LDPC CODE DESIGN FOR GAUSSIAN MULTIPLE-ACCESS CHANNELS USING DYNAMIC EXIT CHART ANALYSIS

Naijun Zheng[†], *Yuxuan He*[†], *Baoming Bai*[†], *Anthony Man-Cho So*[§], *and Kehu Yang*[†]

[†]ISN Lab, Xidian University, Xi'an, China
[§] Dept. of Sys. Engg. & Engg. Mgmt., The Chinese University of Hong Kong, Hong Kong, China

ABSTRACT

We consider the degree distribution design of the low-density parity-check (LDPC) code ensembles for symmetric Gaussian multiple-access channels (GMAC). To characterize the probability density function (PDF) of the message passing in the process of joint decoding, we propose a new scheme to construct the associated Gaussian mixture (GM) distribution, where each GM component is assigned according to the corresponding signal group transmitted by the users. By tracking the variation of the GM components in the iterative decoding process, more accurate mutual information can be obtained for the extrinsic information transfer (EXIT) chart analysis. Simulation results show that the performance of our proposed LDPC codes is better than that of the existing methods.

Index Terms— LDPC, EXIT analysis, Gaussian mixture, MAC

1. INTRODUCTION

Recently, network information theory has been widely applied in wireless networks [1], where the applications to multiple-access channels (MACs) are first investigated. It has been shown that the low-density parity-check (LDPC) codes [2] designed under the EXIT chart analysis [3] and the density evolution (DE) [4] can approach the Gaussian channel capacity in point-to-point (P2P) communications using the belief propagation (BP) decoding algorithm [5][6]. However, for multi-user scenarios, since the EXIT chart analysis and the DE cannot be performed straightforwardly, it is difficult to design the LDPC codes based on the decoding process analysis. This means that new approaches are required.

As is known, DE is a powerful tool for designing the degree distribution of LDPC code ensembles. However, DE has higher complexity. The EXIT chart analysis is a method of Gaussian approximation to the DE based on the mutual information measure. Compared with the DE, the EXIT chart method can greatly simplify the PDF computation of the messages in the iterative decoding of the LDPC codes. Furthermore, with the EXIT chart analysis, the problem to optimize the degree distribution can be formulated as a linear program (LP) [7].

However, when applying the EXIT chart analysis to multi-user communications, there are two important issues we must consider: 1) The joint decoding scheme requires the decoders to exchange the message in each decoding iteration, which introduces dynamic features in the decoding process. These dynamic features should be considered in the associated EXIT chart analysis. 2) The PDF of the message passing through the nodes in the decoders is no longer Gaussian [8]. This implies that a new approach to the mutual information computation in the EXIT chart analysis is required.

Considering the above two issues, several methods were proposed to design the LDPC codes for the two-user symmetric GMAC. In [9], an iterative joint decoding scheme is proposed to extend the BP algorithm to multi-user cases, which makes the EXIT chart analysis applicable to the design of the degree distribution for the two-user MAC. In [10], the authors showed that by using the extended DE and EXIT chart analysis, a good LDPC code ensemble with desired degree distribution for GMAC can be obtained. Furthermore, the GM distribution for the EXIT chart analysis is introduced to formulate the PDF of the associated message in the analysis of the decoding process [8].

Motivated by the work of [8, 10], we find that there is still room to improve the estimation accuracy of the PDF with the GM distribution. Here, we propose to estimate the PDF by constructing the components of GM distribution based on the signal groups transmitted by the users. By tracking the parameters of each PDF component, the mutual information for the EXIT chart analysis can be computed with high accuracy.

2. SYSTEM MODEL

2.1. Channel Models and Its Rate Region

Consider an n-user GMAC. Assuming that BPSK signaling is used for each user, the received signal is denoted by

$$Y = \sum_{i=1}^{n} h_i X_i + Z, \qquad (1)$$

where X_i is the input signal for the i-th user with the average energy $E\|X_i\|^2 = 1$, Z is the symmetric white Gaussian noise which follows $\mathcal{N}(0,1)$, and h_i is the channel gain for the i-th user. The signal-to-noise ratio (SNR) for the i-th user at the receiver is thus represented by h_i^2. The associated channel capacity region [11] of an n-user GMAC can be characterized as the convex hull of the rate sets $(R_1, ..., R_n)$ over the product distribution $\prod_{i=1}^{n} p(x_i)$, which is denoted by

$$\sum_{j \in J} R_j \leq I\left(X(J), Y | X(J^c)\right), \forall J \subseteq \{1, \ldots, n\}, \quad (2)$$

where J^c denotes the complementary set of J.

2.2. LDPC Codes and Its Joint Decoding Scheme

The LDPC codes, proposed by R. Gallager in 1962, can be represented by their parity-check matrices. By introducing irregular parity-check matrices, the codes generated can have better performance. These codes are called irregular LDPC codes [2]. The degree distribution of an irregular LDPC code ensemble can be specified by the following two polynomials:

$$\lambda(x) = \sum_{j=2}^{dv} \lambda_j x^{j-1}, \quad \rho(x) = \sum_{j=2}^{dc} \rho_j x^{j-1}, \quad (3)$$

where $\lambda(x)$ is for variable nodes (VNs), $\rho(x)$ is for check nodes (CNs), and d_v and d_c denote the maximum variable degree and the maximum check degree, respectively. The rate of the code can be calculated as

$$R = 1 - \frac{\sum_{i=2}^{dc} \rho_j / j}{\sum_{j=2}^{dv} \lambda_j / j}. \quad (4)$$

In the decoding process, the log-likelihood ratios (LLRs) used in the BP decoding algorithm [12] are often denoted as the messages transferred inside the decoder, represented by

$$m = \log \frac{p(x = +1|y)}{p(x = -1|y)}. \quad (5)$$

However, for an MAC receiver, decoding the signals of several users can be performed simultaneously, which leads to joint decoding. A joint decoding scheme is shown in Fig. 1, where some of the LLRs are shared by the decoders to cancel the interference from each other. Comparing with the single-user decoding process, the additional nodes, which are called state nodes (SNs), are required for the collaboration of all the decoders. The LLRs transferred between the corresponding nodes can be represented by m_{vc} (from VNs to CNs), m_{cv} (from CNs to VNs), m_{vs} (from VNs to SNs), and m_s (from SNs to VNs), respectively. Since m_s contains the channel information and the information sent by other decoders, we call m_s the side information in the sequel. The associated LLRs are updated under the criterion of maximum *a posteriori* in the joint decoding scheme [4, 12].

2.3. EXIT Chart Analysis

The EXIT chart analysis is first introduced by ten Brink to analyze the convergence of turbo codes [13]. The main idea of the EXIT chart analysis is to compute the mutual information $I(x; m)$ between the transmitted signals x and the LLRs m in the iterative coding scheme. If $I(x; m)$ keeps increasing (up to 1) during the decoding process, then the decoding will be considered a successful one. In P2P communications, the PDF of the LLR m_{vc} and m_{cv} can be approximated by the Gaussian distribution with the mean being half of the covariance [14]. This is often called the consistence property, which allows the associated PDF to be characterized by only one parameter (mean or covariance). For a given channel condition and a given degree distribution, the EXIT chart analysis with the associated mutual information computed by $J(\cdot)$ can be used to evaluate the decoding convergence of the code ensembles [3, 15]. However, for the multi-user cases, this property will no longer hold.

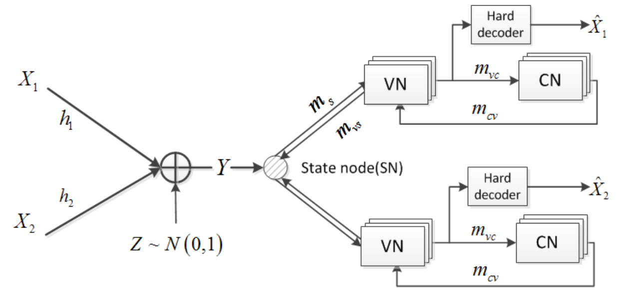

Fig. 1. Receiver structure with joint decoding for 2-user MAC

3. AN APPROACH TO THE LDPC CODE DESIGN

3.1. A New Approach to the Associated PDF

To compute the mutual information $I(x; m)$, the knowledge of the PDF $p(m)$ of the LLRs is required. It is evident that the PDF of the LLRs in the joint decoding scheme is more complicated than that of the single-user decoding. To approach the PDF of the LLRs, the distribution with GM is often exploited [8, 10]. The distribution with GM is often the weighted sum of several Gaussian distributions given by

$$p(m) = \sum_{i=1}^{n} \frac{\omega_i}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(m - \mu_i)^2}{2\sigma_i^2}\right), \quad (6)$$

where n is the number of components, w_i is the weight for the i-th component, and μ_i and σ_i are the mean and covariance of the i-th Gaussian component, respectively.

Different from the existing schemes that use the expected PDF of the received signals, here we propose a new one to approach the PDF, where each of the Gaussian components corresponds to one of the possible group signals transmitted by the users; *i.e.*, assign an independent Gaussian component to each signal group. Without loss of generality, for the i-th decoder related LLRs, we assume that the i-th user transmits only zero codeword; *i.e.*, $x_i = +1$. For the other users' codewords, the information bits follow Bernoulli distribution with equal probability. For each decoder in an n-user synchronized communication system, there are in total 2^{n-1} signal groups

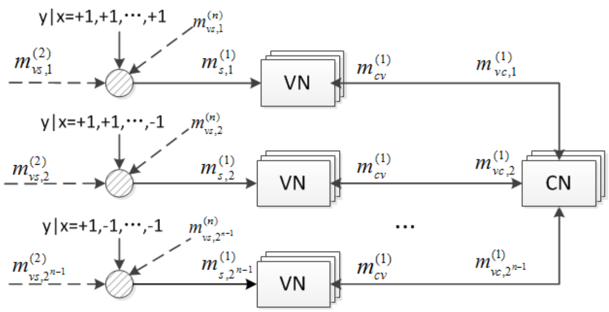

Fig. 2. The LLRs passing between SN and decoders

with equal probability, as shown in Fig. 2.

The PDF of the corresponding LLRs (m_{vc} and the side information m_s) can be approximated by the GM distribution with 2^{n-1} components. For the LLRs sent from CNs to VNs, we consider that all the group signals have the same m_{cv}, which are shown in Fig. 2. The parameters of the k-th component corresponding to m_{cv} received at the VN with degree d can be denoted by

$$\begin{aligned} \mu_{vc,k} &= \mu_{s,k} + (d-1)\,\mu_{cv}, \\ \sigma^2_{vc,k} &= \sigma^2_{s,k} + (d-1)\,\sigma^2_{cv}. \end{aligned} \tag{7}$$

Similar to that of [9, 8], we assume that the PDF of the m_{cv} follows a Gaussian distribution with the mean being half of the covariance.

Considering the transmitted signal groups, the PDF of the side information m_s can be represented by

$$p\,(m_s) = \sum_{k=1}^{2^{n-1}} p\,(m_s|X_k)\,P\,(X_k), \tag{8}$$

where X_k represents the k-th group signal, $p\,(m_s|X_k)$ denotes the PDF component when the k-th group signal is transmitted, and $P(X_k)$ is the probability that the k-th group signal is transmitted. Under the Gaussian assumption for each PDF component, the mean $\mu_{s,k}$ and covariance $\sigma^2_{s,k}$ for the k-th component can be estimated by the Monte Carlo transformation (MCT) [16] according to the following non-linear function:

$$\begin{aligned} m^{(i)}_{s,k} &= \log\frac{p(y_k|x_i=+1)}{p(y_k|x_i=-1)} \\ &= \log\frac{\sum_l p(y_k|x_i=+1,\bar{X}^{(i)}_l)P(\bar{X}^{(i)}_l)}{\sum_l p(y_k|x_i=-1,\bar{X}^{(i)}_l)P(\bar{X}^{(i)}_l)}, \end{aligned} \tag{9}$$

where $\bar{X}^{(i)}_l$ is composed of all the components of X_l except the i-th user signal, and $m^{(i)}_{s,k}$ denotes the side information received at the i-th user decoder.

In the decoding process, m_{vs} sent from VNs to SNs are the sum of m_{cv} passing along the edges connected with the corresponding VNs. Assuming that m_{cv} follows a Gaussian distribution with the mean being half of the covariance, the corresponding PDF $p(m_{vs})$ can be represented by $\mathcal{N}\left(\sigma^2_{vs}/2, \sigma^2_{vs}\right)$, where $\sigma^2_{vs} = \sum_{j=2}^{dv} j\tilde{\lambda}_j \cdot \sigma^2_{cv}$ and $\tilde{\lambda}_j$ is the proportion of the VNs for degree j, denoted by $\tilde{\lambda}_j = \frac{\lambda_j/j}{\sum_{j=2}^{dv}\lambda_j/j}$.

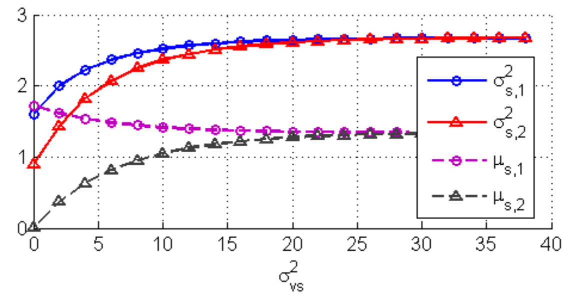

Fig. 3. The trace of the mean and covariance of each m_s PDF component in 2-user Gaussian MAC when SNR=-1.73 dB

In Fig. 3, we plot the curves of the PDF components $p\,(m_s|X_k)$ varying with the covariance σ^2_{cv} for the cases of two-user GMAC, where σ^2_{cv} indicates the progress of the decoding process. By observing the curves in Fig. 3, we see that the mean-to-covariance ratio of each component gradually approaches 0.5 as σ^2_{cv} increases. This suggests that the multi-user joint decoding becomes single-user decoding after a sufficient number of iterations. However, in the initial stage, the mean-to-covariance ratios of all the PDF components are dynamic and different from each other.

3.2. Mutual Information Computation

According to the estimated PDF, we can compute the mutual information between the transmitted bits and the LLRs, which is essential to the EXIT chart analysis. Instead of using the channel adapter in [10], we consider 2^{n-1} kinds of transmitted signal groups with equal probability for n-user cases. Since the symmetric condition [12, Def.1] still holds for joint decoding (*i.e.*, $p(m|x_i=+1,\bar{X}^{(i)}_k) = p(-m|x_i=-1,-\bar{X}^{(i)}_k)$), the mutual information can be computed by

$$I\,(x_i;m) = \frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}} I\left(x_i;m_k|x_i=+1,\bar{X}^{(i)}_k\right). \tag{10}$$

If the mean-to-variance ratio of each component is of the same value D, (10) can be simplified with the k-th component denoted by

$$\begin{aligned} &I\left(x_i;m_k|x_i=+1,\bar{X}^{(i)}_k\right) \\ =&1-\int_{-\infty}^{+\infty} p\left(m|x_i=+1,\bar{X}^{(i)}_k\right)\log\left(1+e^{-2mD}\right)dm. \end{aligned} \tag{11}$$

Notice that when $n=1$ and $D=0.5$, (11) goes to $J(\cdot)$ of [3] for the single-user decoding analysis.

However, as shown in Fig. 3, the mean-to-variance ratio of each component is not constant at the initial stage of the multi-user decoding. If the ratio of each PDF component was considered to be of the same value, there will be larger errors in computing the mutual information using (11), which will make the optimal LDPC code design being deviated. To reduce such errors, we propose to use the exact expression of the mutual information $I(x_i; m_{vc})$ in (10), where the k-th component is numerically computed according to

$$\bar{J}_k\left(\mu_{vc}, \sigma_{vc}^2\right) = 1 - \frac{1}{\sqrt{2\pi}\sigma_{vc}} \int_{-\infty}^{+\infty} e^{-\frac{\left(m-\mu_{vc,k}\right)^2}{2\sigma_{vc,k}^2}} \cdot \log\left(1+\frac{\sum_l \sigma_{vc,l}^{-1} e^{-(m-\mu_{vc,l})^2/2\sigma_{vc,l}^2}}{\sum_l \sigma_{vc,l}^{-1} e^{-(m+\mu_{vc,l})^2/2\sigma_{vc,l}^2}}\right) dm, \qquad (12)$$

in which $\bar{J}_k$ denotes $I(x_i; m_k | x_i = +1, \bar{X}_k^{(i)})$ and $\left(\mu_{vc}, \sigma_{vc}^2\right)$ can be updated according to (7). The mutual information curve I_{Evnd} in the EXIT chart can be obtained, denoted by

$$I_{Evnd}(\lambda, \sigma_{cv}) = I\left(x; m_{cv}\right) = \frac{1}{2^{n-1}} \sum_{j=2}^{d_v} \lambda_j \tilde{J}_j\left(\sigma_{cv}^2\right), \qquad (13)$$

where

$$\tilde{J}_j\left(\sigma_{cv}^2\right) = \sum_{k=1}^{2^{n-1}} \bar{J}_k\left(\mu_s + (j-1)\,\sigma_{cv}^2/2, \sigma_s^2 + (j-1)\,\sigma_{cv}^2\right)$$

and $\left(\mu_s, \sigma_s^2\right)$ can be estimated in (9) for a given σ_{cv}^2. The other curve I_{Acnd} in the EXIT chart is the same as that of [3], and we write it here with σ_{cv} as the input:

$$I_{Acnd}(\sigma_{cv}) = 1 - J(J^{-1}(1 - J(\sigma_{cv}))/\sqrt{d_c - 1}). \qquad (14)$$

3.3. LPDC Code Design by Linear Programming

For simplicity, we only consider a singleton distribution for the check node degree and optimize the variable node degree distribution in the framework of the LP used in [7, Ch. 4]. We assume that all users share the same degree distribution in the symmetric channel; here for simplicity we omit the superscript for user index. The LDPC code design can be formulated as the following optimization problem:

$$\begin{aligned} &\max \sum_{j=2}^{d_v} \frac{\lambda_j}{j} \\ &\text{s. t.} \sum_{j=2}^{d_v} \lambda_j = 1,\ \lambda_j \in [0,1], \\ &\quad \lambda_2\left(d_c - 1\right) \le \exp\left\{h^2\right\} - \varepsilon, \\ &\quad I_{Evnd}(\lambda, \sigma_{cv}) \ge I_{Acnd}\left(\sigma_{cv}\right) + \varepsilon_{ghs}, \sigma_{cv} \in \left[0, \sigma_{cv,mid}\right), or, \\ &\quad I_{Evnd}(\lambda, \sigma_{cv}) \ge I_{Acnd}\left(\sigma_{cv}\right) + \varepsilon, \sigma_{cv} \in \left[\sigma_{cv,mid}, \sigma_{cv,\max}\right], \end{aligned} \qquad (15)$$

where the second constraint is the stability condition [9] with $\varepsilon = 10^{-4}$ and in the third constraint we add a "good head start" gap [3] $\varepsilon_{ghs} = 0.01$ between the curves I_{Evnd} and I_{Acnd} at the initial stage (where the slope of I_{Acnd} in [3, Fig. 3] is less than 1), such that the code rates will be close to the capacity region bound with good BER performance.

Notice that for a given σ_{cv}, $I_{Evnd}(\lambda, \sigma_{cv})$ is a linear function of λ and $I_{Acnd}\left(\sigma_{cv}\right)$ can be directly computed by (14), such that (15) can be solved by an LP solver for a number of points of σ_{cv} within the intervals in (15).

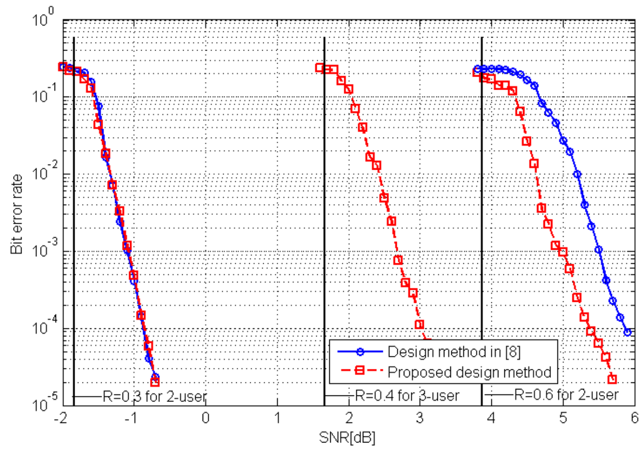

Fig. 4. Simulation results for the designed LDPC codes given in Table 1 for 2-user GMAC and 3-user GMAC

4. NUMERICAL RESULTS

We show the simulation results for the designed LDPC codes using the method we proposed. The corresponding parity check matrices are generated by random constructions using the tool in [17], where all the length-4 cycles are removed. The code block lengths are chosen as 20000, and the maximum number of decoding iterations is set to 200. The optimized degree distributions are shown in Table 1. For the 2-user GMAC, we compare the performance of the LDPC code designed by our proposed method with that by [10] for $R = 0.3$ and $R = 0.6$, respectively. As can be seen from Fig. 4, two curves are overlapping for $R = 0.3$, while for $R = 0.6$, using the method we proposed, there is an improvement of 0.5dB over that by [10] at BER $= 10^{-4}$. For reference, the performance of the designed codes for the 3-user GMAC with $R = 0.4$ is also shown in Fig. 4, where we can see that the designed code has a decoding threshold close to the Shannon limit.

Table 1. Optimized degree distribution for symmetric GMAC

	2-GMAC				3-GMAC	
Rate	0.3		0.6		0.4	
d_c	6		9		6	
$\lambda(x)$	λ_j	j	λ_j	j	λ_j	j
	0.2699	2	0.4629	2	0.5127	2
	0.2414	3	0.1173	3	0.0074	3
	0.0371	10	0.2773	52	0.2666	16
	0.2163	13	0.0618	55	0.2133	100
	0.2353	100	0.0807	100		
Shannon limit	-1.832 dB		3.857 dB		1.665 dB	

5. REFERENCES

[1] D. Culler, D. Estrin, and M. Srivastava, "Overview of sensor networks," *IEEE Computer*, vol. 37, no. 8, pp. 41 – 49, 2004.

[2] R G Gallager, "Loiv-density parity-check codes," *Information Theory Ire Transactions on*, vol. 8, 1962.

[3] S. Ten Brink, G. Kramer, and A. Ashikhmin, "Design of low-density parity-check codes for modulation and detection," *IEEE Transactions on Communications*, vol. 52, no. 4, pp. 670–678, 2004.

[4] Thomas J. Richardson, M. Amin Shokrollahi, and Rdiger L. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 619–637, 2001.

[5] Sae Young Chung, G. D. Forney, T. J. Richardson, and R. Urbanke, "On the design of low-density parity-check codes within 0.0045 db of the shannon limit," *IEEE Communications Letters*, vol. 5, no. 2, pp. 58–60, 2001.

[6] T. Richardson and R. Urbanke, "The renaissance of gallager's low-density parity-check codes," *Communications Magazine IEEE*, vol. 41, no. 8, pp. 126–131, 2003.

[7] Tom Richardson and Ruediger Urbanke, "Modern coding theory," 2008.

[8] Iqbal Shahid and Pradeepa Yahampath, "Distributed joint source-channel code design for gmac using irregular ldpc codes," *Eurasip Journal on Wireless Communications & Networking*, vol. 2014, no. 1, pp. 1–15, 2014.

[9] Aline Roumy and David Declercq, "Characterization and optimization of ldpc codes for the 2-user gaussian multiple access channel," *Eurasip Journal on Wireless Communications & Networking*, vol. 2007, no. 1, pp. 211–216, 2007.

[10] S. Sharifi, A. K. Tanc, and T. M. Duman, "Ldpc code design for the two-user gaussian multiple access channel," *IEEE Transactions on Wireless Communications*, vol. 15, no. 4, pp. 2833–2844, 2016.

[11] Abbas A. El Gamal and Young Han Kim, *Network Information Theory*, Cambridge University Press, 2011.

[12] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 599–618, 2001.

[13] S. Ten Brink, "Designing iterative decoding schemes with the extrinsic information chart," *AEU - International Journal of Electronics and Communications*, vol. 54, no. 6, pp. 389–398, 2000.

[14] Sae Young Chung, Thomas J. Richardson, and Rdiger L. Urbanke, "Analysis of sum-product decoding of low-density parity-check codes using a gaussian approximation," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 657–670, 2001.

[15] S. Ten Brink, "Convergence behavior of iteratively decoded parallel concatenated codes," *IEEE Transactions on Communications*, vol. 49, no. 10, pp. 1727–1737, 2001.

[16] Christian P. Robert and George Casella, *Monte Carlo Statistical Methods*, Springer,, 1999.

[17] "Itpp," `http://itpp.sourceforge.net/4.3.1/index.html`.