), Eq. (21), and Eq. (22), we obtain

$$\|\partial_f \hat{\mathcal{L}}_t(f_{t-1})\|_{\mathcal{H}}^2 \leq C^2 k(k+1)c_p^2 + 2Ck(U-1).$$

The bound on R_T in Eq. (12) now follows by summing $\partial_f \hat{\mathcal{L}}_t(f_{t-1})$ over $t \in [T]$. This completes the proof. □

APPENDIX D
PROOF OF THEOREM 2

Proof. The proof is similar to that of Theorem 1. The main difference is to exploit the smoothness of the loss function.

First, since the learning rate $\eta \in (0,1)$ at each trial is chosen to guarantee descent and $\tilde{\mathcal{L}}_t$ is convex, we have

$$\begin{aligned}\tilde{\mathcal{L}}_t(\tilde{f}_t) - \tilde{\mathcal{L}}_t(\tilde{f}^*) &\leq \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) - \tilde{\mathcal{L}}_t(\tilde{f}^*) \\ &\leq \langle \partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}), \tilde{f}_{t-1} - \tilde{f}^* \rangle. \end{aligned} \tag{23}$$

We also have

$$\begin{aligned}&\|\tilde{f}_t - \tilde{f}^*\|_{\mathcal{H}}^2 - \|\tilde{f}_{t-1} - \tilde{f}^*\|_{\mathcal{H}}^2 \\ =&\eta^2 \|\partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1})\|_{\mathcal{H}}^2 - 2\eta \langle \partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}), \tilde{f}_{t-1} - \tilde{f}^* \rangle_{\mathcal{H}}. \end{aligned} \tag{24}$$

Now, we compute

$$\frac{\partial^2 \tilde{\mathcal{L}}_t}{\partial f^2} = I + 2C \sum_{\mathbf{z}_i, \mathbf{z}_j \in N_{t,k}^{-y_t}(\mathbf{z}_t)} \mathbb{I}_{\mathbf{z}_i} \mathbb{I}_{\mathbf{z}_j} \varphi(\mathbf{z}_t, \mathbf{z}_i) \varphi(\mathbf{z}_t, \mathbf{z}_j)^\top, \tag{25}$$

where we denote $\mathbb{I}[\ell_h(\tilde{f}_{t-1}, \mathbf{z}_t, \mathbf{z}_i) > 0]$ by $\mathbb{I}_{\mathbf{z}_i}$ for simplicity. It follows that for any $\tilde{f}, \tilde{g}$,

$$\|\partial_f \tilde{\mathcal{L}}_t(\tilde{f}) - \partial_f \tilde{\mathcal{L}}_t(\tilde{g})\|_{\mathcal{H}} \leq (1+\zeta)\|\tilde{f} - \tilde{g}\|_{\mathcal{H}}, \tag{26}$$

where $\zeta = 2Ck^2 c_p^2$ is obtained by the summation in Eq. (25) and the bound

$$\langle \varphi(\mathbf{z}_t, \mathbf{z}_i), \varphi(\mathbf{z}_t, \mathbf{z}_j) \rangle_{\mathcal{H}} \leq \|\varphi(\mathbf{z}_t, \mathbf{z}_i)\|_{\mathcal{H}} \cdot \|\varphi(\mathbf{z}_t, \mathbf{z}_j)\|_{\mathcal{H}} \leq c_p^2.$$

In particular, suppose that $\tilde{f}_t^*$ minimizes $\tilde{\mathcal{L}}_t$. Then, by the convexity and smoothness of $\tilde{\mathcal{L}}_t$, we have $\partial_f \tilde{\mathcal{L}}_t(\tilde{f}_t^*) = 0$. This, together with Eq. (26) and [31, Theorem 2.1.5], implies that

$$\begin{aligned}\|\partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1})\|_{\mathcal{H}}^2 &= \|\partial_f \tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) - \partial_f \tilde{\mathcal{L}}_t(\tilde{f}_t^*)\|_{\mathcal{H}}^2 \\ &\leq 2(1+\zeta)\left(\tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) - \tilde{\mathcal{L}}_t(\tilde{f}_t^*)\right) \\ &\leq 2(1+\zeta)\,\tilde{\mathcal{L}}_t(\tilde{f}_{t-1}), \end{aligned} \tag{27}$$

where the last inequality is due to $\tilde{\mathcal{L}}_t(\tilde{f}_t^*) \geq 0$.

By combining Eq. (23), Eq. (24), and Eq. (27), we obtain

$$\begin{aligned}&(1-(1+\zeta)\eta)\tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) - \tilde{\mathcal{L}}_t(\tilde{f}^*) \\ \leq&\frac{\|\tilde{f}_{t-1} - \tilde{f}^*\|_{\mathcal{H}}^2 - \|\tilde{f}_t - \tilde{f}^*\|_{\mathcal{H}}^2}{2\eta}. \end{aligned}$$

Upon summing the above inequality over $t \in [T]$ and rearranging, we obtain

$$\begin{aligned}&\sum_{t=1}^{T}(1-(1+\zeta)\eta)\tilde{\mathcal{L}}_t(\tilde{f}_{t-1}) - \sum_{t=1}^{T}\tilde{\mathcal{L}}_t(\tilde{f}^*) \\ \leq&\frac{1}{2\eta}\left(\|\tilde{f}_0 - \tilde{f}^*\|_{\mathcal{H}}^2 - \|\tilde{f}_T - \tilde{f}^*\|_{\mathcal{H}}^2\right) \leq \frac{1}{2\eta}\|\tilde{f}^*\|_{\mathcal{H}}^2. \end{aligned}$$

Here, we use the fact that $\tilde{f}_0 = 0$ and $\|\tilde{f}_T - \tilde{f}^*\|_{\mathcal{H}}^2 \geq 0$. It follows that

$$\begin{aligned}&\sum_{t=1}^{T}\left(\tilde{\mathcal{L}}_t(\tilde{f}_t) - \tilde{\mathcal{L}}_t(\tilde{f}^*)\right) \\ \leq&\frac{1}{1-(1+\zeta)\eta}\left(\frac{1}{2\eta}\|\tilde{f}^*\|_{\mathcal{H}}^2 + (1+\zeta)\eta\sum_{t=1}^{T}\tilde{\mathcal{L}}_t(\tilde{f}^*)\right) \\ \leq&\frac{1}{1-(1+\zeta)\eta}\left(\frac{1}{2\eta}\|\tilde{f}^*\|_{\mathcal{H}}^2 + (1+\zeta)\eta L^* T\right), \end{aligned}$$

as desired. □

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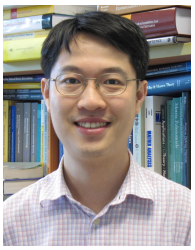

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