Dynamic Regret Bound for Moving Target Tracking Based on Online Time-of-Arrival Measurements

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***Abstract*— The use of online algorithms to track a moving target is gaining attention in the control community for its simpler and faster computation. In this work, we study the dynamic regret of online gradient descent (OGD) for tackling a time-of-arrival (TOA)-based least-squares formulation of the tracking problem. Since the formulation is non-convex, most existing dynamic regret analyses cannot be applied to it directly. To circumvent this difficulty, we proceed in two steps. First, we show that under standard assumptions on the TOA measurement noise, the loss function at each time step will, with high probability, be locally strongly convex at that time step. Moreover, we give an explicit estimate of the size of the strong convexity region. To the best of our knowledge, this result is new and can be of independent interest. Second, we show that under the aforementioned assumptions on the TOA measurement noise and mild assumptions on the target trajectory, the location estimate of the target at each time step will lie in the strong convexity region of the loss function at the next time step with high probability. This allows us to exploit existing analysis for online strongly convex optimization to give the first dynamic regret bound of OGD for the TOA-based target tracking problem. Simulation results are presented to illustrate our theoretical findings.**

I. Introduction

The problem of moving target tracking is of great interest due to its significance in autonomous systems such as robotics [1] and surveillance system [2]; see [3] and the references therein. Various methods for solving the tracking problem have been proposed in the literature, including detection and classification [4], Kalman filtering [5], [6], and clustering [7]. With a surge of interest in *online learning* in the control community, there have been works that treat the tracking problem as one of sequential decision making [8], [9]. Specifically, at each time step t, a loss function f_t that depends on certain measurement of the target at time t is revealed, and the decision maker needs to find an estimate of the target location $\boldsymbol{x}_{t+1}$ that (approximately) minimizes f_{t+1}, possibly with the help of the information from all previous time steps. The advantage of such an approach is twofold. From a computational point of view, it leads to simpler and faster computation, as an online algorithm can make use of the results from previous rounds. From a theoretical point of view, one can potentially use the well-developed notion of *regret* (see, e.g., [10], [11]) to analyze the performance of an online algorithm.

In this work, we study the target tracking problem under the setting where we do not assume any model for the target trajectory (cf. [6]) but assume that time-of-arrival (TOA) measurements of the target are available at each time step [12] (e.g., when tracking via impulse-radio UWB (IR-UWB) radar in indoor environment [13], [14]). We consider a least-squares loss function based on the TOA measurements at each time step, which corresponds to the maximum-likelihood (ML) function when the TOA measurement noise is Gaussian. To estimate the target location at each time step, we use the *online gradient descent* (OGD) method, which has low computational complexity. Our goal is to analyze the *dynamic regret* of the OGD method. The notion of dynamic regret captures the changes in the optimal solution to the loss minimization problem at each time step and hence is suitable for the target tracking setting. However, since the loss function at each time step is in general *non-convex*, most existing dynamic regret analyses (such as those in [15], [16], [8]) cannot be applied directly to our setting.

The major contribution of this work is the development of the first non-trivial dynamic regret bound for the OGD method when applied to the non-convex TOA-based loss functions. To achieve this, we proceed in two steps. First, we elucidate the local geometry of the loss functions by showing that under standard assumptions on the TOA measurement noise, the loss function at each time step will, with high probability, be locally strongly convex at that time step. Moreover, we give an explicit estimate of the size of the strong convexity region. We remark that a similar result has previously been established for TDOA-based loss functions [17]. However, to the best of our knowledge, the result for TOA-based loss functions is new and can be of independent interest. Then, we show that as long as the aforementioned assumptions on the TOA measurement noise are satisfied and the distance between the true locations of the target at consecutive time steps is sufficiently small, the location estimate of the target at each time step t will lie in the strong convexity region of the loss function at time $t+1$ with high probability. This allows us to exploit existing results in [15] for online strongly convex optimization to establish the desired dynamic regret bound for the OGD method. Lastly, we present some simulation results to illustrate our theoretical findings.

The rest of the paper is organized as follows. In Section II, we formulate the target tracking problem as an online optimization problem. In Section III, we establish the local strong convexity of the static TOA-based localization problem. In Section IV, we give the dynamic regret rate of OGD for tracking a moving target using TOA measurements. In Section V, simulation results are presented to illustrate

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our theoretical findings. Unless otherwise specified, $\|\cdot\|$ denotes the Euclidean norm and $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue.

II. Problem Formulation and Preliminaries

In this work, we are interested in tracking a moving target using TOA measurements. Let $\boldsymbol{x}_t^\star \in \mathbb{R}^n$ be the unknown true location of the moving target at time t, where $t = 1, \ldots, T$ and T is the time horizon of interest. Furthermore, let $\boldsymbol{a}_i \in \mathbb{R}^n$ ($i = 1, \ldots, m$) be the known location of the ith anchor and suppose that the vectors $\{\boldsymbol{a}_i - \boldsymbol{a}_1\}_{i=2}^m$ span $\mathbb{R}^n$. We consider the following model for TOA-based range measurements:

$$r_i^t = \|\boldsymbol{x}_t^\star - \boldsymbol{a}_i\| + w_i^t, \quad i = 1, \ldots, m;\ t = 1, \ldots, T.$$

Here, w_i^t is the measurement noise, which is assumed to be Gaussian with mean zero, variance σ^2 and independent of the noise at other anchors and at other time steps; r_i^t is the noisy TOA measurement between the target and the ith anchor at time t. We assume that $|w_i^t| \ll \|\boldsymbol{x}_t^\star - \boldsymbol{a}_i\|$ for $i = 1, \ldots, m$ and $t = 1, \ldots, T$, which is standard in the localization literature (see, e.g., [18]). As the target is moving, we would like to estimate the trajectory in an online fashion by updating the estimate upon each short move. Towards that end, recall that the ML estimate of the target location at time t is given by an optimal solution to the following problem [19]:

$$\min_{\boldsymbol{x}} \quad f_t(\boldsymbol{x}) := \sum_{i=1}^m (\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i^t)^2. \tag{1}$$

We generate an estimate of the target location $\boldsymbol{x}_t \in \mathbb{R}^n$ at time t using the OGD update

$$\boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t), \quad t = 0, \ldots, T-1, \tag{2}$$

where $\eta_t > 0$ is the step size. To evaluate the performance of the sequence $\{\boldsymbol{x}_t\}_{t=0}^T$, we employ the notion of *dynamic regret*, which is defined as

$$\text{regret}(T) := \sum_{t=1}^T f_t(\boldsymbol{x}_t) - \sum_{t=1}^T f_t(\hat{\boldsymbol{x}}_t)$$

with $\hat{\boldsymbol{x}}_t \in \arg\min_{\boldsymbol{x}} f_t(\boldsymbol{x})$. Such a notion captures the *accumulative loss* associated with the deviation of our estimate of the target location from the ML estimate at each time step. One immediate obstacle to establishing a bound on the dynamic regret of the OGD method (2) is the non-convexity of the loss function f_t in (1). Nevertheless, as we will show in the next section, when the power of the measurement noise σ^2 is sufficiently small, the loss function f_t is locally strongly convex at the ML estimate $\hat{\boldsymbol{x}}_t$. This will allow us to invoke the existing bound in [15] on the dynamic regret of OGD for online strongly convex optimization.

III. Local Strong Convexity of TOA-Based Static Target Localization

Consider a fixed time t. Then, Problem (1) reduces to the classic TOA-based source localization problem [19], in which the target can be regarded as static. For notational simplicity, we drop the index t and write Problem (1) as

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x}) := \sum_{i=1}^m (\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2, \tag{3}$$

where $r_i = \|\boldsymbol{x}^\star - \boldsymbol{a}_i\| + w_i$ and w_i is a Gaussian random variable with mean zero and variance σ^2 with $|w_i| \ll \|\boldsymbol{x}^\star - \boldsymbol{a}_i\|$. Let $\hat{\boldsymbol{x}}$ denote an optimal solution to Problem (3), which is an ML estimate of $\boldsymbol{x}^\star$. The following proposition, which plays a crucial role in our subsequent development, shows that the ML estimate $\hat{\boldsymbol{x}}$ and the true target location $\boldsymbol{x}^\star$ are close when the power of the measurement noise vector $\boldsymbol{w}$ is small.

Proposition 1: Suppose that $\|\boldsymbol{w}\| \leq c_0\sqrt{m}\sigma$ for some constant $c_0 > 0$. Then, there exist constants $K_1,\ K_2 > 0$, which are determined by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ and $\boldsymbol{x}^\star$, such that

$$\|\hat{\boldsymbol{x}} - \boldsymbol{x}^\star\| \leq K_1\sqrt{m}\sigma + K_2 m\sigma^2.$$

By the concentration properties of a Gaussian random vector, the assumption on $\|\boldsymbol{w}\|$ in Proposition 1 will be satisfied with high probability by choosing c_0 appropriately. Now, using Proposition 1, we can prove the following theorem, which establishes the local strong convexity of f at $\hat{\boldsymbol{x}}$ and the size of the strong convexity region around $\hat{\boldsymbol{x}}$. This constitutes our first main result in this paper.

Theorem 1: Under the setting of Proposition 1, let $\delta > 0$ be such that $\|\boldsymbol{x}^\star - \boldsymbol{a}_i\| > K_1\sqrt{m}\sigma + K_2 m\sigma^2 + \delta$ for $i = 1, \ldots, m$. Then, we have $\nabla^2 f(\hat{\boldsymbol{x}}) \succ \boldsymbol{0}$ whenever

$$\sigma < \frac{-b + \sqrt{b^2 + 4aW}}{2a}, \tag{4}$$

where

$$a := K_2(N+m)m, \quad b := (N+m)K_1\sqrt{m} + c_0 m,$$
$$W := \Lambda \cdot \lambda_{\min}\left(\sum_{i=1}^m \left(\frac{\boldsymbol{x}^\star - \boldsymbol{a}_i}{\|\boldsymbol{x}^\star - \boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}^\star - \boldsymbol{a}_i}{\|\boldsymbol{x}^\star - \boldsymbol{a}_i\|}\right)^T\right)$$

for some constants $N, \Lambda > 0$ determined by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, $\boldsymbol{x}^\star$, and δ. Moreover, we have $\nabla^2 f(\hat{\boldsymbol{x}} + \boldsymbol{\epsilon}) \succ \boldsymbol{0}$ whenever $\boldsymbol{\epsilon}$ satisfies $\|\boldsymbol{\epsilon}\| \leq \min\{\delta, \Theta\}$ with

$$\Theta := \frac{W - (N+m)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) - c_0 m\sigma}{N+m}. \tag{5}$$

The assumption that the target is sufficiently far from the anchors, together with the result in Proposition 1, implies that the loss function f is smooth at the ML estimate $\hat{\boldsymbol{x}}$. Under such a setting, Theorem 1 shows that when the noise power σ^2 is sufficiently small, the loss function f is locally strongly convex at $\hat{\boldsymbol{x}}$, even though f itself is non-convex in general. As can be seen from (4), the level of noise power is determined by the number of anchors and the locations of anchors and the target. It is worth noting that when the number of anchors is greater than the dimension of the space

n, the vectors $\{\boldsymbol{x}^\star - \boldsymbol{a}_i\}_{i=1}^m$ generically span the whole space $\mathbb{R}^n$, which implies that the right-hand sides of (4) and (5) are positive.

IV. Dynamic Regret of OGD for TOA-Based Target Tracking

The results in Section III suggest that for the target tracking problem, as long as the iterate $\boldsymbol{x}_t$ generated by the OGD method (2) lies in the strong convexity region of the loss function f_{t+1} at every time step t, we are essentially performing online optimization of a sequence of strongly convex functions. Such a setting has been well studied in the literature. In particular, we can utilize existing results in [15] to establish a bound on the dynamic regret of OGD for the target tracking problem.

To realize the above idea, let us first collect some consequences of the results in Section III. Let $f_0(\boldsymbol{x}) := \sum_{i=1}^m (\|\boldsymbol{x} - \boldsymbol{a}_i\| - \|\boldsymbol{x}_0^\star - \boldsymbol{a}_i\|)^2$, where $\boldsymbol{x}_0^\star$ is known. Under the setting of Theorem 1, we know that for each time step t, there exists a constant $\epsilon_t > 0$ such that the loss function f_t is strongly convex over the ball $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$. Hence, there exist constants $\mu, L, G > 0$ such that for $t = 0, 1, \ldots, T$,

1) f_t is μ-strongly convex over $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$—i.e. for any $\boldsymbol{x}, \boldsymbol{y} \in B(\hat{\boldsymbol{x}}_t, \epsilon_t)$,
$$f_t(\boldsymbol{x}) \geq f_t(\boldsymbol{y}) + \nabla f_t(\boldsymbol{y})^T(\boldsymbol{x} - \boldsymbol{y}) + \frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2; \tag{6}$$
2) ∇f_t is L-Lipschitz continuous over $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$—i.e. for any $\boldsymbol{x}, \boldsymbol{y} \in B(\hat{\boldsymbol{x}}_t, \epsilon_t)$,
$$\|\nabla f_t(\boldsymbol{y}) - \nabla f_t(\boldsymbol{x})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|; \tag{7}$$
3) $\|\nabla f_t\|$ is bounded above by G over $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$—i.e. for any $\boldsymbol{x} \in B(\hat{\boldsymbol{x}}_t, \epsilon_t)$,
$$\|\nabla f_t(\boldsymbol{x})\| \leq G. \tag{8}$$

Indeed, the inequality (6) is simply a characterization of the strong convexity of f_t over $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$. On the other hand, both (7) and (8) hold since $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$ is compact and ∇f and $\nabla^2 f$ are continuous over $B(\hat{\boldsymbol{x}}_t, \epsilon_t)$.

Next, we are interested in estimating the progress of the OGD method by bounding the distance between the iterate $\boldsymbol{x}_t$ and the ML estimate $\hat{\boldsymbol{x}}_t$ of the target location at time t. Towards that end, let us first recall the following well-known result, which concerns the linear convergence of the gradient descent method when applied to minimize a strongly convex function.

Fact 1: (cf. [20, Theorem 2.1.15]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function that is μ-strongly convex and L-gradient Lipschitz continuous on a closed convex set $\mathcal{X} \subseteq \mathbb{R}^n$. Let $\hat{\boldsymbol{x}} \in \arg\min_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x})$ and suppose that $\hat{\boldsymbol{x}} \in \operatorname{int}(\mathcal{X})$. Then, the sequence $\{\boldsymbol{x}_k\}_{k \geq 0}$ generated by the gradient descent method
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \eta \nabla f(\boldsymbol{x}_k)$$
with the initial point $\boldsymbol{x}_0 \in \operatorname{int}(\mathcal{X})$ and step size $0 < \eta \leq 2/(\mu + L)$ satisfies
$$\|\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}\|^2 \leq \left(1 - \frac{2\eta\mu L}{\mu + L}\right)\|\boldsymbol{x}_k - \hat{\boldsymbol{x}}\|^2.$$

When $\boldsymbol{x}_t$ lies in the strong convexity region of $\hat{\boldsymbol{x}}_t$, using Fact 1 and Proposition 1, we have
$$\begin{aligned}
&\|\boldsymbol{x}_{t+1} - \hat{\boldsymbol{x}}_{t+1}\| \\
\leq &\|\boldsymbol{x}_{t+1} - \hat{\boldsymbol{x}}_t\| + \|\hat{\boldsymbol{x}}_t - \hat{\boldsymbol{x}}_{t+1}\| \\
\leq &\left(1 - \frac{2\eta_t\mu L}{\mu + L}\right)^{1/2}\|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\| + v_t \\
&\quad + \|\hat{\boldsymbol{x}}_t - \boldsymbol{x}_t^\star\| + \|\boldsymbol{x}_{t+1}^\star - \hat{\boldsymbol{x}}_{t+1}\| \\
\leq &\left(1 - \frac{2\eta_t\mu L}{\mu + L}\right)^{1/2}\|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\| + v_t \\
&\quad + (K_1^t + K_1^{t+1})\sqrt{m}\sigma + (K_2^t + K_2^{t+1})m\sigma^2,
\end{aligned} \tag{9}$$
where K_1^t and K_2^t correspond to the constants K_1 and K_2 in Proposition 1 for the loss function f_t and $v_t := \|\boldsymbol{x}_{t+1}^\star - \boldsymbol{x}_t^\star\|$ denotes the variation in the true target location between time t and $t+1$. Now, we are ready to estimate the progress of the OGD method (2) for tackling Problem (1).

Proposition 2: Suppose that

1) $\boldsymbol{x}_0 = \boldsymbol{x}_0^\star$;
2) the measurement noise $\{\boldsymbol{w}^t\}_{t=1}^T$ satisfies $\|\boldsymbol{w}^t\| \leq c_0\sqrt{m}\sigma$ for some constant $c_0 > 0$;
3) there exists a constant $\zeta > 0$ such that $\|\boldsymbol{x}_t^\star - \boldsymbol{a}_i\| > 2K_1^t\sqrt{m}\sigma + 2K_2^t m\sigma^2 + \zeta$ for each time step t, where K_1^t and K_2^t are the constants in Proposition 1 for the loss function f_t.

Let μ and L be as in (6) and (7), respectively. Furthermore, set $\rho_t := \left(1 - \frac{2\eta_t\mu L}{\mu + L}\right)^{1/2}$ and note that $\rho_t \in (0, 1)$. Assuming that σ and $\{v_t\}_{t=0}^{T-1}$ are sufficiently small, the following hold for $t = 1, \ldots, T$:

(i) $\boldsymbol{x}_t$ lies in the strong convexity region of $\hat{\boldsymbol{x}}_t$.

(ii) We have
$$\begin{aligned}
&\|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\| \\
&\leq \left(\frac{(1 - \rho^t)(\rho + 1)}{1 - \rho} + 1\right)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) \\
&\quad + \left(\frac{1 - \rho^{t-1}}{1 - \rho} + 1\right)v,
\end{aligned}$$
where
$$\rho := \max_{t \in \{1,\ldots,T\}} \rho_t, \quad v := \max_{t \in \{1,\ldots,T\}} v_t,$$
$$K_1 := \max_{t \in \{1,\ldots,T\}} K_1^t, \quad K_2 := \max_{t \in \{1,\ldots,T\}} K_2^t.$$

Proof: We proceed by induction on t. By Proposition 1, we have
$$\|\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0\| = \|\boldsymbol{x}_0^\star - \hat{\boldsymbol{x}}_0\| = 0 \leq K_1\sqrt{m}\sigma + K_2 m\sigma^2.$$

By definition of $\boldsymbol{x}_0$, it can be seen that $\boldsymbol{x}_0 \in B(\hat{\boldsymbol{x}}_0, \epsilon_0)$. Using Fact 1, we have

$$\begin{aligned}\|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1\| &\leq \|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_0\| + \|\hat{\boldsymbol{x}}_0 - \hat{\boldsymbol{x}}_1\| \\ &\leq \rho\|\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0\| + 2K_1\sqrt{m}\sigma + 2K_2 m\sigma^2 + v \\ &\leq (\rho+2)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) + v.\end{aligned}$$

Moreover, for sufficiently small σ and v, we have $\boldsymbol{x}_1 \in B(\hat{\boldsymbol{x}}_1, \epsilon_1)$. Now, by (9) and the inductive hypothesis, we compute

$$\begin{aligned}&\|\boldsymbol{x}_{t+1} - \hat{\boldsymbol{x}}_{t+1}\| \\ \leq&\rho\|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\| + 2K_1\sqrt{m}\sigma + 2K_2 m\sigma^2 + v \\ \leq&\rho\left(\frac{\rho(1-\rho^t)(\rho+1)}{1-\rho} + \rho + 2\right)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) \\ &\quad + \left(\rho\left(\frac{1-\rho^{t-1}}{1-\rho} + 1\right) + 1\right)v \\ =&\left(\frac{(1-\rho^{t+1})(\rho+1)}{1-\rho} + 1\right)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) \\ &\quad + \left(\frac{1-\rho^t}{1-\rho} + 1\right)v.\end{aligned}$$

Moreover, for sufficiently small σ and v, we have $\boldsymbol{x}_{t+1} \in B(\hat{\boldsymbol{x}}_{t+1}, \epsilon_{t+1})$. This completes the proof. ∎

To derive the desired regret bound for the OGD method (2), it remains to show that for some suitable choice of parameters, the iterate $\boldsymbol{x}_t$ lies in the strong convexity region of f_{t+1}. This is made precise in the following theorem, which constitutes our second main result in this paper.

Theorem 2: Let G be as in (8). Under the setting of Proposition 2, denote $N = \max_{t\in\{1,\ldots,T\}} N_t$. Then, the sequence $\{\boldsymbol{x}_t\}_{t=1}^T$ generated by the OGD method (2) lies in the strong convexity region $B(\hat{\boldsymbol{x}}_{t+1}, \epsilon_{t+1})$ of the loss function f_{t+1} whenever

$$\begin{aligned}&\Lambda_{t+1}\cdot\lambda_{\min}\left(\sum_{i=1}^m\left(\frac{\boldsymbol{x}^\star_{t+1} - \boldsymbol{a}_i}{\|\boldsymbol{x}^\star_{t+1} - \boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}^\star_{t+1} - \boldsymbol{a}_i}{\|\boldsymbol{x}^\star_{t+1} - \boldsymbol{a}_i\|}\right)^T\right) \\ \geq&(N+m)\left(\frac{(1-\rho^t)(\rho+1)}{1-\rho} + 4\right)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) \\ &\quad + c_0 m\sigma + (N+m)\left(\frac{1-\rho^{t-1}}{1-\rho} + 2\right)v, \qquad (10)\end{aligned}$$

where Λ_{t+1} corresponds to the constant Λ in Theorem 1 and

$$\begin{aligned}\zeta \geq& \left(\frac{(1-\rho^t)(\rho+1)}{1-\rho} + 2\right)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) \\ &+ \left(\frac{1-\rho^{t-1}}{1-\rho} + 2\right)v. \qquad (11)\end{aligned}$$

In particular, if the above holds for $t = 0, 1, \ldots, T-1$, then the dynamic regret of the OGD method (2) is bounded by

$$\mathrm{regret}(T) \leq G\left(M_1\sum_{t=1}^{T-1}\|\boldsymbol{x}^\star_{t+1} - \boldsymbol{x}^\star_t\| + M_2 T\sigma + M_3\right), \qquad (12)$$

where

$$\begin{aligned}M_1 &= \frac{3K_1\sqrt{m}\sigma + 3K_2 m\sigma^2 + v_0}{1-\nu}, \\ M_2 &= 2M_1(K_1\sqrt{m} + K_2 m\sigma), \\ M_3 &= \frac{1}{1-\nu}, \quad \nu = \min_t(1-\mu\eta_t)^{1/2}.\end{aligned}$$

Proof: Using the fact that

$$\begin{aligned}\|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_{t+1}\| \leq& \|\boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\| + \|\hat{\boldsymbol{x}}_t - \boldsymbol{x}^\star_t\| \\ &+ \|\boldsymbol{x}^\star_t - \boldsymbol{x}^\star_{t+1}\| + \|\boldsymbol{x}^\star_{t+1} - \hat{\boldsymbol{x}}_{t+1}\|\end{aligned}$$

and applying Proposition 1, Theorem 1, and Proposition 2, the condition (10) follows. With Assumption 3 in Proposition 2, we plug $\delta = K_1\sqrt{m}\sigma + K_2 m\sigma^2 + \zeta$ into Theorem 1 and thereby get the condition (11).

Now, given that for $t = 0, 1, \ldots, T-1$, the iterate $\boldsymbol{x}_t$ lies in the strong convexity region $B(\hat{\boldsymbol{x}}_{t+1}, \epsilon_{t+1})$ of the loss function f_{t+1}, we can invoke [15, Corollary 1] (setting $\mathcal{X} = \mathbb{R}^n$, $h = 1$, and $\gamma \geq \max_{t\in\{1,\ldots,T\}}\eta_t^{-1}$ in [15, Algorithm 1]) to get

$$\mathrm{regret}(T) \leq G\left(H_1\sum_{t=2}^T\|\hat{\boldsymbol{x}}_t - \hat{\boldsymbol{x}}_{t-1}\| + H_2\right),$$

where

$$\begin{aligned}&H_1 = \frac{\|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1\| - \nu\|\boldsymbol{x}_T - \hat{\boldsymbol{x}}_T\|}{1-\nu}, \quad H_2 = \frac{1}{1-\nu}, \\ &\nu = \min_t(1-\mu\eta_t)^{1/2}.\end{aligned}$$

Upon noting that

$$\begin{aligned}&\sum_{t=2}^T\|\hat{\boldsymbol{x}}_t - \hat{\boldsymbol{x}}_{t-1}\| \\ \leq&\sum_{t=2}^T(\|\hat{\boldsymbol{x}}_t - \boldsymbol{x}^\star_t\| + \|\boldsymbol{x}^\star_t - \boldsymbol{x}^\star_{t-1}\| + \|\boldsymbol{x}^\star_{t-1} - \hat{\boldsymbol{x}}_{t-1}\|) \\ \leq&\sum_{t=2}^T(2K_1\sqrt{m}\sigma + 2K_2 m\sigma^2 + v_{t-1}) \\ \leq&\sum_{t=1}^{T-1}v_t + 2K_1\sqrt{m}T\sigma + 2K_2 mT\sigma^2\end{aligned}$$

and

$$\begin{aligned}&\|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1\| - \nu\|\boldsymbol{x}_T - \hat{\boldsymbol{x}}_T\| \\ \leq&\|\boldsymbol{x}_1 - \hat{\boldsymbol{x}}_1\| \\ \leq&(\rho_0+1)(K_1\sqrt{m}\sigma + K_2 m\sigma^2) + K_1\sqrt{m}\sigma \\ &\quad + K_2 m\sigma^2 + v_0 \\ \leq&3K_1\sqrt{m}\sigma + 3K_2 m\sigma^2 + v_0,\end{aligned}$$

the proof is complete. ∎

As can be seen from (12), even if the cumulative path variation $\sum_{t=1}^{T-1}\|\boldsymbol{x}^\star_{t+1} - \boldsymbol{x}^\star_t\|$ grows sublinearly in T, as long as σ remains constant, the regret bound will eventually be dominated by the linearly-growing term $M_2 T\sigma$. This is also observed in our numerical results, where the regret eventually becomes linear when the time horizon T is sufficiently large.

Recall that the regret can be interpreted as the accumulative loss associated with the deviation of our estimate of the target location from the ML estimate at each time step. This, together with the fact that the ML estimate is close to the true target location with high probability (see Proposition 1), implies that our algorithm is performing well if the regret bound is small. In particular, this can be achieved when both the path variations and measurement noise power are small.

The conditions (10) and (11) suggest how the parameters could be set to keep each iterate $\boldsymbol{x}_t$ in the strong convexity region of the loss function f_{t+1}, so that the regret bound (12) holds. Without knowing $\lambda_{\min}\left(\sum_{i=1}^{m}\left(\frac{\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{a}_i}{\|\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{a}_i}{\|\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{a}_i\|}\right)^T\right)$ in advance, an intuitive choice would be to reduce the path variations and the power of the measurement noise. However, in our experiments, we discover that the result in Theorem 2 is robust against large path variations and noise power. Specifically, even if we have path variations that violate (10), the regret grows sublinearly at the beginning (when $\sum_{t=1}^{T-1}\|\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{x}_t^{\star}\|$ dominates $T\sigma$) and linearly eventually (when $T\sigma$ dominates $\sum_{t=1}^{T-1}\|\boldsymbol{x}_{t+1}^{\star}-\boldsymbol{x}_t^{\star}\|$). As a future work, it would be interesting to understand whether a similar regret bound can be obtained without the assumptions on the path variations and the power of the measurement noise.

V. Simulations

In this section, we illustrate our theoretical findings by evaluating the performance of OGD for tracking a moving target. Our setup is similar to that in [9], in which the tracking problem is formulated as a non-convex online optimization problem and solved using the online Newton method (ONM). The computational cost of OGD is smaller than that of ONM, but the regret bounds and numerical performance of both methods are similar. We refer the interested readers to [9] for a comparison of the results.

In the following, we assume that the location of the target evolves as $\boldsymbol{x}_{t+1}^{\star}=\boldsymbol{x}_t^{\star}+\boldsymbol{d}_t$ for $t=0,1,\ldots,T-1$, where $\boldsymbol{d}_t$ will be specified in the different scenarios below. We set the initial true location of the target as $\boldsymbol{x}_0^{\star}=\begin{bmatrix}2 & 1\end{bmatrix}^T$, while the anchors are located at $\boldsymbol{a}_1=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}^T$, $\boldsymbol{a}_2=\begin{bmatrix}0 & \frac{1}{2}\end{bmatrix}^T$, and $\boldsymbol{a}_3=\begin{bmatrix}\frac{1}{2} & 0\end{bmatrix}^T$. We assume that the initial location of the target is known, so that we can set $\boldsymbol{x}_0=\boldsymbol{x}_0^{\star}$. In all examples, we set $T=10000$ and run 1000 Monte Carlo simulations. The step size in OGD is set to be $\eta_t=1/\sqrt{t}$ for all simulations. We evaluate the performance of OGD via the notion of regret defined in Section II.

Example 1. In this example and the next, we set $\boldsymbol{d}_t=\frac{(-1)^{b_t}0.0025}{\sqrt{2t}}\mathbf{1}$, where $b_t\sim\text{Bernoulli}(0.5)$ and $\mathbf{1}\in\mathbb{R}^2$ is the all-one vector. In this example, we set $\sigma=0.0001$. Figure 1 presents the numerical regret. As can be seen, the regret grows sublinearly with t at first. This can be justified by Theorem 2. Indeed, as σ is negligible compared to $v_t=\|\boldsymbol{d}_t\|$, the first term in (12) dominates. However, since v_t diminishes as $t\to\infty$, the regret rate eventually becomes linear as v_t no longer dominates σ. Again, this corroborates with Theorem 2.

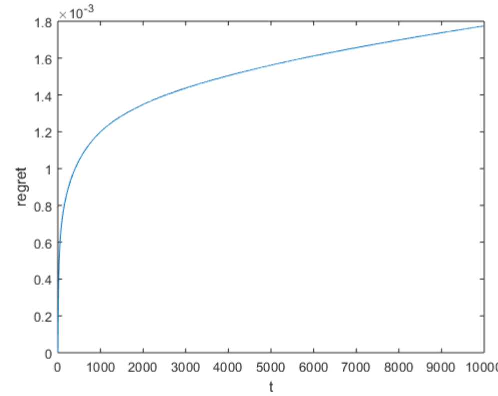

Fig. 1. Regret rates when $\sigma=0.0001$ and $\boldsymbol{d}_t=\frac{(-1)^{b_t}0.0025}{\sqrt{2t}}\mathbf{1}$, where $b_t\sim\text{Bernoulli}(0.5)$

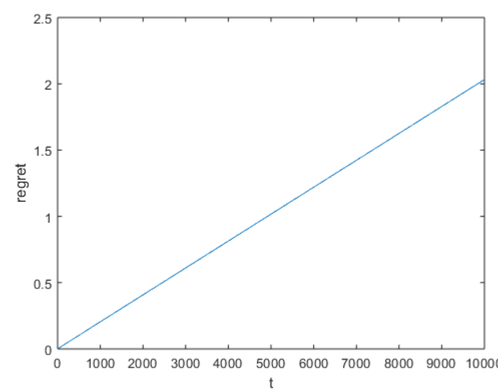

Fig. 2. Regret rates when $\sigma=0.01$ and $\boldsymbol{d}_t=\frac{(-1)^{b_t}0.0025}{\sqrt{2t}}\mathbf{1}$, where $b_t\sim\text{Bernoulli}(0.5)$

Example 2. In this example, we set $\sigma=0.01$. Figure 2 shows the performance of OGD. It is not hard to see that the regret grows linearly with t. Again, this can be justified by Theorem 2, since v_t is negligible compared to σ this time and hence the second term in (12) dominates.

Example 3. In the previous two examples, the random transition $\boldsymbol{v}_t$ follows a discrete distribution. In this example, we draw $\boldsymbol{d}_t$ uniformly from $[-0.0025,0.0025]\times[-0.0025,0.0025]$. Figures 3 and 4 show the performance when $\sigma=0.0001$ and $\sigma=0.01$, respectively. It can be seen that the results are similar to Examples 1 and 2, though the generation method of the trajectory has been changed. This confirms the findings of Theorem 2.

VI. Conclusion and Future Work

This paper studies the dynamic regret of OGD for tracking a moving target based on TOA measurements. Although the TOA-based loss function at each time step is non-convex, by making standard assumptions on the TOA measurement noise, we showed that the loss function at each time step will, with high probability, be locally strong convex at that time step. Furthermore, we showed that the iterate generated by the OGD method at time t will lie in the strong convexity region of the loss function at time $t+1$. Consequently, we

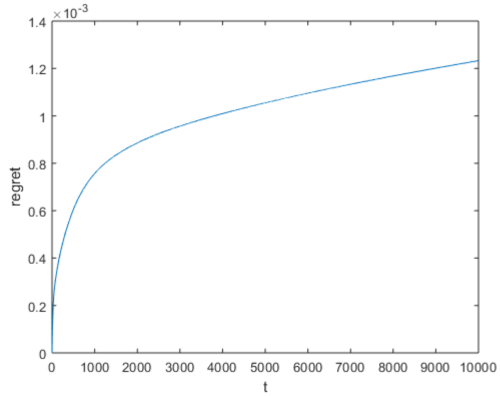

Fig. 3. Regret rates when $\sigma = 0.0001$ and $\boldsymbol{d}_t$ is distributed uniformly on $[-0.0025, 0.0025] \times [-0.0025, 0.0025]$

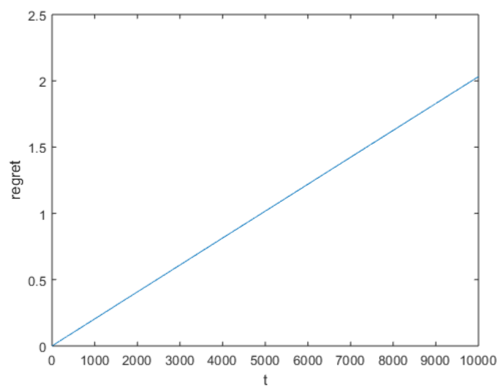

Fig. 4. Regret rates when $\sigma = 0.01$ and $\boldsymbol{d}_t$ is distributed uniformly on $[-0.0025, 0.0025] \times [-0.0025, 0.0025]$

were able to reduce the TOA-based target tracking problem to that of online strongly convex optimization, for which a dynamic regret bound can be derived from existing results. The regret bound we obtained depends on both the path variations and measurement noise power. Our simulation results support the theoretical findings.

A possible future direction is to extend our analysis to the TDOA setting—where there is no synchronization requirement between the target and anchors—by exploiting the result in [17]. Besides, it would be interesting to design good algorithms for solving the TOA-based target tracking problem.

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APPENDIX

Proof of Proposition 1: The proof makes use of another easy-to-compute estimate of the target location (see [21]), which is constructed as follows. Recall that we are interested in finding $\boldsymbol{x}$ such that

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| \approx r_i^2, \quad i = 1, \ldots, m.$$

By subtracting the ith equation from the $(i+1)$st, where $i = 1, \ldots, m-1$, we obtain

$$2(\boldsymbol{a}_{i+1} - \boldsymbol{a}_i)^T \boldsymbol{x} \approx \|\boldsymbol{a}_{i+1}\|^2 - \|\boldsymbol{a}_i\|^2 + r_i^2 - r_{i+1}^2.$$

This suggests the following least-squares formulation for estimating $\boldsymbol{x}$:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2. \tag{13}$$

Here,

$$\boldsymbol{A} := \begin{bmatrix} (\boldsymbol{a}_2 - \boldsymbol{a}_1)^T \\ \vdots \\ (\boldsymbol{a}_m - \boldsymbol{a}_{m-1})^T \end{bmatrix} \quad \text{and}$$

$$\boldsymbol{b} := \frac{1}{2} \begin{bmatrix} \|\boldsymbol{a}_2\|^2 - \|\boldsymbol{a}_1\|^2 + r_1^2 - r_2^2 \\ \vdots \\ \|\boldsymbol{a}_m\|^2 - \|\boldsymbol{a}_{m-1}\|^2 + r_{m-1}^2 - r_m^2 \end{bmatrix}. \tag{14}$$

Since the vectors $\{\boldsymbol{a}_i - \boldsymbol{a}_1\}_{i=2}^m$ span $\mathbb{R}^n$, so do $\{\boldsymbol{a}_{i+1} - \boldsymbol{a}_i\}_{i=1}^{m-1}$. Hence, the solution to (13) is given by

$$\boldsymbol{x}_{LS} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}.$$

Now, consider the actual target location $\boldsymbol{x}^\star$ and the actual distance between the target and the ith anchor $r_i^\star = \|\boldsymbol{x}^\star - \boldsymbol{a}_i\|$, where $i = 1, \ldots, m$. By repeating the above derivation and letting $\boldsymbol{b}^\star$ to be the vector obtained by replacing r_i with $r_i^\star$ in (14), we have

$$\|\boldsymbol{x}_{LS} - \boldsymbol{x}^\star\| = \|(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T(\boldsymbol{b} - \boldsymbol{b}^\star)\|.$$

Using the fact that $r_i = r_i^\star + w_i$ for $i = 1, \ldots, m$, we have $r_i^2 - (r_i^\star)^2 = 2r_i^\star w_i + w_i^2$ and hence

$$\|\boldsymbol{b} - \boldsymbol{b}^\star\| \leq C_0\|\boldsymbol{w}\| + \frac{1}{2}\|\tilde{\boldsymbol{w}}\| \tag{15}$$

for some constant $C_0 > 0$, where

$$\tilde{\boldsymbol{w}} := \begin{bmatrix} w_1^2 - w_2^2 \\ w_2^2 - w_3^2 \\ \vdots \\ w_{m-1}^2 - w_m^2 \end{bmatrix}.$$

Since $\|\tilde{\boldsymbol{w}}\| \leq \|\tilde{\boldsymbol{w}}\|_1 \leq 2\|\boldsymbol{w}\|^2$, our assumption on $\|\boldsymbol{w}\|$ yields

$$\|\boldsymbol{x}_{LS} - \boldsymbol{x}^\star\| \leq C_1\sqrt{m}\sigma + C_2 m\sigma^2 \tag{16}$$

for some constants $C_1, C_2 > 0$.

Next, denote $\hat{r}_i = \|\hat{\boldsymbol{x}} - \boldsymbol{a}_i\|$ for $i = 1, \ldots, m$ and let $\hat{\boldsymbol{b}}$ be the vector obtained by replacing r_i with $\hat{r}_i$ in (14). By repeating the same argument as in (15) and noting that

$$\begin{aligned} r_i^2 - \hat{r}_i^2 &= 2r_i(r_i - \hat{r}_i) - (r_i - \hat{r}_i)^2 \\ &= 2(r_i^\star + w_i)(r_i - \hat{r}_i) - (r_i - \hat{r}_i)^2, \end{aligned}$$

$$f(\hat{\boldsymbol{x}}) = \sum_{i=1}^m (r_i - \hat{r}_i)^2 \leq f(\boldsymbol{x}^\star) = \|\boldsymbol{w}\|^2 \leq c_0^2 m\sigma^2,$$

we have

$$\begin{aligned} \|\boldsymbol{b} - \hat{\boldsymbol{b}}\| &\leq C_3\sqrt{f(\hat{\boldsymbol{x}})} + \frac{1}{2}\|\tilde{\boldsymbol{r}}\| \leq C_3\sqrt{f(\hat{\boldsymbol{x}})} + C_4 f(\hat{\boldsymbol{x}}) \\ &\leq C_5\sqrt{m}\sigma + C_6 m\sigma^2 \end{aligned}$$

for some constants $C_3, C_4, C_5, C_6 > 0$, where

$$\tilde{\boldsymbol{r}} := \begin{bmatrix} (r_2 - \hat{r}_2)^2 - (r_1 - \hat{r}_1)^2 \\ (r_3 - \hat{r}_3)^2 - (r_2 - \hat{r}_2)^2 \\ \vdots \\ (r_m - \hat{r}_m)^2 - (r_{m-1} - \hat{r}_{m-1})^2 \end{bmatrix}.$$

This gives

$$\|\boldsymbol{x}_{LS} - \hat{\boldsymbol{x}}\| \leq C_7\sqrt{m}\sigma + C_8 m\sigma^2 \tag{17}$$

for some constant $C_7, C_8 > 0$. The desired result then follows by applying the triangle inequality to (16) and (17). ∎

Next, we state two lemmas, which will be used in the proof of Theorem 1.

Lemma 1: For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$,

$$\begin{aligned} \lambda_{\min}(\boldsymbol{u}\boldsymbol{u}^T - \boldsymbol{v}\boldsymbol{v}^T) &= \frac{\|\boldsymbol{u}\|^2 - \|\boldsymbol{v}\|^2 - \|\boldsymbol{u} - \boldsymbol{v}\|\|\boldsymbol{u} + \boldsymbol{v}\|}{2} \\ &= \frac{\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle - \|\boldsymbol{u} - \boldsymbol{v}\|\|\boldsymbol{u} + \boldsymbol{v}\|}{2}. \end{aligned}$$

Proof: Let λ and $\boldsymbol{w}$ be an eigenvalue and respective eigenvector of $\boldsymbol{u}\boldsymbol{u}^T - \boldsymbol{v}\boldsymbol{v}^T$. Then by definition, it satisfies

$$(\boldsymbol{u}\boldsymbol{u}^T - \boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{w} = \lambda\boldsymbol{w}.$$

If $\boldsymbol{w} \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}^\perp$, then $\lambda = 0$. For eigenvectors in $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$, let $\boldsymbol{w} = a\boldsymbol{u} + b\boldsymbol{v}$. We compute

$$\begin{aligned} &(\boldsymbol{u}\boldsymbol{u}^T - \boldsymbol{v}\boldsymbol{v}^T)(a\boldsymbol{u} + b\boldsymbol{v}) \\ =&(a\|\boldsymbol{u}\|_2^2 + b(\boldsymbol{u}^T\boldsymbol{v}))\boldsymbol{u} - (a(\boldsymbol{u}^T\boldsymbol{v}) + b\|\boldsymbol{v}\|^2)\boldsymbol{v} \\ =&\lambda a\boldsymbol{u} + \lambda b\boldsymbol{v}. \end{aligned}$$

By comparing terms, we then obtain

$$\lambda = \frac{a\|\boldsymbol{u}\|^2 + b(\boldsymbol{u}^T\boldsymbol{v})}{a} = \frac{-a(\boldsymbol{u}^T\boldsymbol{v}) - b\|\boldsymbol{v}\|^2}{b}. \tag{18}$$

Solving the quadratic equation

$$(\boldsymbol{u}^T\boldsymbol{v})a^2 + (\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2)ab + (\boldsymbol{u}^T\boldsymbol{v})b^2 = 0,$$

which can always be done because $|\boldsymbol{u}^T\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$, we can derive a relationship between a and b. Plugging this relationship into (18) yields

$$\begin{aligned}\lambda =&\frac{(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2) \pm \sqrt{(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)^2-4(\boldsymbol{u}^T\boldsymbol{v})^2}}{2}-\|\boldsymbol{v}\|^2\\ =&\frac{\|\boldsymbol{u}\|^2-\|\boldsymbol{v}\|^2 \pm \|\boldsymbol{u}-\boldsymbol{v}\|\|\boldsymbol{u}+\boldsymbol{v}\|}{2}\\ =&\frac{\langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle \pm \|\boldsymbol{u}-\boldsymbol{v}\|\|\boldsymbol{u}+\boldsymbol{v}\|}{2}.\end{aligned}$$

The negative root gives a non-positive eigenvalue by the Cauchy-Schwarz inequality, thus the desired result follows. ■

Lemma 2: For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have

$$\left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\| \leq \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\min\{\|\boldsymbol{x}\|, \|\boldsymbol{y}\|\}}.$$

Proof: If $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$, then the lemma is trivial and equality is obtained; else if $\|\boldsymbol{x}\| \neq \|\boldsymbol{y}\|$, we may assume without loss of generality that $\|\boldsymbol{x}\| < \|\boldsymbol{y}\|$, which implies that

$$\left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right\|^2 = \frac{1}{\|\boldsymbol{x}\|^2} \left\| \boldsymbol{x} - \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{y}\|}\boldsymbol{y} \right\|^2.$$

Writing $\alpha = \|\boldsymbol{x}\|/\|\boldsymbol{y}\|$, where $0 < \alpha < 1$, we can see that

$$\begin{aligned}&\|\boldsymbol{x}-\alpha\boldsymbol{y}\|^2\\ =&\|\boldsymbol{x}\|^2 - 2\boldsymbol{x}^T\boldsymbol{y} + \|\boldsymbol{y}\|^2 + 2(1-\alpha)\boldsymbol{x}^T\boldsymbol{y} + (\alpha^2-1)\|\boldsymbol{y}\|^2\\ =&\|\boldsymbol{x}-\boldsymbol{y}\|^2 + (1-\alpha)(2\boldsymbol{x}^T\boldsymbol{y} - (\alpha+1)\|\boldsymbol{y}\|^2)\end{aligned}$$

and

$$\begin{aligned}2\boldsymbol{x}^T\boldsymbol{y} - \left(\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{y}\|}+1\right)\|\boldsymbol{y}\|^2 &\leq 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| - (\|\boldsymbol{x}\|+\|\boldsymbol{y}\|)\|\boldsymbol{y}\|\\ &= \|\boldsymbol{y}\|(\|\boldsymbol{x}\|-\|\boldsymbol{y}\|) < 0.\end{aligned}$$

Thus the desired inequality is obtained. ■

Proof of Theorem 1: Whenever $\boldsymbol{x} \neq \boldsymbol{a}_i$ for $i = 1, \ldots, m$, the Hessian of the loss function f can be computed as

$$\begin{aligned}\nabla^2 f(\boldsymbol{x}) = 2\sum_{i=1}^m \Bigg\{ &\frac{r_i}{\|\boldsymbol{x}-\boldsymbol{a}_i\|^3}(\boldsymbol{x}-\boldsymbol{a}_i)(\boldsymbol{x}-\boldsymbol{a}_i)^T\\ &+\left(1-\frac{r_i}{\|\boldsymbol{x}-\boldsymbol{a}_i\|}\right)\boldsymbol{I}\Bigg\}.\end{aligned}$$

Our goal is to prove that $\nabla^2 f(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}) \succ \boldsymbol{0}$ for all $\boldsymbol{\epsilon}$ within some ball. In particular, this would imply that $\nabla^2 f(\hat{\boldsymbol{x}}) \succ \boldsymbol{0}$. To show this, we see that the minimum eigenvalue of $\nabla^2 f(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon})$ can be lower bounded by

$$\begin{aligned}&\lambda_{\min}(\nabla^2 f(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}))\\ \geq\ & 2\cdot\lambda_{\min}\left(\sum_{i=1}^m \frac{r_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|^3}(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i)(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i)^T\right)\\ &+2\sum_{i=1}^m\left(1-\frac{r_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}\right)\\ \geq\ & 2\cdot\frac{\lambda_{\min}\Big(\sum_{i=1}^m r_i\Big((\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|})(\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|})^T - (\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|})(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|})^T\Big)\Big)}{\max_{i\in\{1,\ldots,m\}}\{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|\}}\\ &\sum_{i=1}^m(\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|-r_i)+\\ &+2\cdot\frac{\lambda_{\min}\Big(\sum_{i=1}^m r_i(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|})(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|})^T\Big)}{\max_{i\in\{1,\ldots,m\}}\{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|\}}. \qquad (19)\end{aligned}$$

Using Lemma 1 and by simple computation, it is not hard to see that

$$\begin{aligned}&\lambda_{\min}\left(\sum_{i=1}^m r_i\Bigg(\left(\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}\right)\left(\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}\right)^T - \left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)^T\Bigg)\right)\\ \geq&-\frac{1}{2}\cdot\sum_{i=1}^m r_i\left\|\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}-\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right\|\cdot\left\|\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}+\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right\|\\ \geq&-\sum_{i=1}^m r_i\left\|\frac{\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i}{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|}-\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right\|\\ \geq&-\sum_{i=1}^m\frac{r_i\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{x}^\star\|}{\min\{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|, \|\boldsymbol{x}^\star-\boldsymbol{a}_i\|\}}\\ \geq&-N\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{x}^\star\| \qquad (20)\end{aligned}$$

for some constant $N > 0$, which is well-defined when $\|\boldsymbol{\epsilon}\| < \delta$. The third inequality follows from Lemma 2.

Now, since $|w_i| \ll \|\boldsymbol{x}^\star-\boldsymbol{a}_i\|$ for $i = 1, \ldots, m$, we have

$$\begin{aligned}&\lambda_{\min}\left(\sum_{i=1}^m r_i\left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)^T\right)\\ \geq&\Lambda\cdot\lambda_{\min}\left(\sum_{i=1}^m\left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)\left(\frac{\boldsymbol{x}^\star-\boldsymbol{a}_i}{\|\boldsymbol{x}^\star-\boldsymbol{a}_i\|}\right)^T\right)\end{aligned}$$

for some constant $\Lambda > 0$. Putting the result back to (19) and recalling the definition of W, we obtain

$$\begin{aligned}&\lambda_{\min}(\nabla^2 f(\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}))\\ \geq\ &2\cdot\frac{-N\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{x}^\star\|+\sum_{i=1}^m(\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|-r_i)+W}{\max_{i\in\{1,\ldots,m\}}\{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|\}}\\ \geq\ &2\cdot\frac{-N\|\hat{\boldsymbol{x}}-\boldsymbol{x}^\star\|-N\|\boldsymbol{\epsilon}\| - \sum_{i=1}^m(\|\hat{\boldsymbol{x}}+\boldsymbol{x}^\star\|+\|\boldsymbol{\epsilon}\|+w_i)+W}{\max_{i\in\{1,\ldots,m\}}\{\|\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}-\boldsymbol{a}_i\|\}}. \qquad (21)\end{aligned}$$

Since

$$\sum_{i=1}^m w_i \leq \|\boldsymbol{w}\|_1 \leq \sqrt{m}\|\boldsymbol{w}\| \leq c_0 m\sigma$$

by assumption, upon applying Proposition 1 and setting (21) to be non-negative yield

$$\|\boldsymbol{\epsilon}\| \leq \frac{W-(N+m)(K_1\sqrt{m}\sigma+K_2 m\sigma^2)-c_0 m\sigma}{N+m}.$$

The desired result then follows by requiring the right-hand side to be positive. ■