On Approximating Complex Quadratic Optimization Problems via Semidefinite Programming Relaxations*

Anthony Man–Cho So[1], Jiawei Zhang[2], and Yinyu Ye[3]

[1] Department of Computer Science, Stanford University, Stanford, CA 94305, USA.
manchoso@cs.stanford.edu

[2] Department of Information, Operations, and Management Sciences, Stern School of Business, New York University, New York, NY 10012, USA.
jzhang@stern.nyu.edu

[3] Department of Management Science and Engineering and, by courtesy, Electrical Engineering, Stanford University, Stanford, CA 94305, USA.
yinyu-ye@stanford.edu

Abstract. In this paper we study semidefinite programming (SDP) models for a class of discrete and continuous quadratic optimization problems in the complex Hermitian form. These problems capture a class of well–known combinatorial optimization problems, as well as problems in control theory. For instance, they include Max–3–Cut where the Laplacian matrix is positive semidefinite (in particular, some of the edge weights can be negative). We present a generic algorithm and a unified analysis of the SDP relaxations which allow us to obtain good approximation guarantees for our models. Specifically, we give an $(k\sin(\frac{\pi}{k}))^2/(4\pi)$–approximation algorithm for the discrete problem where the decision variables are k–ary and the objective matrix is positive semidefinite. To the best of our knowledge, this is the first known approximation result for this family of problems. For the continuous problem where the objective matrix is positive semidefinite, we obtain the well–known $\pi/4$ result due to [2], and independently, [12]. However, our techniques simplify their analyses and provide a unified framework for treating these problems. In addition, we show for the first time that the integrality gap of the SDP relaxation is precisely $\pi/4$. We also show that the unified analysis can be used to obtain an $O(1/\log n)$–approximation algorithm for the continuous problem in which the objective matrix is not positive semidefinite. ...

1 Introduction

Following the seminal work of Goemans and Williamson [6], there has been an outgrowth in the use of semidefinite programming (SDP) for designing approximation algorithms. Recall that an α–approximation algorithm for a problem $\mathscr{P}$ is a polynomial–time algorithm such that for every instance I of $\mathscr{P}$, it delivers

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a solution that is within a factor of α of the optimum value [5]. It is well–known that SDPs can be solved in polynomial time (up to any prescribed accuracy) via interior–point algorithms (see, e.g., [10]), and they have been used very successively in the design of approximation algorithms for, e.g., graph partitioning, graph coloring, and quadratic optimization problems [4, 11].

In this paper, we consider a class of discrete and continuous quadratic optimization problems in the complex Hermitian form. Specifically, we consider the following problems:

$$\begin{array}{ll} \text{maximize} & z^H Q z \\ \text{subject to} & z_j \in \{1, \omega, \ldots, \omega^{k-1}\} \qquad j = 1, 2, \cdots, n \end{array} \tag{1}$$

and

$$\begin{array}{ll} \text{maximize} & z^H Q z \\ \text{subject to} & |z_j| = 1 \qquad j = 1, 2, \cdots, n \\ & z \in C^n \end{array} \tag{2}$$

where $Q \in C^{n \times n}$ is a Hermitian matrix, ω is the principal k–th root of unity, and z^H denotes the conjugate transpose of the complex vector $z \in C^n$. The difference between (1) and (2) lies in the values that the decision variables are allowed to take. In problem (1), we have discrete decision variables, and such variables can be conveniently modelled as roots of unity. On the other hand, in problem (2), the decision variables are constrained to lie on the unit circle, which is a continuous domain. Such problems arise from many applications. For instance, the MAX–3–CUT problem where the Laplacian matrix is positive semidefinite can be formulated as an instance of (1). On the other hand, (2) arises from the study of robust optimization as well as control theory [9, 2].

It is known that both of these problems are NP–hard, and thus we will settle for approximation algorithms. Previously, various researchers have considered SDP relaxations for (1) and (2). However, approximation guarantee is known only for the continuous problem [2, 12], and to the best of our knowledge, no such guarantees are known for the discrete problem.

Our main contribution is to present a generic algorithm and a unified treatment of the two seemingly very different problems (1) and (2) using their natural SDP relaxations, and to give the first known approximation result for the discrete problem. Specifically, we are able to achieve an $(k \sin(\frac{\pi}{k}))^2/(4\pi)$–approximation ratio for the discrete problem[4]. As a corollary, we obtain an 0.537–approximation algorithm for the MAX–3–CUT problem where the Laplacian matrix is positive semidefinite. This should be contrasted with the 0.836–approximation algorithm of Goemans and Williamson [7] for MAX–3–CUT with *non–negative* edge weights. For this particular case, our result might be seen as a generalization of Nesterov's result [8] that gives an $2/\pi$–approximation for the MAX–CUT problem where the Laplacian matrix is positive semidefinite.

[4] Recently, Zhang and Huang [13] have informed us that by extending their analysis in [12], they are able to obtain the same approximation ratio for the discrete problem.

For the continuous problem, our analysis also achieves the $\pi/4$ guarantee of [2, 12]. However, our analysis is simpler than that of [2, 12], and it follows the same framework as that of the discrete problem. Moreover, we give a tight example showing that the integrality gap of the SDP relaxation is precisely $\pi/4$. In addition, we show that the unified analysis can be used to obtain an $O(1/\log n)$–approximation algorithm for the continuous problem in the case where the objective matrix is not positive semidefinite. This result also provides an alternative analysis of the algorithm by Charikar and Wirth [3] for the (real) quadratic optimization problem.

One apparent difficulty in analyzing SDP relaxation–based algorithms for problems (1) and (2) is that the usual Goemans–Williamson analysis [6, 7] (and its variants thereof) only provides a term–by–term estimate of the objective function and does not provide a global estimate. Although global techniques for analyzing (real) SDP relaxations exist [8], it is not clear how they can be applied to our problems. Our analysis is mainly inspired by a recent result of Alon and Naor [1], who proposed three different methods for analyzing (real) SDP relaxations in a global manner using results from functional analysis. One of these methods uses averaging with Gaussian measure and the simple fact that $\sum_{i,j} q_{ij}(v_i \cdot v_j) \geq 0$ if the matrix $Q = (q_{ij})$ is positive semidefinite and $v_i \cdot v_j$ is the inner product of two vectors v_i and v_j in some Hilbert space. Our results for (1) and (2) in the case where Q is positive semidefinite are motivated by this method. Although the assumption that Q is positive semidefinite is essential to make the analyses go through, we manage to analyze our algorithm in a unified way for the case where Q is not positive semidefinite as well.

2 Complex Quadratic Optimization

Let $Q \in C^{n\times n}$ be a Hermitian matrix. Given an integer $n \geq 1$, consider the following discrete quadratic optimization problem:

$$\begin{array}{ll} \text{maximize} & z^H Q z \\ \text{subject to} & z_j \in \{1, \omega, \ldots, \omega^{k-1}\} \qquad j = 1, 2, \cdots, n \end{array} \tag{3}$$

where ω is the principal k–th root of unity. We note that as k goes to infinity, the discrete problem (3) becomes a continuous optimization problem:

$$\begin{array}{ll} \text{maximize} & z^H Q z \\ \text{subject to} & |z_j| = 1 \qquad j = 1, 2, \cdots, n \\ & z \in C^n \end{array} \tag{4}$$

Although problems (3) and (4) are quite different in nature, the following *complex* semidefinite program provides a relaxation for both of them:

$$\begin{array}{ll} \text{maximize} & Q \bullet Z \\ \text{subject to} & Z_{jj} = 1 \qquad j = 1, 2, \cdots, n \\ & Z \succeq 0 \end{array} \tag{5}$$

We use w_{SDP} to denote the optimal value of the SDP relaxation (5).

Our goal is to get a near optimal solution for problem (3) and (4). Below we present a generic algorithm that can be used to solve both (3) and (4). Our algorithm is quite simple, and it is similar in spirit to the algorithm of Goemans and Williamson [6, 7].

Algorithm

STEP 1. Solve the SDP relaxation (5) and obtain an optimal solution Z^*. Since Z^* is positive semidefinite, we can obtain a Cholesky decomposition $Z^* = VV^H$, where $V = (v_1, v_2, \cdots, v_n)$.

STEP 2. Generate two independent normally distributed random vector $x \in R^n$ and $y \in R^n$ with mean 0 and covariance matrix $\frac{1}{2}I$. Let $r = x + yi$.

STEP 3. For $j = 1, 2, \cdots, n$, let $\hat{z}_j = f(v_j \cdot r)$, where the function $f(\cdot)$ depends on the structure of the problem and will be fixed later. Let $\hat{z} = (\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_n)$ be the resulting solution.

In order to prove the performance guarantee of our algorithm, we are interested in analyzing the quantity:

$$\hat{z}^H Q \hat{z} = Q \bullet \hat{z}\hat{z}^H = \sum_{l,m} Q_{lm} \hat{z}_l \overline{\hat{z}_m} = \sum_{l,m} Q_{lm} f(v_l \cdot r) \overline{f(v_m \cdot r)}$$

Since our algorithm is randomized, we compute the expected objective value given solution $\hat{z}$. By linearity of expectation, we have:

$$\mathrm{E}[\hat{z}^H Q \hat{z}] = \sum_{l,m} Q_{lm} \mathrm{E}[f(v_l \cdot r) \overline{f(v_m \cdot r)}]$$

To that end, it would be sufficient to compute the quantity $\mathrm{E}[f(v_l \cdot r) \overline{f(v_m \cdot r)}]$ for any l, m, and this will be the main concern of our analysis. The analysis, of course, depends on the choice of the function $f(\cdot)$. However, the following Lemma will be useful and it is independent of the function $f(\cdot)$. Recall that for two vectors $b, c \in C^n$, we have $b \cdot c = \sum_{j=1}^n b_j \overline{c_j}$.

Lemma 1 *For any pair of vectors $b, c \in C^n$, $\mathrm{E}[(b \cdot r)\overline{(c \cdot r)}] = b \cdot c$, where $r = x + yi$ and $x \in R^n$ and $y \in R^n$ are two independent normally distributed random vector with mean 0 and co-variance matrix $\frac{1}{2}I$.*

Proof. This follows from a straightforward computation:

$$\mathrm{E}[(b \cdot r)\overline{(c \cdot r)}] = \mathrm{E}\left[\left(\sum_{j=1}^n b_j \overline{r}_j\right)\left(\sum_{k=1}^n \overline{c}_k r_k\right)\right] = \sum_{j,k=1}^n b_j \overline{c}_k \mathrm{E}[\overline{r}_j r_k] = \sum_{j=1}^n b_j \overline{c}_j$$

where the last equality follows from the fact that the entries of x and y are independent normally distributed with mean 0 and variance $1/2$.

3 Discrete Problems where Q is Positive Semidefinite

In this section, we assume that Q is Hermitian and positive semidefinite. We consider the discrete complex quadratic optimization problem (3).

In this case, in the generic algorithm presented in Section 2, we specify the function $f(\cdot)$ as follows:

$$f(z) = \begin{cases} 1 & \text{if } \arg(z) \in [-\pi/k, \pi/k) \\ \omega & \text{if } \arg(z) \in [\pi/k, 3\pi/k) \\ \vdots & \vdots \\ \omega^{k-1} & \text{if } \arg(z) \in [(2k-3)\pi/k, (2k-1)\pi/k) \end{cases} \tag{6}$$

This way, we guarantee that $\hat{z}_j \in \{1, \omega, \dots, \omega^{k-1}\}$ for $j = 1, 2, \cdots, n$, i.e. $\hat{z}$ is a feasible solution of problem (3).

Then, we can establish the following lemma:

Lemma 2

$$\mathrm{E}[(b \cdot r)\overline{f(c \cdot r)}] = \frac{k \sin(\pi/k)}{2\sqrt{\pi}}(b \cdot c)$$

Proof. By rotation invariance, we may assume without loss of generality that $b = (b_1, b_2, 0, \dots, 0)$ and $c = (1, 0, \dots, 0)$. Then, we have:

$$\begin{aligned} \mathrm{E}[(b_1\overline{r_1} + b_2\overline{r_2})\overline{f(\overline{r_1})}] &= b_1 \mathrm{E}[\overline{r_1}\overline{f(\overline{r_1})}] \\ &= \frac{b_1}{\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} (x - iy)\overline{f(x - iy)} \exp\{-(x^2 + y^2)\}\, dxdy \\ &= \frac{b_1}{\pi}\int_0^{\infty}\int_0^{2\pi} \rho^2 e^{-i\theta}\overline{f(\rho e^{-i\theta})} e^{-\rho^2}\, d\theta d\rho \end{aligned}$$

Now, observe that, for $j = 1, \dots, k$, we have:

$$\int_{(2j-3)\pi/k}^{(2j-1)\pi/k} \overline{f(\rho e^{-i\theta})} e^{-i\theta}\, d\theta = \omega^{j-1}\int_{(2j-3)\pi/k}^{(2j-1)\pi/k} e^{-i\theta}\, d\theta = 2\sin(\pi/k)$$

In particular, the above quantity is independent of j. Moreover, since we have:

$$\int_0^{\infty} \rho^2 e^{-\rho^2}\, d\rho = \frac{\sqrt{\pi}}{4}$$

it follows that:

$$\mathrm{E}[(b_1\overline{r_1} + b_2\overline{r_2})\overline{f(\overline{r_1})}] = \frac{k\sin(\pi/k)}{2\sqrt{\pi}} b_1 = \frac{k\sin(\pi/k)}{2\sqrt{\pi}}(b \cdot c)$$

We are now ready to prove the main result of this section.

Theorem 1 *When Q is positive semidefinite and Hermitian, there exists an $\frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}$–approximation algorithm for (3).*

Proof. It follows from Lemma 1 and Lemma 2 that:

$$\begin{aligned} &\mathrm{E}\left[\left\{(b\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(b\cdot r)\right\}\left\{\overline{(c\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(c\cdot r)}\right\}\right] \\ &=-(b\cdot c)+\frac{4\pi}{(k\sin(\frac{\pi}{k}))^2}\mathrm{E}[f(b\cdot r)\overline{f(c\cdot r)}] \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \mathrm{E}[\hat{z}^H Q\hat{z}] &= \frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} q_{lm}(v_l\cdot v_m) \\ &+\frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} q_{lm}\mathrm{E}\left[\left\{(v_l\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(v_l\cdot r)\right\}\left\{\overline{(v_m\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(v_m\cdot r)}\right\}\right] \\ &\geq \frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}\sum_{l=1}^{n}\sum_{m=1}^{n} q_{lm}(v_l\cdot v_m)=\frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}w_{SDP} \end{aligned} \tag{7}$$

The last inequality is true since

$$\mathrm{E}\left[\left\{(v_l\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(v_l\cdot r)\right\}\left\{\overline{(v_m\cdot r)-\frac{2\sqrt{\pi}}{k\sin(\frac{\pi}{k})}f(v_m\cdot r)}\right\}\right]$$

is an inner product of two vectors in a Hilbert space, which together with the fact that Q is positive semidefinite shows that:

$$\sum_{l=1}^{n}\sum_{m=1}^{n} q_{lm}\mathrm{E}\left[\left\{(v_l\cdot r)-\frac{2}{\sqrt{\pi}}f(v_l\cdot r)\right\}\left\{\overline{(v_m\cdot r)-\frac{2}{\sqrt{\pi}}f(v_m\cdot r)}\right\}\right]\geq 0$$

It follows that our algorithm gives an $\frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}$–approximation.

Now, let us consider problem (4) when Q is positive semidefinite. This problem can be seen as a special case of (3) by letting $k\to\infty$. In this case, the function $f(\cdot)$ defined in (6) is as follows:

$$f(t)=\begin{cases}\frac{t}{|t|} & \text{if } |t|>0\\ 0 & \text{if } t=0\end{cases} \tag{8}$$

Note that as $k\to\infty$, we have $\frac{(k\sin(\frac{\pi}{k}))^2}{4\pi}\to\pi/4$. This establishes the following result, which has been proved independently by Ben–Tal, Nemirovski and Roos [2], and Zhang and Huang [12]. However, our proof is quite a bit simpler.

Corollary 1 *When Q is positive semidefinite and Hermitian, there exists an $\frac{\pi}{4}$–approximation algorithm for (4).*

Next, we show that our analysis is in fact tight for the continuous complex quadratic optimization problem (4). We give a family of examples which shows that the natural SDP relaxation for the above problem has an integrality gap arbitrarily close to $\pi/4$. We begin with a technical lemma.

Lemma 3 *Let u, v be two random, independent vectors on the unit sphere of C^p. Then, we have:*

$$\mathrm{E}[|u \cdot v|^2] = \frac{1}{p}; \qquad \mathrm{E}[|u \cdot v|] = \left(\frac{\sqrt{\pi}}{2} + o(1)\right) \frac{1}{\sqrt{p}}$$

Proof. Omitted in this extended abstract.

To construct the tight example, let p and $n \gg p$ be fixed. Let $v_1, \ldots, v_n$ be independent random vectors chosen uniformly according to the normalized Haar measure on the unit sphere of C^p. We define $A = (a_{ij})$ by $a_{ij} = \frac{1}{n^2}\overline{(v_i \cdot v_j)}$. By construction, the matrix A is positive semidefinite and Hermitian. Moreover, we have:

$$\sum_{i,j} a_{ij}(v_i \cdot v_j) = \frac{1}{n^2} \sum_{i,j} |v_i \cdot v_j|^2$$

By taking $n \to \infty$, the right–hand side converges to the average of the square of the inner product between two random vectors on the unit sphere of C^n. By Lemma 3, this value is $1/p$, and hence the optimal value of the SDP relaxation is at least $1/p$.

Now, let $z_i \in C$ be such that $|z_i| = 1$. Then, we have:

$$z^H A z = \sum_{i,j} a_{ij} \overline{z_i} z_j = \left| \frac{1}{n} \sum_{i=1}^{n} z_i v_i \right|^2$$

Hence, the value of the original SDP is the square of the maximum possible modulus of a vector $\frac{1}{n}\sum_{i=1}^{n} z_i v_i$. If we somehow know that the direction of this optimal vector is given by the unit vector c, then we must set $z_i = \overline{f(v_i \cdot c)}$ in order to maximize the modulus. It then follows that:

$$\frac{1}{n} \sum_{i=1}^{n} z_i v_i \cdot c = \left| \frac{1}{n} \sum_{i=1}^{n} z_i v_i \right|$$

by the Cauchy–Schwarz inequality. Moreover, this quantity converges to the average value of $|v \cdot c|$ as $n \to \infty$. By letting n arbitrarily large and choosing an appropriate ϵ–net of directions on the sphere, we conclude that with high probability, the value of the original SDP is at most $[(\sqrt{\pi}/2 + o(1))/\sqrt{p}]^2 = (\pi/4 + o(1))/p$, which yields the desired result.

4 Continuous Problems where Q is not Positive Semidefinite

In this section, we deal with problem (4) where the matrix Q is not positive semidefinite. However, for convenience, we assume that $w_{SDP} > 0$ such that the standard definition of approximation algorithm makes sense for our problem. It is clear that $w_{SDP} > 0$ as long as all the diagonal entries of Q are zeros.

Again, we use our generic algorithm presented in Section 2. In this case, we specify the function $f(\cdot)$ as follows:

$$f(t) = \begin{cases} \frac{t}{T} & \text{if } |t| \le T \\ \frac{t}{|t|} & \text{if } |t| > T \end{cases} \tag{9}$$

where T is a parameter which will be fixed later. If we let $z_j = f(v_j \cdot r)$, the solution $z = (z_1, \ldots, z_n)$ obtained by this rounding may not be feasible, as the point may not have unit modulus. However, we know that $|z_j| \le 1$. Thus, we can further round the solution as follows:

$$\hat{z} = \begin{cases} z/|z| & \text{with probability } (1 + |z|)/2 \\ -\bar{z}/|z| & \text{with probability } (1 - |z|)/2 \end{cases}$$

We then have the following:

Fact. For $i \neq j$, $\mathrm{E}[\hat{z}_i \overline{\hat{z}_j}] = \mathrm{E}[z_i \overline{z_j}]$.

This shows that the expected value of the solution on the circle equals that of the "fractional" solution obtained by applying $f(\cdot)$ to the SDP solution. Therefore, we could still restrict ourselves to the rounding function $f(\cdot)$.

Define

$$g(T) = \frac{1}{T} - \frac{1}{T} e^{-T^2} + \sqrt{\pi}(1 - \Phi(\sqrt{2}T))$$

where $\Phi(\cdot)$ is the probability distribution function of $N(0, 1)$.

Lemma 4 $\mathrm{E}[(b \cdot r)\overline{f(c \cdot r)}] = g(T)(b \cdot c)$

Proof. Again, without loss of generality, we assume that $c = (1, 0, \ldots, 0)$ and $b = (b_1, b_2, 0, \ldots, 0)$. Then, we have:

$$\begin{aligned}
&\mathrm{E}[(b \cdot r)\overline{f(c \cdot r)}] \\
&= \mathrm{E}\left[(b_1\bar{r}_1 + b_2\bar{r}_2)\frac{r_1}{T}\Big| |r_1| \le T\right] + \mathrm{E}\left[(b_1\bar{r}_1 + b_2\bar{r}_2)\frac{r_1}{|r_1|}\Big| |r_1| > T\right] \\
&= \frac{1}{T}\mathrm{E}\left[b_1|\bar{r}_1|^2 \Big| |r_1| \le T\right] + \mathrm{E}\left[b_1|\bar{r}_1| \Big| |r_1| > T\right] \\
&= \frac{b_1}{T} \cdot \frac{1}{\pi} \int_{x^2+y^2 \le T^2} (x^2 + y^2)\ \exp(-(x^2 + y^2))dxdy \\
&\quad + \frac{b_1}{\pi} \int_{x^2+y^2 > T^2} \sqrt{x^2 + y^2}\ \exp(-(x^2 + y^2))dxdy \\
&= \frac{b_1}{\pi T} \int_0^{2\pi} \int_0^T \rho^3 \exp(-\rho^2)\ d\rho d\theta + \frac{b_1}{\pi} \int_0^{2\pi} \int_T^{\infty} \rho^2 \exp(-\rho^2)\ d\rho d\theta \\
&= g(T) b_1
\end{aligned}$$

where the last equality follows from the facts:

$$\int_0^T \rho^3 \exp(-\rho^2)d\rho = \frac{1}{2}\left(1-(T^2+1)\exp(-T^2)\right)$$

and

$$\int_T^\infty \rho^2 \exp(-\rho^2)d\rho = \frac{1}{2}\left(T\exp(-T^2)+\sqrt{\pi}(1-\Phi(\sqrt{2}T))\right)$$

Lemma 5 $\mathrm{E}[f(c\cdot r)\overline{f(c\cdot r)}] = \frac{1}{T^2} - \frac{1}{T^2}\exp(-T^2)$

Proof. The proof is similar to that of Lemma 2. We again assume that $c = (1,0,\dots,0)$.

$$\begin{aligned}
&\mathrm{E}[f(c\cdot r)\overline{f(c\cdot r)}]\\
&= \mathrm{E}\left[\frac{\bar{r}_1}{T}\frac{r_1}{T}\Big||r_1|\le T\right] + \mathrm{E}\left[\frac{\bar{r}_1}{|r_1|}\frac{r_1}{|r_1|}\Big||r_1|>T\right]\\
&= \frac{1}{T^2}\cdot\frac{1}{\pi}\int_{x^2+y^2\le T^2}(x^2+y^2)\exp(-(x^2+y^2))\ dxdy\\
&\quad+\frac{1}{\pi}\int_{x^2+y^2>T^2}\exp(-(x^2+y^2))\ dxdy\\
&= \frac{1}{T^2}\cdot\frac{1}{\pi}\int_0^{2\pi}\int_0^T \rho^3\exp(-\rho^2)\ d\rho d\theta + \frac{1}{\pi}\int_0^{2\pi}\int_T^\infty \rho\exp(-\rho^2)\ d\rho d\theta\\
&= \frac{1}{T^2}(1-(T^2+1)\exp(-T^2)) + \exp(-T^2)\\
&= \frac{1}{T^2} - \frac{1}{T^2}\exp(-T^2)
\end{aligned}$$

Theorem 2 *If $T = 3\sqrt{\ln(n)}$, then we have* $\mathrm{E}[\hat{z}^H Q\hat{z}] \ge \frac{1}{10\ln(n)}\ w_{SDP}$.

Proof. It follows from Lemma 1 and Lemma 4 that:

$$\mathrm{E}[\{(b\cdot r)-Tf(b\cdot r)\}\overline{\{(c\cdot r)-Tf(c\cdot r)\}}] = (1-2Tg(T))(b\cdot c)+T^2\mathrm{E}[f(b\cdot r)\overline{f(c\cdot r)}]$$

Then, we have:

$$\begin{aligned}
\mathrm{E}[\hat{z}^H Q\hat{z}] &= \sum_{k=1}^n\sum_{m=1}^n \frac{2Tg(T)-1}{T^2} q_{km}(v_k\cdot v_m)\\
&+\frac{1}{T^2}\sum_{k=1}^n\sum_{m=1}^n q_{km}\mathrm{E}[\{(v_k\cdot r)-Tf(v_k\cdot r)\}\overline{\{(v_m\cdot r)-Tf(v_m\cdot r)\}}]
\end{aligned}$$

Again, the quantity $\mathrm{E}[\{(b\cdot r)-Tf(b\cdot r)\}\overline{\{(c\cdot r)-Tf(c\cdot r)\}}]$ can be seen as an inner product of two vectors in a Hilbert space. Moreover, by letting $b = c$

and using Lemma 5, we know that the norm of an Euclidean unit vector in this Hilbert space is:

$$2 - 2Tg(T) - \exp(-T^2) = \exp(-T^2) - 2T\sqrt{\pi}(1 - \Phi(\sqrt{2}T))$$

It follows that:

$$\begin{aligned}
&\frac{1}{T^2}\sum_{k=1}^{n}\sum_{m=1}^{n} q_{km}\mathrm{E}[\{(v_k\cdot r) - Tf(v_k\cdot r)\}\cdot\{\overline{(v_m\cdot r) - Tf(v_m\cdot r)}\}] \\
&\geq -\frac{\exp(-T^2) - 2T\sqrt{\pi}(1-\Phi(\sqrt{2}T))}{T^2}\sum_{k=1}^{n}\sum_{m=1}^{n}|q_{km}|
\end{aligned}$$

On the other hand, one can show that $w_{SDP} \geq \frac{1}{6n^3}\sum_{k,m}|q_{km}| > 0$. Thus, we have:

$$\begin{aligned}
&\frac{1}{T^2}\sum_{k=1}^{n}\sum_{m=1}^{n} q_{km}\mathrm{E}[\{(v_k\cdot r) - Tf(v_k\cdot r)\}\cdot\{\overline{(v_m\cdot r) - Tf(v_m\cdot r)}\}] \\
&\geq -\frac{\exp(-T^2) - 2T\sqrt{\pi}(1-\Phi(\sqrt{2}T))}{T^2}\,6n^3\,w_{SDP}
\end{aligned}$$

from which it follows that:

$$\begin{aligned}
\mathrm{E}[\hat{z}^H Q\hat{z}] &\geq \left(\frac{2Tg(T)-1}{T^2} - \frac{\exp(-T^2) - 2T\sqrt{\pi}(1-\Phi(\sqrt{2}T))}{T^2}6n^3\right) w_{SDP} \\
&\geq \frac{1-(2+6n^3)\exp(-T^2)}{T^2}\; w_{SDP}
\end{aligned}$$

By letting $T = 3\sqrt{\ln n}$, we have $\mathrm{E}[\hat{z}^H Q\hat{z}] \geq \frac{1}{10\ln n}w_{SDP}$ when $n \geq 3$ as desired.

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