SDR APPROXIMATION BOUNDS FOR THE ROBUST MULTICAST BEAMFORMING PROBLEM WITH INTERFERENCE TEMPERATURE CONSTRAINTS

Sissi Xiaoxiao Wu[†], Man-Chung Yue[‡], Anthony Man-Cho So[‡] and Wing-Kin Ma[⋆]

[†] School of Electrical, Computer and Energy Engineering, Arizona State Univ., Tempe, AZ, USA

[‡] Department of Sys. Eng. & Eng. Mgmt., CUHK, Hong Kong, [⋆] Department of Elec. Eng., CUHK, Hong Kong

ABSTRACT

In this work, we consider the robust beamforming design for secondary downlink multicasting channels, where primary users are present with norm-bounded channel errors. In particular, the max-min-fair formulation is considered and the resulting design problem is a quadratically constrained quadratic program (QCQP) with a set of semi-infinite constraints, which is NP-hard in general. As a remedy, we apply the semidefnite relaxation (SDR) technique and $\mathcal{S}$-lemma to approximate the problem into a tractable form. The key contribution of this paper is to study the approximation quality. Our analytical results show that, the SDR solution achieves an objective value that is at least $\Omega(\frac{1}{MN \log J})$ times the optimal objective value, where M is the number of secondary users, J is the number of primary users, and N is the number of antennas at the secondary base station. This is a fundamentally new result for SDR applied to robust QCQPs. Practically, it provides a performance guarantee for the robust beamforming design. All these results are verified by our numerical simulations.

Index Terms— MISO, multicast, semidefinite relaxation (SDR), $\mathcal{S}$-lemma, approximation bounds, robust, beamforming.

1. INTRODUCTION

The design of multi-user multi-antenna communication systems has attracted much attention in recent years. Therein, sophisticated beamforming strategies, together with spectrum sharing techniques can help accommodate more users in the same system while guaranteeing users' quality-of-service (QoS). In this work, we study the robust transmit beamforming design [1–4] for physical-layer multicasting channels with primary users. Due to the nature of the existing communication protocols, the channel state information (CSI) of the primary users (PUs) is usually difficult to be fully perceived at the secondary base station (SBS). Hence, our beamformer design should be robust against the primary channel errors.

In this work, we consider the worst-case robust design; i.e., the channel error lies in a norm bounded ball, under a max-min-fair design criterion. In particular, we aim at maximizing the worst secondary users' (SUs') signal-to-noise ratio (SNR), subject to the total power constraint at the SBS, as well as the interference temperature (IT) constraints (subject to channel uncertainty) associated with the PUs. The resulting problem is a quadratically constrained quadratic program (QCQP) with semi-infinite constraints, which is generally an NP-hard problem [1,2]. Nevertheless, the QCQP can be tackled in a unified manner by the semidefinite relaxation (SDR) technique [5] (see [6] for an overview of SDR), while the semi-infinite constraints can be reformulated as linear matrix inequalities (LMIs). The outcome of the above framework is a rank-one SDR solution, which is generally sub-optimal. Empirical results from prior works [1, 6, 7]

have suggested that SDR approximation has high approximation accuracy. Thus, a natural question is whether one can provide a provable performance guarantee for the beamforming strategy by deriving SDR approximation bounds for robust QCQPs.

There are some prior works that give provable bounds on the approximation accuracy of SDR solutions. The first is due to Chang *et al.* [8], who considered the transmit beamforming scheme for multi-group multicast and showed that the approximation accuracy of the SDR solution is on the order of $1/M$, where M is the number of users. The second is due to the authors in [9], which further stated that the approximation quality with the number of power constraints J is on the order of $1/\log J$. However, none of these results can be applied to robust cases. The difficulty lies in the presence of semi-infinite constraints, which prevents a straightforward application of the union bound in evaluating the constraint-violation probabilities. In this work, we propose to make use of the ϵ-net technique [10–12] circumvent this difficulty. Our results show that for the robust QCQP we considered, the SDR solution achieves an objective value that is at least $\Omega(\frac{1}{MN \log J})$ times the optimal objective value, where N is the number of antennas at the secondary base station. Compared with the previous bounds [8,9], the N factor here is due to the semi-infinite constraints. The intuition here is that as the dimension of the the uncertainty set increases, it is less likely that a rank-one solution can account for all channel uncertainty. To our best knowledge, this is the first provable result of SDR approximation bounds for robust QCQPs. Numerical results are provided to validate our analysis and algorithms. The organization of this paper is as follows. The system model and problem formulation are provided in Section 2. In Section 3, based on the Gaussian randomization algorithm, we establish the approximation bounds under the robust IT constraints. Numerical results are provided in Section 4 and the conclusions are given in Section 5.

Our notation is standard: $\mathbb{C}^n$ stands for the sets of complex n-dimensional vectors, respectively; $\boldsymbol{X} \succeq \boldsymbol{0}$ means that $\boldsymbol{X}$ is a complex Hermitian matrix that is positive semidefinite; $\| \cdot \|$ is the vector Euclidean norm; $|\cdot|$ is the cardinality operator; $\text{Re}\{\cdot\}$ stands for the real part; $(\cdot)^H$ denotes the Hermitian transpose; $\text{Tr}(\boldsymbol{X})$ stands for the trace of $\boldsymbol{X}$; $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{W})$ denotes the circularly symmetric complex Gaussian distribution, with mean vector $\boldsymbol{0}$ and covariance matrix $\boldsymbol{W}$.

2. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a physical-layer multicasting cognitive radio system, where the SBS, equipped with N antennas, aims to transmit a common signal to M single-antenna SUs while restricting their interference to PUs. In particular, we assume that the SUs' channels are perfectly known while the PUs' channels are partially known at the

SBS. The designed strategy should be robust against the erroneous primary channels.

2.1. Problem Formulation

Suppose that all the channels are quasi-static. We then consider the transmit design for each code block and thus we shall omit the time index from the notations henceforth. Our design criterion is to maximize the worst SU's SNR while suppressing IT to PUs, subject to the total power constraint at the SBS. For $i = 1, \ldots, M$, let $\boldsymbol{h}_i \in \mathbb{C}^N$ denote the perfectly estimated channel between the SBS and SU i. To tackle the effect of imperfect CSI, we model the actual channel between the SBS and PU j as

$$\boldsymbol{g}_j = \boldsymbol{a}_j + \boldsymbol{f}_j, \quad j = 1, \ldots, J, \tag{1}$$

where $\boldsymbol{a}_j \in \mathbb{C}^N$ is the estimated channel vector and $\boldsymbol{f}_j \in \mathbb{C}^N$ is the channel error vector. We only assume that $\boldsymbol{f}_j$ is *bounded in norm*; i.e., $\|\boldsymbol{f}_j\| \leq \delta_j$ for some given parameter $\delta_j \geq 0$. Such an assumption is rather standard for robust designs and has been adopted in many previous works; see, e.g., [13–20] and the references therein.

In this paper, we consider the transmit beamforming scheme and model the received signal of secondary user i is as

$$y_i = \boldsymbol{h}_i^H \boldsymbol{w} s + n_i, \quad \forall i = 1, 2, ..., M, \tag{2}$$

where $P > 0$ is the average transmit power of $\boldsymbol{x}$, $\boldsymbol{w} \in \mathbb{C}^N$ is the transmit beamforming vector satisfying $\|\boldsymbol{w}\|^2 \leq P$, s is a stream of data symbols with unit power (i.e., $\mathbb{E}[|s|^2] = 1$), n_i is a complex Gaussian noise with mean zero and variance σ_i^2, and $\boldsymbol{x} \in \mathbb{C}^N$ is the transmit signal. We should mention that the beamfomer $\boldsymbol{w}$ in (2) is assumed to be fixed during the whole frame length, just like most of the existing transmit beamforming studies; see, e.g., [1,13,21]. Based on the model (2), the SNR at SU i is given by $\gamma_i = |\boldsymbol{h}_i^H \boldsymbol{w}|^2 / \sigma_i^2$. At the meanwhile, the maximum IT at PU j may respectively be calculated as $\max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{w} \boldsymbol{w}^H (\boldsymbol{a}_j + \boldsymbol{f}_j)$. Then, our design problem is formulated as

$$\max_{\boldsymbol{W}} \min_{i=1,\ldots,M} \boldsymbol{h}_i^H \boldsymbol{W} \boldsymbol{h}_i$$
$$\text{s.t.} \quad \max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W} (\boldsymbol{a}_j + \boldsymbol{f}_j) \leq \eta_j, \quad j = 1, \ldots, J, \tag{3a}$$

$$\text{Tr}(\boldsymbol{W}) \leq P, \quad \boldsymbol{W} \succeq \boldsymbol{0}, \tag{3b}$$

$$\text{rank}(\boldsymbol{W}) \leq 1, \tag{3c}$$

where η_j is the prescribed IT tolerance level for PU j.

2.2. The SDR and $\mathcal{S}$-lemma Techniques

Problem (3) is NP-hard, as it encapsulates the NP-hard problem of MMF transmit beamforming for multigroup multicasting [2]. In fact, (3) describes a class of QCQP problems, for which some of the power constraints (e.g., the IT constraints) are subject to errors. Solving this problem has been well studied in the literature [15,21]. Usually, the first step is to drop the rank constraint (3c) by using the SDR. Then, at the second step, we observe that for a given j, constraint (3a) can be expressed as

$$\forall \|\boldsymbol{f}_j\|^2 \leq \delta_j^2, \quad \left(\boldsymbol{f}_j^H \boldsymbol{W} \boldsymbol{f}_j + 2\text{Re}\left\{ \boldsymbol{c}_j^H \boldsymbol{f}_j \right\} + \zeta_j \right) \leq \eta_j, \tag{4a}$$

where $\boldsymbol{c}_j = \boldsymbol{W} \boldsymbol{a}_j$, $\zeta_j = \boldsymbol{a}_j^H \boldsymbol{W} \boldsymbol{a}_j$. Then, a natural idea is to apply the $\mathcal{S}$-lemma and convert the relaxed problem into a system of linear

matrix inequalities (LMIs). As such, we can reformulate (3) as

$$\boldsymbol{W}^\star = \arg \max_{\boldsymbol{W}, \gamma, \kappa_j} \gamma \tag{5}$$

$$\text{s.t.} \quad \boldsymbol{h}_i^H \boldsymbol{W} \boldsymbol{h}_i \geq \gamma, \quad i = 1, \ldots, M,$$

$$\begin{bmatrix} \kappa_j \boldsymbol{I}_N - \boldsymbol{W} & -\boldsymbol{c}_j \\ -\boldsymbol{c}_j^H & \eta_j - \zeta_j - \delta_j^2 \kappa_j \end{bmatrix} \succeq \boldsymbol{0}, j = 1, \ldots, J,$$

$$\kappa_j \geq 0, \quad j = 1, \ldots, J,$$

$$(3a) - (3b) \quad \text{satisfied.}$$

Note that Problem (5) is convex, and actually an SDP, which is polynomial-time solvable [22]. However, since (5) is a relaxation of (3), the optimal solution $\boldsymbol{W}^\star$ could violate the rank constraint in (3c), especially when M is large. Therefore, we need a procedure that can convert an optimal solution to Problem (5) into a feasible solution to Problem (3). One simple yet powerful strategy for designing such procedure is to employ Gaussian randomization; see, e.g., [6]. Algorithm 1 shows our proposed Gaussian randomization procedure. It takes as input of the optimal solution to Problem (5) and outputs a feasible solution, as can be easily verified from Steps 2 and 6. It is worth mentioning that in Step 4, by using the triangular inequality, we can obtain ι_r^j with a closed-form expression:

$$\iota_j^\ell = \max_{\|\boldsymbol{f}_j\| \leq \delta_j} \left| (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^\ell \right|^2 = \left(\left| \boldsymbol{a}_j^H \widehat{\boldsymbol{\xi}}^\ell \right| + \delta_j \left\| \widehat{\boldsymbol{\xi}}^\ell \right\| \right)^2,$$

where the maximum is attained at $\boldsymbol{f}_j^\star = \delta_j \cdot \widehat{\boldsymbol{\xi}}^\ell / \left\| \widehat{\boldsymbol{\xi}}^\ell \right\|$.

Algorithm 1 Gaussian Randomization Procedure for Problem (3)

1: **input**: an optimal solution $\boldsymbol{W}^\star$ to Problem (5), number of randomizations NR ≥ 1
2: **for** $\ell = 1, \ldots,$ NR **do**
3:      generate $\boldsymbol{\xi}^\ell \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^\star)$
4:      set $\widetilde{\boldsymbol{\xi}}^\ell = \widehat{\boldsymbol{\xi}}^\ell / \sqrt{\max\left\{ \pi^\ell, \max_{j=1,\ldots,J} \left\{ \iota_j^\ell \right\} \right\}}$, where

$$\pi^\ell = \text{Tr}(\widehat{\boldsymbol{W}}_j)/P$$
$$\iota_j^\ell = \max_{\|\boldsymbol{f}_j\| \leq \delta_j} (\boldsymbol{a}_j + \boldsymbol{f}_j)^H \widehat{\boldsymbol{\xi}}^\ell (\widehat{\boldsymbol{\xi}}^\ell)^H (\boldsymbol{a}_j + \boldsymbol{f}_j)/\eta_j$$

5: **end for**
6: let $\ell^\star = \arg \max_{\ell=1,\ldots,\text{NR}} |\boldsymbol{h}_i^H \widetilde{\boldsymbol{\xi}}^\ell|^2$
7: **output**: a feasible solution $\hat{\boldsymbol{w}} = \widetilde{\boldsymbol{\xi}}^{\ell^\star}$

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3. APPROXIMATION ACCURACY ANALYSIS

Now, we are ready to consider the key problem in this work, which is to evaluate the quality of the solution $\hat{\boldsymbol{w}}$ returned by Algorithm 1. Although there are some works studying SDR approximation bounds for QCQPs [5,8,9,23], none of them is applicable to imperfect CSIs, i.e., with constraints in the form of (3a), to our best knowledge. In this paper, we fill this gap by proving the following theorem.

Theorem 1. *Considering Problem (5) and Algorithm 1, we have*

$$\Pr \left(\min_{i=1,\ldots,M} \boldsymbol{h}_i^H \hat{\boldsymbol{w}} \hat{\boldsymbol{w}}^H \boldsymbol{h}_i = \Omega\left(\frac{1}{MN \log J} \right) \min_{i=1,\ldots,M} \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i \right)$$
$$\geq 1 - (3/4)^{\text{NR}}, \tag{6}$$

where NR *is the number of randomizations, M is the number of SU, J is the number of PU, and N is the number of antennas.*

The proof of this theorem is Section 3.1 and Section 3.2. This result is insightful, since it implies that the proposed SDR method can guarantee an SNR performance, which in the worst case scales with M, N and J with orders of $1/M$, $1/N$ and $\log J$, respectively. It is worth noting that for the perfect CSI cases, the SDR bound only scales with M and J [9, 23]. Now we strengthen those results by showing that when CSI error is present, the number of antennas also influences the quality of the solution.

3.1. Proof of Theorem 1

We can prove (6) by determining parameters $\beta \in (0,1)$ and $\gamma_1, \gamma_2 > 1$ such that

$$
\begin{aligned}
&\Pr\Big(\min_i \big|\boldsymbol{h}_i^H \widehat{\boldsymbol{\xi}}^\ell\big|^2 \geq \beta \min_i \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i \\
&\bigcap \big|(\widehat{\boldsymbol{\xi}}^\ell)^H \widehat{\boldsymbol{\xi}}^\ell\big|^2 \leq \gamma_1 \mathrm{Tr}(\boldsymbol{W}^\star) \bigcap \max_{\|\boldsymbol{f}_j\| \leq \delta_j} \big|(\widehat{\boldsymbol{\xi}}^\ell)^H(\boldsymbol{a}_j + \boldsymbol{f}_j)\big|^2 \\
&\leq \gamma_2 \max_{\|\boldsymbol{f}_j\| \leq \delta_j}(\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^\star(\boldsymbol{a}_j + \boldsymbol{f}_j), \forall j \Big) \geq p, \quad (7)
\end{aligned}
$$

where $\widehat{\boldsymbol{\xi}}^\ell$ (cf. Step 4) is the randomized solution (may be infeasible) for randomization ℓ. The reason is that (7) implies (6) if we set $\gamma_1 = \pi^\ell$, $\gamma_2 = \max_{j=1,\ldots,J}\{\iota_j^\ell\}$ and $\widetilde{\boldsymbol{\xi}}^\ell = \widehat{\boldsymbol{\xi}}^\ell/\sqrt{\max\{\gamma_1,\gamma_2\}}$, and the resulting approximation ratio would be $\beta/\max\{\gamma_1,\gamma_2\}$, with a probability at least $1 - (1-p)^{NR}$. We now determine β, γ_1 and γ_2 as follows.

Following our previous work in [9,23] and [24, Proposition 2.1], by setting $\beta = (4eM)^{-1}$, $\gamma_1 = \log 64 \approx 4.16$ in (7) and then using the union bounds, we obtain

$$
\Pr\Big(\min_i \big|\boldsymbol{h}_i^H \widehat{\boldsymbol{\xi}}^\ell\big|^2 \leq \beta \min_i \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i \Big) \leq M \cdot e^{1+\log\beta} = 1/4;
$$

$$
\Pr\Big(\big|(\widehat{\boldsymbol{\xi}}^\ell)^H \widehat{\boldsymbol{\xi}}^\ell\big|^2 \geq \gamma_1 \cdot \mathrm{Tr}(\boldsymbol{W}^\star) \Big) \leq e^{-\frac{1}{2}\left(\gamma_1 + 2\log\frac{1}{2}\right)} = 1/4
$$

for the first two events in (7). However, the difficulty in establishing (7) lies in the robust IT constraint, since the union bound cannot be applied to a uncountable set. In this work, we introduce the ϵ-net technique [10–12] to fix this problem.

Definition 1. *[12, Chapter 5] Let $\mathcal{S}$ be a set. A subset $\mathcal{N} \subseteq \mathcal{S}$ is called an ϵ-net of $\mathcal{S}$ if for any point $x \in \mathcal{S}$, there exists a point $z \in \mathcal{N}$ such that $\|z - x\| \leq \epsilon$.*
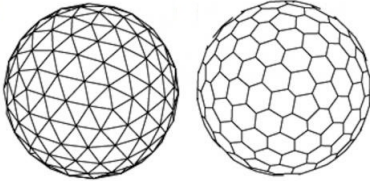
In particular, we make use of the ϵ-net to approximate the uncountably infinite set $\|\boldsymbol{f}\| = \delta$ by a finite set. The following lemma gives a upper bound of the cardinality of the ϵ-net:

Lemma 1. *Let $\mathcal{S}(\delta) \subset \mathbb{C}^n$ define a sphere with radius δ. There exists an $(\delta/2)$-net $\mathcal{N}_\delta^{\delta/2}$ on $\mathcal{S}(\delta)$ with cardinality $|\mathcal{N}_\delta^{\delta/2}| \leq 5^{2n}$.*

Proof: Lemma 1 is a direct consequence of Lemma 5.2 in [12] by putting $\epsilon = \delta/2$ and it gives the minimal cardinality of an ϵ-net of $\mathcal{S}$ (which is also called the covering number of $\mathcal{S}$ at scale ϵ.)

Based on the above results, we can have the following lemma.

Lemma 2. *Let $|\mathcal{N}_\delta^\epsilon|$ be the cardinality of an ϵ-net $\mathcal{N}_\delta^\epsilon$ of the sphere $S = S(\delta)$ of radius δ. Given $\boldsymbol{a} \in \mathcal{C}^n$ and $\boldsymbol{X}^\star \in \mathcal{H}_+^n$, let $\boldsymbol{\xi} \sim \mathcal{CN}(0, \boldsymbol{X}^\star)$. Then, for any $\kappa > 1$, $0 < \epsilon < \delta$, we have*

$$
\Pr\Big(\max_{\|\boldsymbol{f}\|\leq\delta} |\boldsymbol{\xi}^H(\boldsymbol{a}+\boldsymbol{f})| \geq \kappa\Big(\frac{1+\epsilon}{1-\epsilon}\Big)^2 \max_{\|\boldsymbol{f}\|\leq\delta}(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star(\boldsymbol{a}+\boldsymbol{f}) \Big)
$$

$$
\leq (|\mathcal{N}_\delta^\epsilon| + 1)\exp\left(-(\kappa-1)/6\right). \quad (8)
$$

Armed by this lemma, we can choose $\epsilon = \delta_j/2$ and $\gamma_2 = \left(6\log(4J(5^{2N}+1)) + 1\right)\left(\frac{2+\delta_j}{2-\delta_j}\right)^2$ to obtain

$$
\Pr\Big(\max_{\|\boldsymbol{f}_j\|\leq\delta_j} \big|(\widehat{\boldsymbol{\xi}}^\ell)^H(\boldsymbol{a}_j + \boldsymbol{f}_j)\big|^2 \leq
$$

$$
\gamma_2 \max_{\|\boldsymbol{f}_j\|\leq\delta_j}(\boldsymbol{a}_j + \boldsymbol{f}_j)^H \boldsymbol{W}^\star(\boldsymbol{a}_j + \boldsymbol{f}_j), \forall j, \Big) \leq 1/4. \quad (9)
$$

Thus, by further using the union bound, we let $p = 1 - 3/4 = 1/4$ and the approximation bound is $\beta/\max\{\gamma_1,\gamma_2\} = \beta/\gamma_2$ since we always have $\gamma_2 > \gamma_1$ in this case. This immediately leads to (6) in Theorem 1, which completes the proof.

3.2. Proof of Lemma 2

Since for any $\boldsymbol{X}^\star$, the maximum in (8) is attained at a point $\boldsymbol{f}^\star(\boldsymbol{X}^\star)$ with $\|\boldsymbol{f}^\star(\boldsymbol{X}^\star)\| = \delta$, we focus on the set

$$
\mathcal{U} = \{\boldsymbol{a} + \boldsymbol{f} : \|\boldsymbol{f}\| = \delta\}.
$$

For simplicity, let us assume that $\delta = 1$. Fixing $\boldsymbol{u} \in \mathcal{U}$, we have $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{f}(\boldsymbol{u})$ for some $\|\boldsymbol{f}(\boldsymbol{u})\| = 1$. By using the concept of the ϵ-net on the unit sphere $S = S(1)$, there exists an $\boldsymbol{f}_0(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ such that $\|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}_0(\boldsymbol{u})\| \leq \epsilon$, which implies that

$$
\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{f}_0(\boldsymbol{u}) + \epsilon_1(\boldsymbol{u})\tilde{\boldsymbol{f}}(\boldsymbol{u})
$$

for some $\|\tilde{\boldsymbol{f}}(\boldsymbol{u})\| = 1$ and $0 \leq \epsilon_1(\boldsymbol{u}) \leq \epsilon$. In this way, we can express $\boldsymbol{u}$ as

$$
\boldsymbol{u} = \boldsymbol{a} + \sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\boldsymbol{f}_k(\boldsymbol{u}),
$$

where $0 \leq \epsilon_k(\boldsymbol{u}) \leq \epsilon^k$ and $\boldsymbol{f}_k(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ for all $k \geq 0$. Continuing this fashion, by setting

$$
D = \Big(\sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\Big)^{-1},
$$

we can compute

$$
\big|\boldsymbol{u}^H \boldsymbol{\xi}\big| \leq \sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\big|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\big|
$$

and

$$
\big|(D\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\big| \leq \big|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\big| + |1-D|\big|\boldsymbol{f}^H \boldsymbol{\xi}\big|.
$$

It follows that

$$
\big|\boldsymbol{u}^H \boldsymbol{\xi}\big|^2 \leq \Big[\sum_{k\geq 0} \epsilon_k(\boldsymbol{u})\big|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\big| + |(1-D)/D|\big|\boldsymbol{a}^H \boldsymbol{\xi}\big|\Big]^2
$$

$$
\leq 2\Big(\frac{1}{D}\Big)^2 \sup_{k\geq 0}\big|(\boldsymbol{a} + \boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\big|^2 + 2\Big(\frac{1-D}{D}\Big)^2 \big|\boldsymbol{a}^H \boldsymbol{\xi}\big|^2.
$$

Observe that for any $\boldsymbol{f} \in \mathcal{N}_1^\epsilon$, we have

$$\left\{\left|(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{\xi}\right|^2\right\} \leq \kappa \cdot \left\{(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f})\right\}$$

with probability at least $1-\exp\left(-\frac{\kappa-1}{6}\right)$ [24], [9, Lemma 2]. Now, let $\boldsymbol{f}^\star = \arg\max_{\|\boldsymbol{f}\| \leq 1}(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f})$. Since $\boldsymbol{f}_k(\boldsymbol{u}) \in \mathcal{N}_1^\epsilon$ for all $\boldsymbol{u} \in \mathcal{U}$ and $k \geq 0$, the inequalities

$$\sup_{\substack{\boldsymbol{u} \in \mathcal{U} \\ k \geq 0}} \left\{\left|(\boldsymbol{a}+\boldsymbol{f}_k(\boldsymbol{u}))^H \boldsymbol{\xi}\right|^2\right\} \leq \kappa \cdot \max_{\boldsymbol{f} \in \mathcal{N}_1^\epsilon} \left\{(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f})\right\}$$

$$\leq \kappa \cdot (\boldsymbol{a}+\boldsymbol{f}^\star)^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}^\star)$$

hold with probability at least $1-|\mathcal{N}_1^\epsilon|\exp\left(-\frac{\kappa-1}{6}\right)$ for $\kappa > 1$, where the second inequality is due to the optimality of $\boldsymbol{f}^\star$.

Similarly, the inequalities

$$\left|\boldsymbol{f}^H \boldsymbol{\xi}\right|^2 \leq \kappa \cdot \boldsymbol{f}^H \boldsymbol{X}^\star \boldsymbol{f} \leq \kappa \cdot (\boldsymbol{a}+\boldsymbol{f}^\star)^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}^\star)$$

hold with probability at least $1-\exp\left(-\frac{\kappa-1}{6}\right)$ for $\kappa > 1$. Observing that $D^{-1} \leq 1/(1-\epsilon)$ and combining all the pieces together, we have

$$\max_{\|\boldsymbol{f}\|=1}\left|(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{\xi}\right|^2 \leq \kappa \left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \max_{\|\boldsymbol{f}\|=1}(\boldsymbol{a}+\boldsymbol{f})^H \boldsymbol{X}^\star (\boldsymbol{a}+\boldsymbol{f}).$$

It is easy to see that for any $\delta > 0$, we can scale the problem to $\delta = 1$. Hence, the bound (8) holds under our choices of paramters. This completes the proof of Lemma 1.

4. NUMERICAL SIMULATIONS

In this section, we provide numerical simulations to verify the theoretical results. The setup of the experiment is as follows: SU actual channels and PU estimated channels are generated by $\boldsymbol{h}_i, \boldsymbol{a}_j \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})$ independently; the noise power at each user is set to be 1; the PU channel errors are bounded by $\eta_j = 0.1, \forall j$. We solve Problem (5) using the cvx solver [25]. The number of Gaussian randomizations is set to be 1000. In all the figures, we averaged 100 channel realizations to get the plots. If the principal eigenvalue is 10^4 times larger than the second large eigenvalue, we consider the SDR solution as rank-one.

In Figure 1, we show the worst SU's SNR (Left) and the ratio of objectives associated with SDR solution and the optimal solution (Right) scaling with the number of SU, i.e., M, for different number of antennas, i.e., $N = 4$ and $N = 8$, respectively. The transmit power is set to be $P = 20$dB. We assume that there is only one PU in the system and the prescribed IT threshold η_j is set to be 1dB. In the left subfigure, we see that as M increases, the SNR performance degrades and the gap between the SNRs associated with the SDR solution and the optimal solution is enlarged. In the right subfigure, we calculate the ratio by $\frac{\min_{i=1,\ldots,M} \boldsymbol{h}_i^H \hat{\boldsymbol{w}} \hat{\boldsymbol{w}}^H \boldsymbol{h}_i}{\min_{i=1,\ldots,M} \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i}$. It shows that the ratio scales like $1/M$. Moreover, we can tell that the ratio is larger for $N = 8$ than that for $N = 4$. This is consistent with the scaling of N in Theorem 1. In Figure 2, we show the worst SU's SNR scaling with N (Left) and J (Right), respectively. In both subfigures, we set $\eta_j = 1$dB for all j. In the left subfigure, we set $P = 20$dB and $J = 1$. We see that as N increases, SNR becomes better but the gap between the two lines becomes wider. In the right subfigure, we set $P = 5$dB, $N = 4$ and $M = 32$. It shows that as J increases, SNR becomes worse and the gap between the two lines also becomes winder. These observations are consistent with the analytical results in Theorem 1.

5. CONCLUSIONS

To conclude, the robust design for physical-layer multicasting, even with primary users present in the system, has been studied for many years. The classic way is to use the SDR technique and $\mathcal{S}$-lemma, and then resort to a Gaussian randomization algorithm to find an approximate solution. However, there is one important problem remain unsolved; i.e., with the imperfect IT constraints, how to derive the approximation bound for the SDR solution? This paper answers this question by providing a rigorous proof. We have also provided numerical results to verify the analysis.
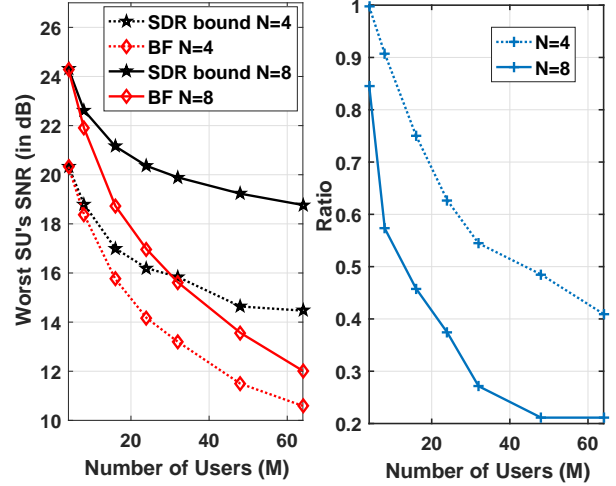

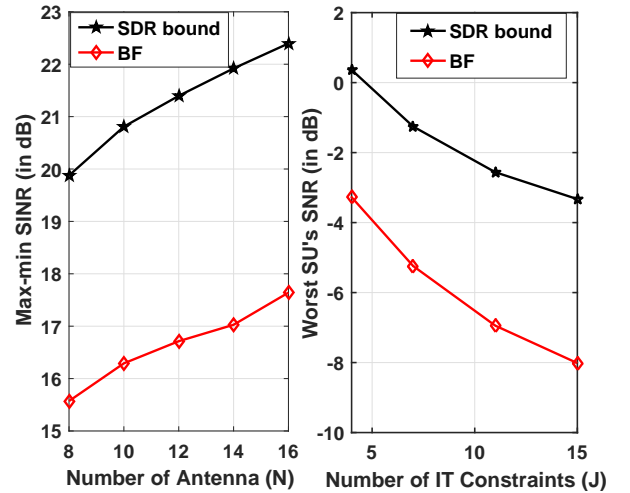**Fig. 1**. The worst SU's SNR and the approximation bound scale with M.

Fig. 2. The worst SU's SNR scales with N and J.

6. REFERENCES

[1] N. D. Sidiropoulos, T. N. Davidson, and Z.-Q. Luo, "Transmit beamforming for physical–layer multicasting," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2239–2251, Jun. 2006.

[2] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, "Quality of service and max–min–fair transmit beamforming to multiple co–channel multicast groups," *IEEE Trans. Signal Process.*, vol. 56, no. 3, pp. 1268–1279, Mar. 2008.

[3] S. X. Wu, W.-K. Ma, and A. M.-C. So, "Physical–layer multicasting by stochastic transmit beamforming and Alamouti space–time coding," *IEEE Trans. Signal Process.*, vol. 61, no. 17, pp. 4230–4245, Sep. 2013.

[4] Y. Huang, Q. Li, W.-K. Ma, and S. Zhang, "Robust multicast beamforming for spectrum sharing-based cognitive radios," *IEEE Trans. Signal Process.*, vol. 60, pp. 527–533, 2012.

[5] S. Ji, S. X. Wu, A. M.-C. So, and W.-K. Ma, "Multi–group multicast beamforming in cognitive radio networks via rank–two transmit beamformed Alamouti space–time coding," in *Proc. 2013 IEEE Int. Conf. Acoust., Speech, and Signal Process. (ICASSP)*, May 2013, pp. 4409–4413.

[6] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 20–34, May 2010.

[7] S. X. Wu, Q. Li, W. K. Ma, and A. M. C. So, "Robust transmit designs for an energy harvesting multicast system," in *2014 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2014, pp. 4748–4752.

[8] T.-H. Chang, Z.-Q. Luo, and C.-Y. Chi, "Approximation bounds for semidefinite relaxation of max–min–fair multicast transmit beamforming problem," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3932–3943, Aug. 2008.

[9] S. X. Wu, Q. Li, A. M.-C. So, and W.-K. Ma, "A stochastic beamformed amplify-and-forward scheme in a multigroup multicast MIMO relay network with per-antenna power constraints," *IEEE Trans. Wireless Commun.*, vol. 15, no. 7, pp. 4973–4986, July 2016.

[10] D. Haussler and E. Welzl, "ϵ-nets and simplex range queries," *Discrete & Computational Geometry*, vol. 2, no. 2, pp. 127–151, 1987.

[11] H. Brönnimann and M. T. Goodrich, "Almost optimal set covers in finite VC-dimension," *Discrete & Computational Geometry*, vol. 14, no. 4, pp. 463–479, 1995.

[12] R. Vershynin, "Introduction to the non-asymptotic analysis of random matrices," in *Compressed Sensing*, Y. C. Eldar and G. Kutyniok, Eds. Cambridge University Press, 2012, pp. 210–268, cambridge Books Online. [Online]. Available: http://dx.doi.org/10.1017/CBO9780511794308.006

[13] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, "Convex optimization–based beamforming: From receive to transmit and network designs," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 62–75, May 2010.

[14] K. T. Phan, S. A. Vorobyov, N. D. Sidiropoulos, and C. Tellambura, "Spectrum sharing in wireless networks via QoS–aware secondary multicast beamforming," *IEEE Trans. Signal Process.*, vol. 57, no. 6, pp. 2323–2335, Jun. 2009.

[15] G. Zheng, K.-K. Wong, and B. Ottersten, "Robust cognitive beamforming with bounded channel uncertainties," *IEEE Trans. Signal Process.*, vol. 57, no. 12, pp. 4871–4881, Dec. 2009.

[16] T.-H. Chang, W.-K. Ma, and C.-Y. Chi, "Worst–case robust multiuser transmit beamforming using semidefinite relaxation: Duality and implications," in *2011 Conf. Rec. 45th Asilomar Conf. Signals, Syst., and Comput. (ASILOMAR)*, Nov. 2011, pp. 1579–1583.

[17] Z. Chen, W. Zhang, and G. Wei, "Robust transmit beamforming for multigroup multicasting," in *Proc. 2012 IEEE Veh. Technology Conf. (VTC Fall)*, Sep. 2012, pp. 1–5.

[18] E. Song, Q. Shi, M. Sanjabi, R.-Y. Sun, and Z.-Q. Luo, "Robust SINR–constrained MISO downlink beamforming: When is semidefinite programming relaxation tight?" *EURASIP J. Wireless Commun. and Netw.*, vol. 2012, no. 243, 2012.

[19] D. Christopoulos, S. Chatzinotas, and B. Ottersten, "Weighted fair multicast multigroup beamforming under per–antenna power constraints," *IEEE Trans. Signal Process.*, vol. 62, no. 19, pp. 5132–5142, Oct. 2014.

[20] M. F. Hanif, L.-N. Tran, A. Tölli, and M. Juntti, "Computationally efficient robust beamforming for SINR balancing in multicell downlink with applications to large antenna array systems," *IEEE Trans. Commun.*, vol. 62, no. 6, pp. 1908–1920, Jun. 2014.

[21] Y. Huang, Q. Li, W.-K. Ma, and S. Zhang, "Robust multicast beamforming for spectrum sharing–based cognitive radios," *IEEE Trans. Signal Process.*, vol. 60, no. 1, pp. 527–533, Jan. 2012.

[22] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, ser. MPS–SIAM Series on Optimization. Philadelphia, PA: SIAM, 2001.

[23] S. X. Wu, A. M.-C. So, J. Pan, and W.-K. Ma, "Semidefinite relaxation and approximation analysis of a beamformed Alamouti scheme for relay beamforming networks," *submitted IEEE Trans. Signal Process.*, 2016, available online at http://arxiv.org/abs/1603.05680.

[24] A. M.-C. So, Y. Ye, and J. Zhang, "A unified theorem on SDP rank reduction," *Math. of Oper. Res.*, vol. 33, no. 4, pp. 910–920, 2008.

[25] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, Mar. 2014.