GEOLOCATION OF UNKNOWN EMITTERS USING TDOA OF PATH RAYS THROUGH THE IONOSPHERE BY MULTIPLE COORDINATED DISTANT RECEIVERS

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ABSTRACT

We consider the problem of unknown emitter geolocation using the time difference of arrival (TDOA) of the path rays through the ionosphere by multiple coordinated distant receivers. We formulate the geolocation in the sense of maximum likelihood with the exact ray expressions for the quasi-parabolic (QP) ionosphere, which is a highly nonlinear and non-convex optimization problem. By carefully studying the characteristic of the group path ray, we propose an efficient procedure to approach the optimal solution of the geolocation. Simulation results show that the geolocation error approaches the associated Cramer-Rao bound when the knowledge of the ionosphere is available. We also performed Monte Carlo runs to evaluate the performance of the geolocation when the knowledge of the ionosphere is not exactly known, *e.g.*, the QP model parameters are perturbed. Simulation results show that the geolocation performance under the perturbation within a given certain range is acceptable.

***Index Terms*—** Geolocation, QP model, TDOA, Nonlinear optimization, Newton method

1. INTRODUCTION

High-frequency geolocation is very useful in a number of civilian and military fields, such as navigation, aviation, maritime search and rescue or support, radio spectrum monitoring and management. However, the geolocation is affected by a number of factors, where the first and the most important factor is that the model of the electron density distribution and its associated parameters of the ionosphere [1][2][3] are not perfectly known. These lead to difficulties to perform the geolocation [10] with high accuracy in practice.

Recently, localization with time differences of arrivals (TDOAs) by employing a synchronized sensor network or multiple coordinated receivers has been widely studied based on the line-of-sight propagation model in the atmosphere and can be performed efficiently [7][8][9].

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However, the path ray of the electromagnetic wave in the ionosphere is a curved line instead of a straight one, which follows the Fermat principle and can be exactly calculated under the quasi-parabolic (QP) model [4][5]. In this case, the TDOA localization methods based on the line-of-sight model will no longer work.

In this paper, we approach the problem of unknown emitter geolocation using TDOA of the rays in the ionosphere by multiple coordinated distant receivers and formulate the problem in the sense of maximum likelihood with the exact ray expressions for the the quasi-parabolic (QP) ionosphere, which is a highly nonlinear and non-convex optimization problem. In addition, we numerically study the effects of QP model perturbations on the geolocation performance.

2. PROBLEM FORMULATION

2.1. Path Ray Model in the Ionosphere

The QP ionosphere model is defined by the equation of a parabola in electron-density distribution versus height. The QP model is given by (see Eqn. (2) in [4])

$$N_e = \begin{cases} N_m \left[1 - \left(\frac{r-r_m}{y_m}\right)^2 \left(\frac{r_b}{r}\right)^2\right], & r_b < r < r_m \left(\frac{r_b}{r_b-y_m}\right) \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

where N_e denotes the electron density with the maximum value N_m, r is the radial distance from earth center (height + earth radius), r_m is the value of r where N_e reaches N_m (h_m + earth radius, h_m is the hight with N_m), r_b is the value of r at the layer base, which is equal to $r_m - y_m$, and y_m is the layer semithickness. By neglecting the effects of the geomagnetic field, the situation that a ray passes through the ionosphere is shown in Fig. 1, where D is the distance traversed and measured along the earth's surface, P' is the group path, β_0 is the ray flying angle, r_0 is the earth radius, $\boldsymbol{x} = [x, y, z]^T \in \mathcal{R}^3$ is the location of the unknown emitter, and $\boldsymbol{S}_i \in \mathcal{R}^3$ is the location of the i-th distant receiver at earth surface. With f denoting the operating frequency and f_c denoting the critical frequency of the ionosphere, the surface distance and the

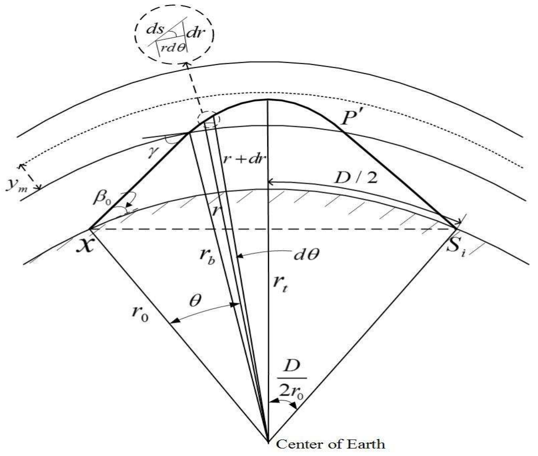

Fig. 1. Ray path geometry

group path can be exactly derived as shown by [4][5]:

$$
\begin{aligned}
D =& 2r_0\Bigg\{(\gamma-\beta_0) - \frac{r_0\cos\beta_0}{2\sqrt{C}}\times \\
& \ln\frac{B^2-4AC}{4C\left(\sin\gamma+\frac{1}{r_b}\sqrt{C}+\frac{1}{2\sqrt{C}}B\right)^2}\Bigg\}, \\
P' =& 2\Bigg\{r_b\sin\gamma - r_0\sin\beta_0 + \frac{1}{A}\Big[-r_b\sin\gamma- \\
& \frac{B}{4\sqrt{A}}\ln\frac{B^2-4AC}{\left(2Ar_b+B+2r_b\sqrt{A}\sin\gamma\right)^2}\Big]\Bigg\}, \qquad (2)
\end{aligned}
$$

where

$$
\begin{aligned}
& F = f/f_c,\ \ A = 1-\frac{1}{F^2}+\left(\frac{r_b}{Fy_m}\right)^2, \\
& B = -\frac{2r_m r_b^2}{F^2 y_m^2},\ \ C = \left(\frac{r_b r_m}{Fy_m}\right)^2 - r_0^2\cos^2\beta_0, \\
& \cos\gamma = \frac{r_0}{r_b}\cos\beta_0.
\end{aligned}
$$

2.2. Geolocation of an Unknown Emitter Using TDOA Measurements

Assuming that the locations of multiple distant receivers with synchronization are known and the knowledge of the parameters of the QP model are available, the geolocation of an unknown emitter, shown in Fig. 1, using TDOA measurements can be straightforwardly formulated as the following nonlinear least square problem under the surface distance constraints:

$$
\begin{aligned}
& \min_{\boldsymbol{x},\beta_i,i=1,\cdots,M} \sum_{i=1}^{M-1}\sum_{j,i<j}^{M}(\tau_i-\tau_j-\tau_{ij})^2 \\
& \text{subject to } \|\boldsymbol{S}_i-\boldsymbol{x}\| = 2r_0\sin\left(\frac{D_i}{2r_0}\right),\ i=1,2,...,M, \\
& \qquad \|\boldsymbol{x}\| = r_0. \qquad (3)
\end{aligned}
$$

where M is the number of receivers, P_j', D_j, and $\tau_j \triangleq P_j'/c$ (c is the light speed) are the group path, the surface distance, and the signal propagation delay from the unknown source to the j-th receivers, respectively, $\tau_{ij}=\tau_i-\tau_j$ is the TDOA between the i-th and the j-th receiver, which can only be measured in practice.

Considering the facts that a ray in the ionosphere follows Fermat principle and there are correlations between TDOA measurement noises, the maximum likelihood estimation of the unknown emitter location $\boldsymbol{x}$ can be written as the following optimization problem:

$$
\begin{aligned}
& \min_{\boldsymbol{x},\boldsymbol{\beta}}\ (\boldsymbol{GP}-\boldsymbol{\tau})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{GP}-\boldsymbol{\tau}) + \delta\sum_{i=1}^{M}P_i', \\
& \text{subject to } \|\boldsymbol{S}_i-\boldsymbol{x}\| = 2r_0\sin\left(\frac{D_i}{2r_0}\right),\ i=1,2,...,M, \\
& \qquad \|\boldsymbol{x}\| = r_0. \qquad (4)
\end{aligned}
$$

where $\boldsymbol{\Sigma} = (c\sigma_n)^2(\mathbf{1}_{N\times N}+\boldsymbol{I}_N)$ ($\mathbf{1}_{N\times N}$ is the matrix with each entry of 1, $\boldsymbol{I}_N$ is the identity matrix, and σ_n^2 is the variance of TDOA measurement noise) with $N=M(M-1)/2$, δ is small positive factor for penalization to all the rays, and

$$
\begin{aligned}
\boldsymbol{G} &= \begin{bmatrix}
1 & -1 & 0 & \cdots & \cdots & 0 \\
\vdots & & \ddots & \ddots & & \vdots \\
1 & 0 & \cdots & \cdots & 0 & -1 \\
0 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & -1
\end{bmatrix}, \\
\boldsymbol{P} &= [P_1', P_2', ..., P_M']^T, \\
\boldsymbol{\beta} &= [\beta_1, \beta_2, ..., \beta_M]^T, \\
\boldsymbol{\tau} &= [\tau_{12}, ..., \tau_{1M}, \tau_{23}, ..., \tau_{2M}, ..., \tau_{(M-1),M}]^T \times c. \qquad (5)
\end{aligned}
$$

Notice that the optimization problem (4) is highly nonlinear and non-convex, which cannot be solved directly. In the following, we propose an efficient approach to solve it.

3. AN EFFICIENT APPROACH TO SOLVE THE GELOCATION PROBLEM

By carefully studying the characteristic of the objective function of (4), we found that the surface distance D and the group path P' versus the flying angle β_0 under a given QP model are all convex within the range we concern. This led us to approach the solution of (4) efficiently by the following procedure:

Step 1: Solving (4) without constraints to find the global flying angles by the coordinate descent algorithm [6]

According to the above analysis, we consider that the objective function of (4) would be convex with respect to all flying angles. In this case, solving (4) by the coordinate descent algorithm [6] will be an efficient way to find the global solution of angles $\boldsymbol{\beta}$. The problem of (4) without constraints becomes

$$\min_{\boldsymbol{\beta}} \ (\boldsymbol{GP}-\boldsymbol{\tau})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{GP}-\boldsymbol{\tau})+\delta\sum_{i=1}^{M}P_i'. \tag{6}$$

Let t represent the objective function of (6). By using the coordinate descent algorithm to (6), each element of $\boldsymbol{\beta}$ is iterated by

$$\beta_i(k+1)=\beta_i(k)+\alpha\frac{dt(\beta_i(k))}{d\beta_i} \tag{7}$$

with a given initial value, where α is the step size.

Step 2: Solving (4) with the constraints w.r.t. $\beta_i,\ i=1,\cdots,M$, to approach the optimal flying angles by the Newton-like method for equality constraints [6]

Solving $\boldsymbol{x}$ from the equality constraints in (4), substituting it to the constraints, and then removing the constraint $\|\boldsymbol{x}\|=r_0$, the problem (4) with the equality constraints with respect only to $\beta_i,\ i=1,\cdots,M$, becomes

$$\min_{\boldsymbol{\beta}} \ (\boldsymbol{GP}-\boldsymbol{\tau})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{GP}-\boldsymbol{\tau})+\delta\sum_{i=1}^{M}P_i'$$
$$\text{subject to } \|\boldsymbol{S}_i-\hat{\boldsymbol{x}}\|=2r_0\sin\left(\frac{D_i}{2r_0}\right),\ i=1,2,...,M, \tag{8}$$

where

$$\widehat{\boldsymbol{x}}=\boldsymbol{A}\begin{bmatrix}\|\boldsymbol{S}_1\|^2+r_0^2\\ \|\boldsymbol{S}_2\|^2+r_0^2\\ \vdots\\ \|\boldsymbol{S}_M\|^2+r_0^2\end{bmatrix}-\boldsymbol{A}\begin{bmatrix}g_1^2\\ g_2^2\\ \vdots\\ g_M^2\end{bmatrix} \tag{9}$$

with $g_i=2r_0\sin\left(\frac{D_i}{2r_0}\right)$, $\boldsymbol{\xi}=2[\boldsymbol{S}_1,\ldots,\boldsymbol{S}_M]^T$, and $\boldsymbol{A}=(\boldsymbol{\xi}^T\boldsymbol{\xi})^{-1}\boldsymbol{\xi}^T$.

By employing the Newton-like method for equality constraints [6] and using the global solution of $\boldsymbol{\beta}$ as the initial point, the optimal angles can be approached by solving (8).

Due to the limited space, the derivation of the iterative equations is omitted here.

Step 3: Solving (4) to find the optimal estimate of $\boldsymbol{x}$ by the Quasi-Newton method [6]

Since the unknown emitter is considered to be located at the earth surface with $\|\boldsymbol{x}\|=r_0$ ($\boldsymbol{x}=[x,y,z]^T$), the coordinate z can be expressed as a function of the other two, *i.e.*, $z=z(x,y)$. On the other hand, the flying angle β_i can also be expressed as a nonlinear function of the coordinates x and y according to the equality constraint equations in (4), *i.e.*, $\beta_i=\beta_i(x,y)$. This implies that (4) can be represented by

$$\min_{x,y} \ (\boldsymbol{GP}-\boldsymbol{\tau})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{GP}-\boldsymbol{\tau})+\delta\sum_{i=1}^{M}P_i'. \tag{10}$$

With the initial point $(x(0),y(0))$ computed by (9) according to the optimal flying angles $\boldsymbol{\beta}_{opt}$ obtained in Step 2, and the derivatives related to the objective function of (10), which include the ones from the equality constraints in (4), $(x(k),y(k))$ is iterated by the Quasi-Newton's method [6] to solve (10). It is noted that in each iteration, $\beta_i(k)$ is computed by minimizing $\left(\|\boldsymbol{S}_i-\boldsymbol{x}(k)\|-2r_0\sin\left(\frac{D_i}{2r_0}\right)\right)^2$ under given $\boldsymbol{x}(k)$, where $z(k)=\pm\sqrt{r_0^2-x(k)^2-y(k)^2}$.

4. NUMERICAL RESULTS AND PERFORMANCE ANALYSIS

4.1. The Cramer-Rao Bound on the Geolocation

The log-likelihood function of the unknown emitter localization, by ignoring the constant term, is given by

$$L=-\frac{1}{2}(\boldsymbol{GP}-\boldsymbol{\tau})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{GP}-\boldsymbol{\tau}). \tag{11}$$

When the parameters in the QP model are known, the Cramer-Rao bound (CRB) for the geolocation can be derived according to the associated Fisher information matrix. Here, the location is defined as $\boldsymbol{\theta}=[x,y]^T$ as the emitter is located on the earth surface. The associated Fisher information matrix is defined and derived by

$$\begin{aligned}J_{\boldsymbol{\theta}}&=-E\left[\frac{\partial^2 L}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right]\\ &=\frac{\partial\boldsymbol{\beta}^T}{\partial\boldsymbol{\theta}}\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{\beta}}^T\boldsymbol{G}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{G}\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{\beta}^T}\frac{\partial\boldsymbol{\beta}}{\partial\boldsymbol{\theta}^T}\end{aligned} \tag{12}$$

The variance of the unknown emitter geolocation $\boldsymbol{\theta}$ is lower bounded by the corresponding diagonal of the inversion of the associated Fisher information matrix:

$$CRB_{\boldsymbol{\theta}}=J_{\boldsymbol{\theta}}{}^{-1}. \tag{13}$$

4.2. Numerical Results

Here, we run Monte Carlo simulations to illustrate the performance of the proposed geolocation method. We assume that the unknown emitter is located on the surface of the earth with the longitude and latitude of $(116.24°, 39.55°)$ and five coordinated distant receivers are available. With the use of the Satellite Tool Kit (STK), it is convenient to determine the longitude and latitude coordinate of five distant receivers on the surface of the earth, which are $(128.72°, 40.55°)$ for receiver S1, $(130.42°, 38.68°)$ for receiver S2, $(132.94°, 33.82°)$ for receiver S3, $(130.90°, 31.84°)$ for receiver S4, and $(129.06°, 35.63°)$ for receiver S5, respectively. The distribution of the receivers is shown in Fig. 2.

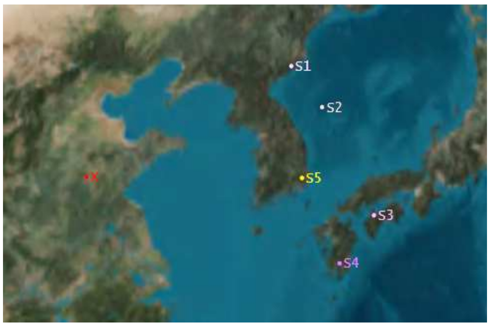

Fig. 2. Geographical distribution of the distant receivers

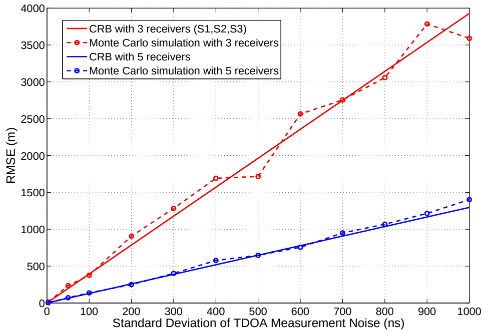

Fig. 3. RMSE of geolocation versus TDOA noise

By assuming that the parameters of the QP model are known, *i.e.* $f_c = 10$MHz, $r_m = 6650$km, and $y_m = 100$km, we perform 50 Monte Carlo runs to calculate the root-mean-square error (RMSE) of the geolocation of the unknown emitter according to the procedure proposed in Section 3, where the operating frequency is set to $f = 15$MHz and the radius of the earth is $r_0 = 6371.2$km. The RMSE of the geolocation for the cases of employing three receivers (S1, S2, S3) and all of five receivers are respectively plotted in Fig. 3. It is seen from Fig. 3 that the RMSE of the proposed geolocation method is close to the associated CRB for both cases of deploying three and five receivers.

Next, we perform Monte Carlo simulations to evaluate the performance of geolocation when the knowledge of the ionosphere is not accurate, *i.e.*, the QP model parameters are perturbed within given range from the true ones. In the simulation, we assume that f_c is uniformly perturbed within $[-0.1\text{MHz}, +0.1\text{MHz}]$, and r_m and y_m are each uniformly perturbed within $[-10\text{km}, +10\text{km}]$ around the true values, and 20 perturbed samples for each parameter are used. The true TDOA is calculated according to the perturbed parameters. We consider the above-mentioned parameters as the estimated one, and perform 50 Monte Carlo runs for the geolocation with three receivers (S1, S2, S3). The simulation results in Fig. 4, Fig. 5, and Fig. 6 show that the effects of perturbed f_c in the QP model on performance is smaller than the other two, and the geolocation performance under the perturbation within a given certain range is acceptable. It is obvious that more knowledge of the ionosphere will help improve the geolocation performance.

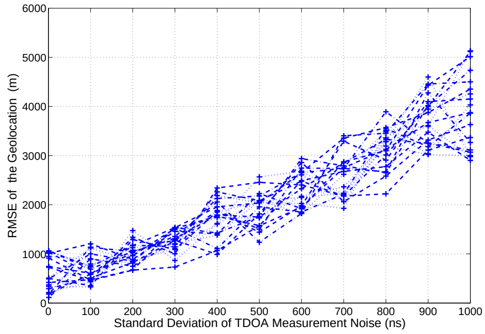

Fig. 4. Perturbed f_c with r_m and y_m unperturbed

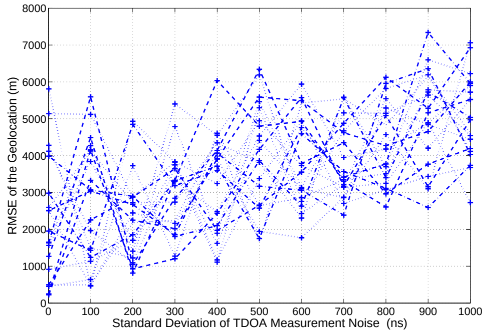

Fig. 5. Perturbed r_m with f_c and y_m unperturbed

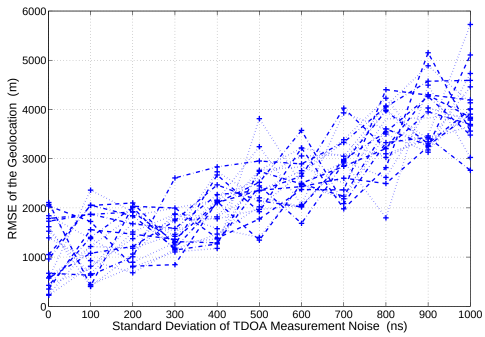

Fig. 6. Perturbed y_m with f_c and r_m unperturbed

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