# TIGHTNESS OF SEMIDEFINITE RELAXATION FOR QUATERNION-BASED ROTATION SYNCHRONIZATION PROBLEMS

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**ABSTRACT**

The rotation synchronization problem, which estimates absolute rotations from relative measurements, is fundamental in computer vision and robotics. Unit quaternions provide a minimal singularity-free representation for 3D rotations, thereby achieving higher computational efficiency and lower storage demands. Assuming that the noise follows the standard quaternion normal distribution, we first formulate the maximum likelihood estimation problem as a non-convex quaternion quadratic program (Q-QP). Then, we derive the recovery accuracy between the Q-QP solution and the ground truth. However, locating the global optimum directly in the quaternion domain remains significantly challenging due to the non-convex landscape induced by the unit-norm constraints. To address this challenge, our main theoretical and algorithmic contribution is the development of a convex relaxation via quaternion semidefinite programming (Q-SDP). We prove that the relaxation is tight—i.e., Q-QP and Q-SDP share the same global optimum—when the noise level is below $\mathcal{O}(n^{1/4})$. For higher noise levels, we show that rank-one solutions certify optimality. Furthermore, we develop an efficient algorithm that solves the proposed Q-SDP in practice. Our numerical experiments demonstrate the efficacy of the proposed method, showing its superior rotation estimation performance.

***Index Terms***— Quaternion-based rotation synchronization, Semidefinite relaxation, Non-convex optimization, Global optimal solution, Tightness

## 1. INTRODUCTION

The rotation synchronization problem [1, 2], also known as rotation averaging [3, 4], aims to estimate a collection of rotations from noisy relative pairwise measurements. This task is fundamental to achieving global consistency in sensor network localization [5] and multi-view geometry [6], and serves as a computational cornerstone for various applications, including structure-from-motion [7], simultaneous localization and mapping (SLAM) for autonomous driving [8], and cryo-electron microscopy [9].

One core challenge in the rotation synchronization problem lies in efficiently resolving inconsistencies among local measurements to recover a globally optimal configuration, leading to a non-convex optimization problem. Various rotation parameterizations [10] necessitate trade-offs between non-convexity and constraint complexities. For instance, Euler angles and axis-angle representations are compact but suffer from singularities that hinder stable optimization. The special orthogonal group $SO(3)$ offers an intuitive matrix representation but is over-parameterized, leading to computational inefficiency in large-scale problems. In addition, the simultaneous enforcement of orthogonality and determinant constraints poses significant computational difficulties. In contrast, unit quaternions ($\mathbb{U}$) [11, 12] provide a minimal singularity-free representation of rotations. They have gained considerable attention in recent years due to their advantages in both computation and storage.

Despite these advantageous properties of quaternions, solving the quaternion-based rotation synchronization directly in the unit quaternion domain remains challenging. The unit quaternion constraint constitutes a non-convex set, making it difficult to locate the global optimum. Without considering the unit-norm constraints, Govindu [13] formulated the problem as a system of linear equations, yielding the maximum likelihood estimate (MLE). As noted in [6, 10], this approach is not structure-preserving since the variables are not constrained on $\mathbb{U}$. Subsequently, Hartley et al. [10] proposed a two-stage method, which first determines the sign of the quaternion and then solves the problem via linear least squares on a relaxed constraint. Nevertheless, this approach is generally also suboptimal. Fredriksson and Olsson [3] eliminated the unit-norm constraints by solving the Lagrangian dual problem via semidefinite programming. Although this approach enables verification of the global optimality of the solution, it lacks theoretical analysis regarding the conditions under which global optimality is guaranteed.

Recently, several classes of algorithms have been developed to tackle rotation synchronization problems over $SO(3)$. The first category is the semidefinite relaxation (SDR) approach [1, 4], which led to breakthrough results in rotation averaging [14, 15] and SLAM applications [8]. Despite strong empirical performance, existing theoretical guarantees for exact recovery via SDR remain limited. The second category is the generalized power method (GPM) [2, 16], which can be regarded as a special form of the projected gradient algorithm. The GPM employs initialization via eigen-decomposition. Although the initial point may be infeasible, the GPM ensures that the iterates remain within a basin of attraction throughout the optimization process until convergence to the global solution. Additionally, Zhu et al. [17] established global optimality guarantees with convergence analysis for the Riemannian gradient method, while Nasiri et al. [18] developed a Gauss–Newton algorithm leveraging trigonometric parameterizations and later analyzed its proximity to the global optimum in [19].

Besides $SO(3)$, SDR and GPM offer strong convergence and global optimality guarantees for synchronization over $SO(2)$ and

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the unit complex numbers, both of which characterize the planar rotation synchronization problem—commonly referred to as angular synchronization. Bandeira et al. [20] proved that the SDR remains tight with high probability for angular synchronization when the noise level is below $\mathcal{O}(n^{1/4})$. Later, Zhong and Boumal [21] significantly improved the upper bound to $\mathcal{O}(n^{1/2})$ and established that GPM converges linearly to the global optimum; cf. [22].

While complex numbers constitute a commutative subalgebra of quaternions (when restricted to $\mathrm{span}\{1, \boldsymbol{i}\}$), the critical distinction lies in the non-commutativity of general quaternion multiplication. This has long hindered the development of theoretical extensions, preventing the direct generalization of results derived from the complex field. Furthermore, the absence of high-dimensional probabilistic tools for quaternion random matrices makes theoretical analysis of global convergence particularly challenging.

**Our Contributions.** Unit quaternions provide an exceptionally elegant and efficient parameterization for 3D rotations. This work establishes a certifiably globally optimal SDR framework for unit quaternion-based rotation synchronization. As far as we know, this is the first SDR algorithm over unit quaternions that formally enjoys tightness under bounded noise regimes, thereby providing rigorous convergence guarantees and explicitly characterizing the admissible noise levels for tight relaxation and the corresponding recovery accuracy. The specific contributions are as follows:

- Under the standard quaternion normal distribution [23], the MLE of the quaternion-based rotation synchronization problem leads to a non-convex quadratic program (Q-QP), which we further relax into a convex quaternion semidefinite program (Q-SDP). Then, we establish the equivalence between (Q-SDP) and a real-valued SDP via the real representation of quaternion matrices and design the corresponding Algorithm 1.
- With new quaternion probabilistic tools and duality theory, we prove that when the noise level is $\mathcal{O}(n^{1/4})$, the non-convex (Q-QP) and the convex (Q-SDP) share the same global optimum. For higher noise levels, rank-one solutions certify optimality. We also derive recovery accuracy between the MLE and ground truth.
- In our numerical experiments, we evaluate the performance of our method by comparing it with two state-of-the-art approaches across varying problem scales and noise levels. The experiment results demonstrate that our algorithm consistently converges to the globally optimal solution under both low and high noise conditions, and it exhibits superior numerical performance.

## 2. PROBLEM FORMULATION

**Quaternion.** A quaternion number $\tilde{q} \in \mathbb{Q}$ [11] has the form $\tilde{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three imaginary units. The conjugate of $\tilde{q}$ is $\tilde{q}^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. The magnitude of $\tilde{q}$ is defined by $|\tilde{q}|^2 = \sum_{i=0}^{3} q_i^2$. The set of all unit quaternions is

$$\mathbb{U} := \{\tilde{q} \in \mathbb{Q} \mid |\tilde{q}| = 1\}.$$

In general, the multiplication of quaternions is not commutative, i.e., $\tilde{p}\tilde{q} \neq \tilde{q}\tilde{p}$ for $\tilde{p},\ \tilde{q} \in \mathbb{Q}$.

A quaternion matrix [24] $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{Q}^{m\times n}$ is denoted by $\tilde{A} = A_0 + A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$, where $A_0,\ A_1,\ A_2,\ A_3 \in \mathbb{R}^{m\times n}$. Its conjugate transpose is $\tilde{A}^* = (a_{ji}^*)$. The sets of quaternion Hermitian matrices and positive semidefinite matrices are denoted by $\mathbb{H}^{n\times n} = \{\tilde{A} \in \mathbb{Q}^{n\times n} \mid \tilde{A} = \tilde{A}^*\}$ and $\mathbb{H}_+^{n\times n} = \{\tilde{X} \in \mathbb{H}^{n\times n} \mid \tilde{X} \succeq 0\}$, respectively. For any $\tilde{A},\ \tilde{B} \in \mathbb{Q}^{m\times n}$, their inner product is defined by $\left\langle \tilde{A}, \tilde{B} \right\rangle = \mathrm{tr}(\tilde{A}^*\tilde{B})$. In general, $\mathrm{tr}(\tilde{A}\tilde{B}) \neq \mathrm{tr}(\tilde{B}\tilde{A})$. The Frobenius norm of $\tilde{A}$ is $\|\tilde{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |\tilde{a}_{ij}|^2}$.

**Synchronization Problem.** The quaternion-based synchronization problem involves recovering a collection of unit quaternions $\tilde{q} = [\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n] \in \mathbb{U}^n$ from corrupted relative measurements [10]

$$\tilde{c}_{ij} = \tilde{q}_i\tilde{q}_j^* + \sigma\tilde{W}_{ij},\ \text{ for } 1 \le i < j \le n, \tag{1}$$

where $\sigma \geq 0$ denotes the noise level and $\{\tilde{W}_{ij}\}_{i<j}$ are independent and identically distributed (i.i.d.) samples from a standard quaternion normal distribution. Defining $\tilde{W}_{ii} = 0$ and $\tilde{W}_{ji} = \tilde{W}_{ij}^*$ for consistency, the generative model (1) for the relative measurements can be written compactly as $\tilde{C} = \tilde{q}\tilde{q}^* + \sigma\tilde{W}$, where $\tilde{C},\ \tilde{W} \in \mathbb{H}^{n\times n}$ are quaternion Hermitian matrices.

Following the definition of a maximum likelihood estimator, problem (1) admits a nonlinear least-squares formulation

$$\underset{\tilde{x}\in\mathbb{U}^n}{\arg\min}\ \|\tilde{C} - \tilde{x}\tilde{x}^*\|_F^2. \tag{2}$$

It follows from

$$\begin{aligned}\|\tilde{C} - \tilde{x}\tilde{x}^*\|_F^2 &= \mathrm{tr}((\tilde{C} - \tilde{x}\tilde{x}^*)^*(\tilde{C} - \tilde{x}\tilde{x}^*)) \\ &= \|\tilde{C}\|_F^2 + \|\tilde{x}\tilde{x}^*\|_F^2 - \mathrm{tr}(\tilde{C}\tilde{x}\tilde{x}^*) - \mathrm{tr}(\tilde{x}\tilde{x}^*\tilde{C}) \\ &= \|\tilde{C}\|_F^2 + n^2 - 2\tilde{x}^*\tilde{C}\tilde{x}\end{aligned}$$

that problem (2) is equivalent to the following quadratically constrained quadratic program:

$$\underset{\tilde{x}\in\mathbb{U}^n}{\arg\max}\ \tilde{x}^*\tilde{C}\tilde{x}. \tag{Q-QP}$$

Noting that $\tilde{C}$ may not be a positive semidefinite matrix and $\mathbb{U}$ can be regarded as a three-dimensional unit sphere, problem (Q-QP) is non-convex and NP-hard in general. However, in practical scenarios [8] where the noise level $\sigma$ is bounded, the problem may exhibit hidden convexity, resulting in a benign optimization landscape that enables efficient attainment of the global optimum. This observation motivates our subsequent work.

## 3. ALGORITHM

### 3.1. Semidefinite Relaxation

Setting $\tilde{X} = \tilde{x}\tilde{x}^*$, we reformulate problem (Q-QP) as

$$\begin{aligned}\max_{\tilde{X}\in\mathbb{H}^{n\times n}}\quad & \mathrm{tr}(\tilde{C}\tilde{X}) + \mathrm{tr}(\tilde{X}\tilde{C}), \\ \text{s.t.}\quad & \mathrm{diag}(\tilde{X}) = \mathbb{1},\ \tilde{X} \succeq 0,\ \mathrm{rank}(\tilde{X}) = 1.\end{aligned} \tag{3}$$

Dropping the rank constraint then yields the following SDR:

$$\begin{aligned}\max_{\tilde{X}\in\mathbb{H}^{n\times n}}\quad & \mathrm{tr}(\tilde{C}\tilde{X}) + \mathrm{tr}(\tilde{X}\tilde{C}), \\ \text{s.t.}\quad & \mathrm{diag}(\tilde{X}) = \mathbb{1},\ \tilde{X} \succeq 0.\end{aligned} \tag{Q-SDP}$$

The relaxation approach involves a quadratic growth in the number of variables compared to the original problem (Q-QP). Nevertheless, as a convex optimization problem, (Q-SDP) guarantees the attainment of global optimizers. Clearly, if (Q-SDP) admits a rank-one optimal solution $\tilde{X}$, then it can be decomposed as $\tilde{X} = \tilde{x}\tilde{x}^*$. In this case, $\tilde{x}$ is also a global minimizer for the original non-convex problem (Q-QP), which certifies the tightness of the SDR.

**Algorithm 1** Quaternion semidefinite relaxation (Q-SDR) algorithm for solving quaternion-based synchronization problem (1).

**Input:** Quaternion Hermitian matrix $\tilde{C} \in \mathbb{H}^{n\times n}$.
1: Generate the real representation $C^R \in \mathbb{R}^{4n\times 4n}$ by (4).
2: Solve problem (R-SDP) via a SDP solver to derive $X^R$.
3: Recover the quaternion matrix $\tilde{X} \in \mathbb{H}^{n\times n}$ from $X^R$.
4: Compute $\tilde{X} = \tilde{x}\tilde{x}^*$ using quaternion SVD.
**Output:** $\tilde{x} \in \mathbb{U}^n$.

### 3.2. The Quaternion SDR Algorithm

As far as we know, there are no readily available solvers or directly applicable algorithms for (Q-SDP). In this subsection, we reformulate the quaternion SDP problem (Q-SDP) as an equivalent SDP in the real field, which can be solved using state-of-the-art solvers efficiently. Given $\tilde{A} \in \mathbb{Q}^{n\times n}$, its real representation [25] is defined by

$$A^R = \begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix}. \tag{4}$$

We define the set $\Omega \subseteq \mathbb{R}^{4n\times 4n}$ as the collection of all matrices of the form $A^R$ and

$$\bar{\Omega} := \{A^R \in \Omega \mid \mathrm{diag}(A_0) = \mathbb{1}, \mathrm{diag}(A_i) = 0, i \neq 0\}.$$

**Theorem 3.1.** *Let $\tilde{C} \in \mathbb{H}^{n\times n}$, $\tilde{X} \in \mathbb{H}^{n\times n}$. Problem* (Q-SDP) *is equivalent to the real SDP*

$$\begin{aligned} \max_{X^R \in \mathbb{R}^{4n\times 4n}} \quad & \left\langle C^R, X^R \right\rangle \\ \text{s.t.} \quad & X^R \in \bar{\Omega},\ X^R \succeq 0, \end{aligned} \tag{R-SDP}$$

*where $C^R, X^R$ are the real representations of $\tilde{C}, \tilde{X}$, respectively.*

The quaternion SDR algorithm is summarized in Algorithm 1. The main computation of Algorithm 1 lies in Step 2 and Step 4. The complexity of the former depends on the chosen SDP solver, while that of the latter depends on the quaternion SVD solver.

## 4. THEORETICAL ANALYSIS

In this section, we first evaluate the accuracy of (Q-QP) by establishing an upper bound on the magnitude of the MLE error between its solution and the ground truth quaternion $\tilde{q}$. Then, we establish the tightness of the SDR, which ensures that the rank-one solution of the convex problem (Q-SDP) yields a global optimum for the nonconvex problem (Q-QP).

### 4.1. Recovery Accuracy

We first characterize the non-adversarial noise matrix $\tilde{W}$ in Definition 4.1, which generalizes the definition in [20]. Then, we show that the characterization will hold with high probability for any quaternion Hermitian random matrices.

**Definition 4.1.** *($\tilde{q}$-discordant matrix) Let $\tilde{q} \in \mathbb{U}^n$. A quaternion matrix $\tilde{A} \in \mathbb{Q}^{n\times n}$ is called $\tilde{q}$-discordant with constants $c_1, c_2 > 0$ if it is Hermitian and satisfies both*

$$\|\tilde{A}\|_2 \leq (2 + c_1)\sqrt{n} \quad \textit{and} \quad \|\tilde{A}\tilde{q}\|_\infty \leq c_2\sqrt{n \log n}.$$

**Theorem 4.1.** *Let $\tilde{A} \in \mathbb{H}^{n\times n}$ be a quaternion Hermitian random matrix with i.i.d. off-diagonal entries following a standard quaternion normal distribution and zeros on the diagonal elements. Namely, $\tilde{a}_{ii} = 0$, $\tilde{a}_{ij} = \tilde{a}_{ji}^*$, $\mathbb{E}\tilde{a}_{ij} = 0$, and $\mathbb{E}|\tilde{a}_{ij}|^2 = 1$ when $i \neq j$. Then, with probability at least $1 - e^{-c_1^2 n/2} - (1 + c_2^2)n^{3-2c_2^2}$, $\tilde{A}$ will be $\tilde{q}$-discordant with constants $c_1, c_2 > 0$.*

In fact, the solution to problem (Q-QP) is not unique. Specifically, for any $\tilde{x} \in \mathbb{U}^n$ and $\tilde{z} \in \mathbb{U}$, it follows from $\tilde{x}^*\tilde{C}\tilde{x} = \tilde{z}^*\tilde{x}^*\tilde{C}\tilde{x}\tilde{z}$ that the available data are insufficient to distinguish $\tilde{x}$ from $\tilde{x}\tilde{z}$. However, one can always find an $\tilde{x}$ such that $|\tilde{x}^*\tilde{q}| = \tilde{x}^*\tilde{q}$. Next, we present the recovery accuracy of (Q-QP) in the following theorem.

**Theorem 4.2.** *Let $\tilde{q} \in \mathbb{U}^n$ be a unit quaternion vector and $\tilde{W} \in \mathbb{H}^{n\times n}$ be a quaternion Hermitian random matrix satisfying Theorem 4.1. Suppose that $\tilde{C} = \tilde{q}\tilde{q}^* + \sigma\tilde{W}$. If $\tilde{x} \in \mathbb{U}^n$ is a global optimizer of* (Q-QP) *that satisfies $|\tilde{x}^*\tilde{q}| = \tilde{x}^*\tilde{q}$, then $\tilde{x}$ will be close to $\tilde{q}$, i.e.,*

$$\|\tilde{x} - \tilde{q}\|_2 \leq 12\sigma,$$
$$\|\tilde{x} - \tilde{q}\|_\infty \leq 4\sigma n^{-1/2}\left(44\sigma + \sqrt{\log n}\right)$$

*with probability at least $1 - e^{-n/2} - 5n^{-5}$.*

When $\sigma = 0$, Theorem 4.2 indicates that the global optimizer is exactly the ground truth $\tilde{q}$. As $n$ grows sufficiently large, the probability approaches 1.

### 4.2. Tightness of the Semidefinite Relaxation

In this subsection, we prove that (Q-SDP) admits a unique rank-one solution under mild conditions. This result implies that Algorithm 1 is able to recover the global optimum of (Q-QP), thereby demonstrating the tightness of the SDR. Our analysis relies primarily on the dual problem of (Q-SDP) and its associated KKT optimality conditions. Specfically, the dual problem of (Q-SDP) is

$$\begin{aligned} \min_{\tilde{Y} \in \mathbb{H}^{n\times n}} \quad & \mathrm{tr}(\tilde{Y} + \tilde{C}) \\ \text{s.t.} \quad & \tilde{Y} + \tilde{C} \text{ is real diagonal},\ \tilde{Y} \succeq 0. \end{aligned} \tag{Q-DSDP}$$

**Theorem 4.3.** *The quaternion Hermitian matrices $\tilde{X}, \tilde{Y} \in \mathbb{H}^{n\times n}$ are the global optimizers of the primal problem* (Q-SDP) *and dual problem* (Q-DSDP)*, respectively, if and only if the following conditions hold:*

*(i) $\mathrm{diag}(\tilde{X}) = \mathbb{1}$;* *(ii) $\tilde{X} \succeq 0$;*
*(iii) $\tilde{Y} + \tilde{C}$ is real diagonal;* *(iv) $\tilde{Y} \succeq 0$;*
*(v) $\tilde{X}\tilde{Y} = 0$ or $\tilde{Y}\tilde{X} = 0$.*

*Furthermore, if $\mathrm{rank}(\tilde{Y}) = n - 1$, then $\tilde{X}$ is of rank one and is the unique global optimizer of* (Q-SDP).

Denote $\tilde{X} = \tilde{x}\tilde{x}^*$, $\tilde{Y} = \mathrm{Re}(\mathrm{ddiag}(\tilde{C}\tilde{x}\tilde{x}^*)) - \tilde{C}$. By employing the second-order optimality conditions of problem (Q-QP), we can show that $\tilde{X}$ and $\tilde{Y}$ satisfy the KKT conditions (i)-(v) in Theorem 4.3. However, due to the potential non-uniqueness of (Q-SDP) solutions, there is no guarantee that the rank-one solution $\tilde{X}$ is attainable. The following theorem shows that when $\sigma = O(n^{1/4})$, the point $\tilde{X}$ is the unique solution to (Q-SDP). Moreover, every second-order critical point for (Q-QP) is a global minimizer.

**Theorem 4.4.** *Let $\tilde{q} \in \mathbb{U}^n$ be a unit quaternion vector and $\tilde{W} \in \mathbb{H}^{n\times n}$ be a quaternion Hermitian random matrix satisfying Theorem 4.1. Suppose that $\tilde{C} = \tilde{q}\tilde{q}^* + \sigma\tilde{W}$ with $\sigma \leq \frac{1}{16}n^{1/4}$. If $\tilde{x} \in \mathbb{U}^n$ is a second-order critical point for* (Q-QP) *such that $\tilde{q}^*\tilde{C}\tilde{q} \leq \tilde{x}^*\tilde{C}\tilde{x}$, then with probability at least $1 - e^{-n/2} - 5n^{-5}$, the point $\tilde{X} = \tilde{x}\tilde{x}^*$ will be the unique solution of* (Q-SDP) *and $\tilde{x}$ will be a global optimizer of* (Q-QP).

## 5. NUMERICAL EXPERIMENTS

In this section, we evaluate the performance of Algorithm 1 (Q-SDR) for solving rotation synchronization problems. The problem (R-SDP) is solved by the SDPT3 solver [26] and the quaternion SVD is computed by using quaternion Householder transformations [27], whose time complexities are upper-bounded by $\mathcal{O}(n^{6.5}\ln(1/\varepsilon))$ and $\mathcal{O}(n^3)$, respectively. As a basis for comparison, we also evaluate the performance of two state-of-the-art approaches, including the chordal relaxation (Chord) method [28, 8] and the Gauss–Newton (RS) method [18, 19]. All experiments are performed using MATLAB on a notebook computer with an Intel i7-10750H CPU.

In our experiments, we employ the circular ring datasets as utilized in [28] and introduce noise through $\tilde{q}_{ij} = \tilde{q}_i^*\tilde{q}_j\tilde{q}_\epsilon$. Specifically, the rotation axis of $\tilde{q}_\epsilon$ is set as a randomly generated unit vector, while the rotation angle follows a normal distribution $\mathcal{N}(0, \sigma_r^2)$, where $\sigma_r$ characterizes the variance of the noise angle. This ensures that the measurements $\tilde{q}_{ij} \in \mathbb{U}$ and the noise level becomes more intuitive to interpret.

**Experiment Settings.** We set the magnitude of rotation noise as $\sigma_r \in (0.1, 2)$. Since the two quaternions $\pm\tilde{q}$ represent the same rotation, opposing signs in the matrix $\tilde{C}$ may lead to the absence of a rank-one solution in problem (Q-SDP). We adopt the spanning tree method from [10] to preprocess the sign of $\tilde{C}$. In addition, we use $C^R$ as an initial point for the SDP solver. The quality of recovery is measured by the relative error (Rel. Err.) and the angular distance (Ang.), which are defined as

$$\text{Rel. Err.} = \frac{\|\tilde{q} - \tilde{q}_0\|}{\|\tilde{q}_0\|} \quad \text{and} \quad \text{Ang.} = \|\operatorname{Log}(R^\top R_0)\|_2,$$

respectively. Here, $\tilde{q}$ (or $R$) is the restored rotation and $\tilde{q}_0$ (or $R_0$) is the ground truth.

**Table 1**: Numerical results for varying dataset sizes and noise levels. Q-SDR is our proposed Algorithm 1.

| | $\sigma_r = 0.1$ | | $\sigma_r = 0.5$ | | $\sigma_r = 1$ | |
|---|---|---|---|---|---|---|
| Algorithm | Rel. Err. | Ang. | Rel. Err. | Ang. | Rel. Err. | Ang. |
| | | | $n = 5$ | | | |
| Chord | 1.31E-03 | 0.31 | 3.21E-02 | 1.60 | 7.88E-02 | 2.12 |
| RS | 1.31E-03 | 0.31 | **2.58E-02** | **1.42** | 6.62E-02 | 2.15 |
| Q-SDR | **1.31E-03** | **0.31** | 2.60E-02 | 1.43 | **6.55E-02** | **2.12** |
| | | | $n = 20$ | | | |
| Chord | 3.56E-04 | 0.71 | 7.30E-03 | 3.16 | 4.31E-02 | 7.39 |
| RS | 3.56E-04 | 0.71 | 6.68E-03 | 3.03 | 2.45E-02 | 5.61 |
| Q-SDR | **3.56E-04** | **0.71** | **6.64E-03** | **3.01** | **2.33E-02** | **5.49** |
| | | | $n = 50$ | | | |
| Chord | 6.32E-05 | 0.73 | 2.63E-03 | 4.65 | 1.73E-02 | 11.65 |
| RS | **6.29E-05** | 0.73 | 2.40E-03 | 4.55 | 8.48E-03 | 8.42 |
| Q-SDR | 6.30E-05 | **0.73** | **2.40E-03** | **4.54** | **8.41E-03** | **8.36** |
| | | | $n = 100$ | | | |
| Chord | 3.98E-05 | 1.17 | 1.65E-03 | 7.52 | 2.43E-02 | 27.24 |
| RS | **3.97E-05** | 1.17 | 1.33E-03 | 6.83 | 3.02E-03 | 10.12 |
| Q-SDR | 3.98E-05 | **1.17** | **1.31E-03** | **6.78** | **2.90E-03** | **9.85** |

**Experiment Results.** The numerical results across different dataset sizes and noise levels are listed in Table 1. Under low-noise level, all three algorithms achieve certified global optimality, with relative error differences maintained at the level of $\mathcal{O}(10^{-7})$. As the noise level increases, the chordal relaxation method can only converge to a suboptimal solution due to constraint discarding during computation. By contrast, both the RS algorithm and the Q-SDR algorithm maintain a high precision. At the noise level $\sigma_r = 1$, Q-SDR outperforms the RS algorithm in both relative error and angular distance metrics.

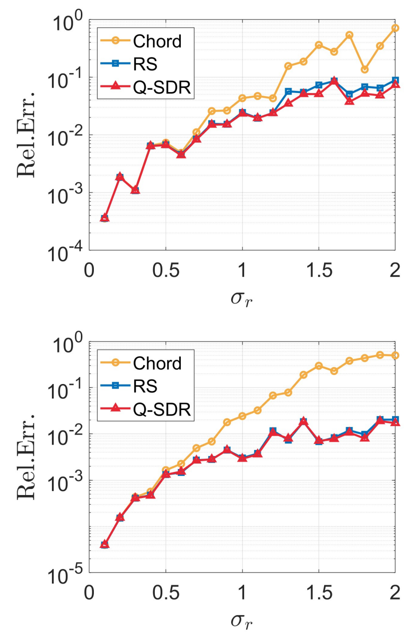


**Fig. 1**: The trend of relative error along with different noise levels. The figures correspond to the sample sizes of $n = 20$ (upper) and $n = 100$ (lower), respectively.

We also compare the recovery accuracy across more noise levels with $n = 20$ and $n = 100$. The trend analysis results are presented in Fig. 1. As the noise level increases, the performance gap among the three algorithms widens, leading to a growing disparity in their relative errors. Notably, the Q-SDR algorithm consistently maintains the lowest relative error across all noise conditions. In fact, throughout all experiments conducted in this study, the solutions obtained by Q-SDR were always tight—that is, the ratio of the second-largest eigenvalue to the largest eigenvalue remained below $10^{-4}$. This tightness guarantees that the relaxation is exact and confirms that Q-SDR consistently converges to the global optimal solution.

## 6. CONCLUSION

We have proposed a convex quaternion SDR approach for quaternion-based rotation synchronization, proving tightness under moderate noise level (below $\mathcal{O}(n^{1/4})$)—i.e., the non-convex (Q-QP) and the convex (Q-SDP) share the same global optimum. Otherwise, rank-one solutions certify optimality. We have further developed an efficient algorithm for solving the Q-SDP via its real-valued representation. Theoretically, we have derived the recovery accuracy between the estimator and the ground truth. Empirically, we have shown the superior performance of the proposed method on datasets with different noise levels. An open question remains: Is $\mathcal{O}(n^{1/4})$ the tightest possible noise bound for certifying the tightness of the relaxation?

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