American Option Sensitivities Estimation via a Generalized IPA Approach

Nan Chen

Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong,
nchen@se.cuhk.edu.hk

Yanchu Liu

Lingnan (University) College, Sun Yat-sen University, Guangzhou, China, yanchuliu@gmail.com

In this paper, we develop efficient Monte Carlo methods for estimating American option sensitivities. The problem can be re-formulated as how to perform sensitivity analysis for a stochastic optimization problem with model uncertainty. We introduce a generalized infinitesimal perturbation analysis (IPA) approach to resolve the difficulty caused by discontinuity of the optimal decision with respect to the underlying parameter. The IPA estimators are unbiased if the optimal decisions are explicitly known. To quantify the estimation bias caused by untractable exercising policies in the case of pricing American options, we also provide an approximation guarantee which relates the sensitivity under the optimal exercise policy to that computed under a suboptimal policy. The price-sensitivity estimators yielded from this approach demonstrate significant advantages numerically in both high-dimensional environments and various process settings. We can easily embed them into many of the most popular pricing algorithms without extra simulation effort to obtain sensitivities as a by-product of the option price. Our generalized approach also casts new insights on how to perform sensitivity analysis using IPA: we do not need pathwise continuity to apply it.

Key words: finance: asset pricing: American option; price sensitivities; simulation: applications; dynamic programming: applications: optimal stopping.

History:

1. Introduction

An American option entitles the holder to a right to buy or sell a certain amount of underlying assets at a pre-specified price any time up to contract maturity. It is one of the most important path-dependent options: nearly all exchange-traded stock options are American-style and many other derivatives, such as callable interest rate exotics, have this feature. Price sensitivities of such options, or Greeks as they are known in market jargon, reflect how much the option price will

2

Chen and Liu: *Monte Carlo and American Option Sensitivities*
Article submitted to *Operations Research*; manuscript no. (Please, provide the manuscript number!)

change in response to changes of market parameters. They play a vital role in risk management on options. For instance, the risk in a short position on an option can be offset effectively by the strategy that the option seller holds *delta* units of underlying assets (see, e.g., Chapter 17 of Hull (2009)), where the delta is simply the first-order partial derivative of the option price with respect to the current price of the underlying asset. Another example is the *vega*, which is defined as the rate of value change of the option with respect to the underlying asset's volatility. Traders typically rely on some calibration methods to obtain volatility estimates. Therefore, the vega measures the magnitude of the pricing error caused by a wrong estimation of the underlying volatility. Whereas option prices can often be observed in the market, their sensitivities cannot. The accurate calculation of sensitivities is arguably even more important than the calculation of prices in this sense.

American option pricing and hedging can be viewed as a typical application of optimal control theory. In a broader class of optimal control problems, a decision maker needs to determine control policies for a complex stochastic system on the basis of some parametric models for which the modeling parameters are estimated through empirical data and subject to considerable statistical error. The sensitivity of the system's expected performance with respect to the estimated parameter characterizes the effect of such "model risk" on the quality of her decision. With the help of this information, she can identify the most important system parameters and prioritize related analysis on them.

Aiming to establish American option sensitivity estimators, we develop an efficient Monte Carlo method in this paper for estimating sensitivities for stochastic control problems. The standard simulation procedure to obtain sensitivities takes differentiation on the payoff along each sample path with respect to the parameter of interest. People usually refer to this method as the *infinitesimal perturbation analysis* (IPA); see Ho and Cao (1983), Suri and Zazanis (1988), Glasserman (1991, 2004), and Asmussen and Glynn (2007) for instance. The continuity of the performance value with respect to the parameter of interest turns out to be a vital precondition to warrant the success of this method; see p. 396 of Glasserman (2004), p. 600 of Fu (2006), and Remark 7.2.6 of Asmussen and Glynn (2007). However, we find that it is no longer the case in American options: sometimes the optimal exercising policy, one crucial component in determining the eventual payoff along each sample path, changes discontinuously even under a small parameter perturbation!

To resolve this difficulty, we investigate a generalized IPA approach for studying the sensitivity analysis of the performance of a stochastic system in the presence of decision variables, especially focusing on those cases in which the optimal decisions are not sufficiently smooth in the underlying parameters. The key observation is that we can still obtain IPA-like sensitivity estimators, as long as we can show that the *expected value change* for those sample paths on which the pathwise

continuity fails is a higher order infinitesimal than the parameter's change. This observation extends the traditional IPA approach significantly: we do not need pathwise continuity to apply it. In a stochastic optimization problem, the expected value of the optimal performance is typically less sensitive to a change in the underlying parameter than the optimal decision itself. Accordingly, the expected value change resulting from a discontinuous change in the optimal decision can still be negligible. On the basis of the previous discussions, we develop sensitivity estimators for the stochastic system with decision variables, even though the parameter perturbation does create dramatic changes in the decision. We show our IPA sensitivity estimators are unbiased if the used decision is optimal, using an example from the dynamic inventory management theory.

We further proceed to apply this approach to the case of American options. Under some very mild regularity conditions, we manage to derive IPA estimators to the first-order sensitivities in the current underlying price and model parameters, respectively. For most cases of practical interest, the optimal exercising policies of American-type options are untractable. All pricing algorithms in the existing literature are subject to simulation bias and they can only produce "good" sub-optimal exercising policies at best. We should acknowledge that our IPA estimators suffer from such policy bias because they are based on these exercising policies obtained from one pricing algorithm.

We therefore perform an error analysis to attempt to quantify the magnitude of policy bias relative to the quality of input exercising policies. The analysis indicates that the IPA estimators can also achieve asymptotic unbiasedness when we increase the computational effort to infinity. Observing some existing literature about price convergence when the quality of approximate optimal exercising policy is improved (see, e.g., Clément, Lamberton and Protter (2002), Glasserman and Yu (2004), Stentoft (2004), and Belomestny (2011)), we find that convergence to the correct sensitivity values also requires an additional increment of computational effort in approximation improvement, along with the growth of the number of simulated paths. The numerical examples in the paper demonstrate that our estimators perform accurately and efficiently across a wide range of problem dimensionality and underlying asset price models.

There are two main contributions of the paper. It contributes to the literature of financial engineering first, extending the Monte Carlo Greeks proposed by Broadie and Glasserman (1996), which mainly handle European-style derivatives, to American-style ones. A huge body of literature has accumulated around American option Monte Carlo pricing; see Chapter 8 in Glasserman (2004) and the references therein for a comprehensive overview. In contrast to the full-fledged pricing literature, the research on estimating price sensitivities for American options is somewhat underdeveloped. We aim to fill this gap in the existing literature of financial engineering in this paper.

4

Chen and Liu: *Monte Carlo and American Option Sensitivities*
Article submitted to *Operations Research*; manuscript no. (Please, provide the manuscript number!)

Several papers are closely related with the main theme of this paper. Piterbarg (2003, 2004) discusses sensitivity estimators for American-style swaptions, a special kind of options based on interest rate swaps. These two papers have already developed IPA-like estimators for estimating some Greeks of American options. However, their justification, in particular the derivation of vega, is informal and highly dependent on the structure of swaptions. We improve upon these earlier works by providing theoretical justification and extend them to a broader class of optimal control problems. Belomestny, Milstein, and Shoemakers (2010) follow the works of Piterbarg to construct delta estimators based on the least-square and the finite-difference methods. Gobet (2004) develops a delta estimator under continuous-time diffusion models. His result relies on a key assumption that the option price function satisfies the smooth pasting condition; that is, it is differentiable over the exercise boundary. The current paper weakens this requirement significantly. We focus on a general discrete-time framework that is more suitable for the discussion of Monte Carlo simulation. Kaniel, Tompaidis, and Zemlianov (2008) explore using the dual pricing approach to develop confidence interval estimation for the delta and gamma. This method is appealing because it can provide true bounds on sensitivities. But like all other likelihood-ratio (LR) based methods, its application is limited by two features: it requires explicit knowledge of the relevant probability densities and its estimates often have large variance. In addition, it is very time consuming when implementing. Wang and Caflisch (2010) propose an approach based on the celebrated Longstaff-Schwartz regression method, which can be viewed as a variation of the finite-difference approximation to option sensitivities. Our numerical examples show that their method is prone to large simulation bias.

The proposed approach in this paper treats sensitivity analysis for stochastic optimization problems and therefore is more general than these alternative methods. In this sense, the results in this paper should be of interest to an audience beyond the field of option pricing. Moreover, it enjoys significant advantages over these competing methods in terms of bias, variance, and computational time. Our estimators can be easily embedded to various popular pricing methods to produce sensitivities simultaneously as a by-product of prices, no matter whether the pricing procedure is based on value approximations or direct specification of an exercise strategy. In contrast, some existing methods are only applicable for the cases in which value approximations are available.

The second contribution of this paper is to the methodology of simulation. The generalized IPA approach leads to pathwise sensitivity estimates for the system's performance when we adopt the optimal policy made under the estimated parameter value. It helps the decision maker to make use of Monte Carlo simulation to assess the effectiveness of the control in the presence of model error. The aforementioned case of vega exemplifies this situation. The underlying asset price volatility is not directly observable from the market. A large vega may raise a flag to warn the trader of a risk of model mis-specification and call for hedging needs using other financial instruments.

Several works in the simulation literature relate to this paper. L'Ecuyer (1995) (Section 3.2) and Asmussen and Glynn (2007) (Section 7.4) perform sensitivity analysis on a stochastic process stopped by a random time. Constrained by the traditional IPA framework, they assume that the random time has no pathwise dependence on the parameter of interest. Some recent papers, such as Liu and Hong (2010) and Wang, Fu, and Marcus (2009), investigate sensitivity estimation for some financial applications in which sample pathwise differentiability is absent. In particular, they are interested in barrier options whose payoffs are defined by the random time when an underlying asset price crosses a pre-specified level. This paper complements the preceding literature by adding another dimension that shows how to develop sensitivity estimators when the pathwise continuity is lost due to the discontinuity of an optimal decision.

The remainder of the paper is organized as follows. Section 2 introduces the notations to set up the American option pricing problem. We present the main results in Sections 3 and 4. Section 3 consists of two parts. The first part is about the generalized IPA approach in a generic setting and the second part uses one example from dynamic inventory management to illustrate the unbiasedness of the IPA sensitivity estimators when the optimal decision is tractable. In Section 4, we apply the approach to develop sensitivity estimators for American options and also discuss some implementation issues, especially an analysis of the error due to suboptimal exercising policy approximation. Some numerical experiments are presented in Section 5. We defer the proofs of all main theorems to the E-Companion. For completeness, another important sensitivity estimation method, the LR method, is investigated rigorously in the E-Companion. The LR method is very helpful for establishing unbiased estimators for second-order sensitivities. There we introduce a mixed approach based on a combination of the LR method and our generalized IPA approach to avoid a non-smooth difficulty when we derive the second-order sensitivities.

2. Formulation of the American Option Pricing Problem

In this section, we set up the notations for the American option pricing problem. From now on, we use $\mathbb{R}_+^d$ to denote $[0, +\infty)^d$. For any pair of vectors $a, b \in \mathbb{R}_+^d$, denote their inner product to be $a^{\mathtt{T}} \cdot b$, where the superscript $\mathtt{T}$ indicates the transpose of a vector. Given a multivariate differentiable function $f(x) : \mathbb{R}_+^d \to \mathbb{R}$, denote its gradient vector by ∇f, i.e., $\nabla f(x) = (\partial f / \partial x_1, \cdots, \partial f / \partial x_d)^{\mathtt{T}}$.

Consider an American option issued at time 0 and maturing at time $T > 0$. The option holder is allowed to exercise at a fixed set of dates: $0 < t_1 < ... < t_N = T$. Without loss of generality, we assume that all exercise times are evenly distributed on $[0, T]$, i.e., $t_i - t_{i-1} = \Delta t = T/N$. Let $X^{\theta} = \{X_i^{\theta}, 0 \le i \le N\}$ be a Markov process valued on $\mathbb{R}_+^d$, representing the values of d underlying assets over the time grid. A generic parameter θ affects the evolution of X and we are interested in the option price sensitivities with respect to θ and the current underlying price X_0^{θ}, respectively.

6

Chen and Liu: *Monte Carlo and American Option Sensitivities*
Article submitted to *Operations Research*; manuscript no. (Please, provide the manuscript number!)

Denote $\mathcal{F} = \{\mathcal{F}_i, 0 \leq i \leq N\}$ to be the information filtration generated by X. Suppose that $h : \{1, \cdots, N\} \times \mathbb{R}_+^d \to \mathbb{R}_+$ is the (discounted) payoff function to the option holder from exercising. Then, the problem of pricing the American option can be formulated as

$$Q_0(x; \theta) = \sup_{\tau \in \mathcal{T}} E\left[h(\tau, X_\tau^\theta)|X_0^\theta = x\right] \tag{1}$$

for $x \in \mathbb{R}_+^d$, where $\mathcal{T}$ is a class of stopping times valued at $\{t_1, \cdots, t_N\}$ adaptive to $\mathcal{F}$. Some literature would like to call the above security *Bermudan* options and restrict the term *American* to those with continuously exercisable features. We do not stress this subtle difference between the two terms because Monte Carlo simulation in principle can only cope with the former.

The theoretical foundation for a variety of simulation methods is the following dynamic-programming characterization of the option value. Let $Q_i(x; \theta)$ be the option value at t_i given $X_i^\theta = x$, assuming that the option has not been exercised previously. Then, we have a recursion as follows to determine $Q_i's$:

$$Q_N(x; \theta) = h(N, x); \tag{2}$$

$$Q_i(x; \theta) = \max\{h(i, x), E[Q_{i+1}(X_{i+1}^\theta; \theta)|X_i^\theta = x]\}, \ 1 \leq i \leq N - 1; \tag{3}$$

$$Q_0(x; \theta) = E[Q_1(X_1^\theta; \theta)|X_0^\theta = x]. \tag{4}$$

In words, Eq. (2) states that the option value at maturity should equal the payoff at that time because no further exercising opportunities are left; Eq. (3) states that in an intermediate time step, the option value is the maximum of the immediate exercise payoff and the expected present value of continuing. In Eq. (4) we assume that there is no exercise at the initial time 0. Furthermore, an optimal stopping rule can be constructed if we specify that

$$\tau^* = \min\{i \in \{1, 2, \cdots, N\} : h(i, X_i^\theta) \geq C_i(x; \theta)\}, \tag{5}$$

where $C_i(x; \theta) := E[Q_{i+1}(X_{i+1}^\theta; \theta)|X_i^\theta = x], \ 1 \leq i \leq N - 1$, and $C_N(x; \theta) = 0$. Eq. (5) simply asserts that the option holder should exercise her option the first time the payoff exceeds what she can get by continuing to hold it.

To solve Q_0 through the recursions (2-4) using Monte Carlo, one key step lies in how to evaluate the continuation values $C_i(x; \theta)$ in an efficient manner. Some algorithms start from approximations of the option value function. They use a pilot program to generate $\tilde{C}_i$, an approximate continuation value function at t_i, for all $1 \leq i \leq N$, and then determine the value function Q_i approximately by substituting $\tilde{C}_i$ in (3). See Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999, 2001), and Broadie and Glasserman (2004) for the representatives of this approach. Sometimes, it may be possible to develop a good exercising policy independently of continuation value approximations.

For instance, Andersen and Andreasen (2001) find that the suboptimal exercise strategies derived from a best-fit single-factor model result in only a very insignificant loss for Bermudan swaptions. Svenstrup (2005) uses relatively crude low-dimensional approximations in finite difference grids to produce excellent exercise policies for Bermudan swaptions in a multifactor world.

In light of these two approaches widely used in the literature of American option Monte Carlo pricing, our sensitivity estimators demonstrate a great flexibility because they are compatible with both of them, as shown in Section 4. One can easily implement the estimators as a complementary part of the pricing algorithm to produce sensitivities estimates without re-simulation at multiple parameter values.

We need to impose some additional structures on X^θ to discuss Monte Carlo sensitivity estimators. Suppose that $\{R_i, 1 \leq i \leq N\}$ are independent random variables in $\mathbb{R}^n$. The parameter θ is valued in $\Theta \subseteq \mathbb{R}$. Given X_{i-1}^θ, there exists a function $F_i : \mathbb{R}_+^d \times \mathbb{R}^n \times \Theta \to \mathbb{R}_+^d$ such that the value of X_i^θ is defined recursively by

$$X_i^\theta = F_i(X_{i-1}^\theta, R_i; \theta) = \left(F_{i,1}(X_{i-1}^\theta, R_i; \theta), \cdots, F_{i,d}(X_{i-1}^\theta, R_i; \theta)\right)^{\mathsf{T}} \tag{6}$$

for all $1 \leq i \leq N$. Furthermore, we require the differentiability of F:

Assumption 2.1 *The function F_i is Lipschitz continuous in x and θ, respectively. In other words, at any fixed $\theta \in \Theta$ and $r \in \mathbb{R}^n$, there exists a constant $K_i(r, \theta)$ such that*

$$\|F_i(x, r; \theta) - F_i(y, r; \theta)\| \leq K_i(r, \theta)\|x - y\|$$

for all $x, y \in \mathbb{R}^d$; at any fixed $x \in \mathbb{R}_+^d$ and $r \in \mathbb{R}^n$, there exists a constant $G_i(x, r)$ such that

$$\|F_i(x, r; \theta_1) - F_i(x, r; \theta_2)\| \leq G_i(x, r)|\theta_1 - \theta_2|$$

for all $\theta_1, \theta_2 \in \Theta$. The Lipschitz constants satisfy that $E[\sup_{\theta \in \Theta} K_i(R_i, \theta)] < +\infty$ and $E[\sup_{\theta \in \Theta} G_i(X_{i-1}^\theta, R_i)] < +\infty$ for any $\theta \in \Theta$. In addition, the partial derivatives $\partial F_{i,j}(x, y; \theta)/\partial x_k$ and $\partial F_{i,j}(x, y; \theta)/\partial \theta$ exist for all $1 \leq i \leq N$ and $1 \leq j, k \leq d$.

The settings in many financial applications satisfy the preceding assumption. Take the geometric Brownian motion (GBM), a benchmark model in finance, as an example. It follows a stochastic differential equation $dS_t/S_t = \mu dt + \sigma dW_t$, $S_0 = s$, where $\{W_t\}$ is a standard Brownian motion. Over the time grid $\{t_0, t_1, \cdots, t_N\}$, the recursive dynamic of S is given by

$$S_{t_i} = S_{t_{i-1}} \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}})\right). \tag{7}$$

In this case, choose $R_i := W_{t_i} - W_{t_{i-1}} \sim N(0, \Delta t)$ and $F_i(x, r; \mu, \sigma) = x \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma r\right)$. It is straightforward to verify that Assumption 2.1 covers this model.

In contrast to pure diffusion models represented by GBM, Assumption 2.1 also puts some popular pure jump or jump-diffusion mix models under its umbrella. An example is the variance gamma (VG) process proposed by Madan, Carr, and Chang (1998). It is a pure jump process and allows for more flexible skewness and kurtosis than GBM does. The discrete version of this process is given by

$$S_{t_i} = S_{t_{i-1}} \exp\left(\mu\Delta t + \theta G_i + \sigma\sqrt{G_i}V_i\right), \quad 1 \le i \le N, \tag{8}$$

where $G_i \sim \text{Gamma}(\Delta t/\beta, \beta)$, a gamma distributed random number with scale $\Delta t/\beta$ and shape β, and $V_i \sim N(0, 1)$. The accompanying function F_i under this model is $F_i(x, g, v; \mu, \sigma, \theta) = x\exp\left(\mu\Delta t + \theta g + \sigma\sqrt{g}v\right)$, which maps the current price $S_{t_{i-1}}$, a random vector $R_i = (G_i, V_i)$, and two parameters μ and σ into S_{t_i}. We will discuss more underlying models in the numerical experiments of Section 5.

With a realization of $\{R_i, 1 \le i \le N\}$ held fixed, think of X^θ as a function that maps θ and the initial value x to a stochastic process. Assumption 2.1 implies that this mapping is smooth with respect to θ and x. By the chain rule of differentiation,

$$\frac{\partial X_{i,j}^\theta}{\partial x_k} = \sum_{l=1}^{d} \frac{\partial F_{i,j}}{\partial x_l}(X_{i-1}^\theta, R_i; \theta) \cdot \frac{\partial X_{i-1,l}^\theta}{\partial x_k} \tag{9}$$

and

$$\frac{\partial X_{i,j}^\theta}{\partial \theta} = \sum_{l=1}^{d} \frac{\partial F_{i,j}}{\partial x_l}(X_{i-1}^\theta, R_i; \theta) \cdot \frac{\partial X_{i-1,l}^\theta}{\partial \theta} + \frac{\partial F_{i,j}}{\partial \theta}(X_{i-1}^\theta, R_i; \theta) \tag{10}$$

for $1 \le i \le N$, $1 \le j \le d$, and $1 \le k \le d$. When $i = 0$, we have $\partial X_{0,j}^\theta / \partial x_k = \delta_{jk}$ and $\partial X_{0,j}^\theta / \partial \theta = 0$, where δ_{jk} is the Kronecker symbol, equal to 1 if $j = k$ and 0 otherwise.

Denote

$$Y_i^k := \left(\frac{\partial X_{i,1}^\theta}{\partial x_k}, \cdots, \frac{\partial X_{i,d}^\theta}{\partial x_k}\right)^\mathsf{T} \quad \text{and} \quad Z_i := \left(\frac{\partial X_{i,1}^\theta}{\partial \theta}, \cdots, \frac{\partial X_{i,d}^\theta}{\partial \theta}\right)^\mathsf{T}.$$

The two processes $Y^k = \{Y_i^k, 0 \le i \le N\}$ and $Z = \{Z_i, 0 \le i \le N\}$ record how the impacts of value changes in x and θ will be propagated over time along each sample path of X^θ. We call these two processes the *derivative processes* of X^θ from now on. They constitute an essential component in the estimators we will build up later. Similar processes have been used in the literature to obtain sensitivity for models involving continuous-time stochastic differential equations; see Example 7.2.7 of Asmussen and Glynn (2007), Fournié et al. (1999), and Chen and Glasserman (2007). Processes Y and Z here can be viewed as their counterparts in a discrete-time setting.

The recursions in (9) and (10) are very helpful in generating Y and Z along each sample path of X^θ with little extra effort. We can generate $\{R_i, 1 \le i \le N\}$ one by one and substitute them into (6) to produce a sample path of X^θ; at the same time, plugging the obtained $\{R_i\}$ and $\{X_i^\theta\}$ into (9) and (10) will yield paths of Y and Z.

3. A Generalized IPA Approach in the Presence of a Decision Variable

This section is devoted to presenting how to obtain unbiased Monte Carlo estimators for the price sensitivities with respect to x and θ, i.e.,

$$\frac{\partial Q_0}{\partial x_j}(x;\theta) = \frac{\partial}{\partial x_j} E\left[h(\tau^*, X_{\tau^*}^\theta)\Big| X_0 = x\right] \quad \text{and} \quad \frac{\partial Q_0}{\partial \theta}(x;\theta) = \frac{\partial}{\partial \theta} E\left[h(\tau^*, X_{\tau^*}^\theta)\Big| X_0 = x\right],$$

where τ^* is the optimal exercise rule given by (5). As noted in the introduction, we find that the optimal stopping time τ^* changes discontinuously in response to small perturbations in parameters. This poses a fundamental difficulty for us to apply directly the IPA approach in the traditional simulation literature. To resolve it, we develop a generalized theoretical framework in Section 3.1 to investigate the issue of sensitivity analysis in the presence of decision variables, especially for those cases in which the optimal decisions change discontinuously to the underlying parameters. We use an example from the dynamic inventory management theory in Section 3.2 to show our IPA sensitivity estimators are unbiased if the optimal decision is explicitly known.

To demonstrate the difficulty in a more concrete way, we use the delta of an American put under the GBM model (7) as a showcase. The holder receives $\max\{K - S_t, 0\}$ when she chooses to exercise the option at t. From this payoff, it is apparent to see that the holder has a strong incentive to exercise if the option is deeply in-the-money, i.e., S_t is sufficiently low. Figure 1 shows the optimal exercise boundary in this case.
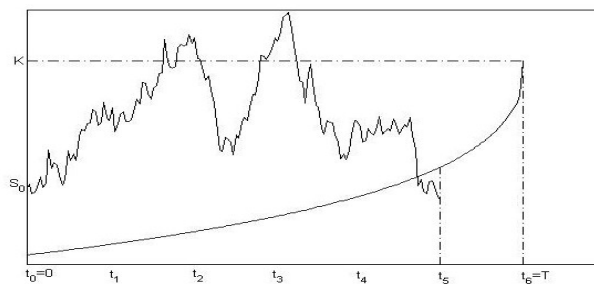

Figure 1   The exercise boundary for an American put. It is defined as the critical value B_i^* satisfying that $\max\{K - B_i^*, 0\} = E[Q_{i+1}(S_{t_{i+1}})|S_i = B_i^*]$ for all $1 \le i \le N - 1$. The holder should exercise the option at $\tau^* = \min\{i \in \{1, \ldots, N\} : S_{t_i} \le B_i^*\}$. The optimal exercising time $\tau^* = t_5$ for the sample path shown in the figure.

Fixing a realization of the Brownian motion $\{W_t, 0 \le t \le T\}$, we perturb the initial price S_0 with a small quantity ΔS_0. Figure 2 illustrates two possible consequences of the small perturbation to the optimal exercising time τ^*. For some sample paths of W as shown in the left plot, it changes not the value of τ^*, but the value of S at τ^*. For such W as the one in the right plot, the small perturbation ΔS_0 results in a value change even for τ^*. The discontinuity on τ^* in the right plot
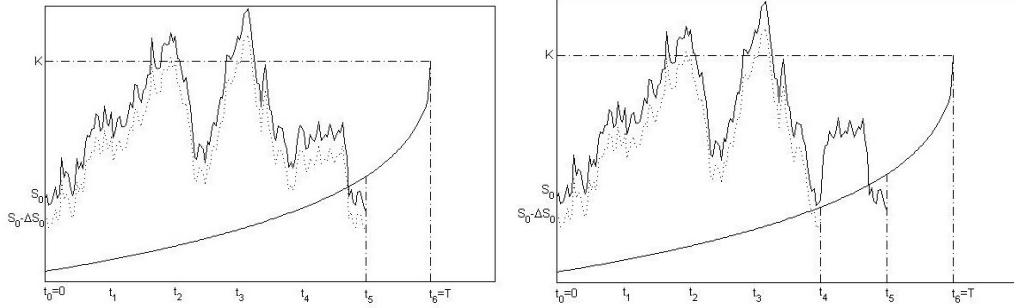

Figure 2   The effects of a small perturbation on S_0. In the left plot, it changes the payoff value only. The perturbed underlying process (shown by the dotted curve) has the same optimal exercising time as the unperturbed one (the solid curve). The right plot shows that the perturbation can change τ^* for some sample paths. Originally the holder should exercise the option at t_5. But, with a small perturbation ΔS_0, she should exercise the option at t_4.

prevents us from taking differentiation inside the expectation directly to obtain

$$\frac{d}{dS_0} E\left[\max\{0, K - S_{\tau^*}\} \Big| S_0\right] = E\left[\frac{d}{dS_0} \max\{0, K - S_{\tau^*}\} \Big| S_0\right].$$

A more general approach is therefore needed to address this issue.

3.1. The Estimators

In view of the aforementioned obstacle, we develop a generalized IPA approach in this subsection to estimate value sensitivities in the presence of a decision variable. Suppose that we have a probability space $(\Omega, \mathcal{F}, P)$ and an action space Ψ (it may or may not be Euclidean). Consider a system whose performance measure, $L(\psi, \xi, \theta) : \Psi \times \mathbb{R}^n \times \Theta \to \mathbb{R}$, depends on a random factor ξ defined on the probability space, a model parameter θ, and more importantly, a decision variable $\psi \in \Psi$. One decision maker solves the following maximization problem

$$\alpha(\theta) := \sup_{\psi \in \mathcal{A}} E[L(\psi, \xi, \theta)] \tag{11}$$

by selecting an optimal ψ^*, where $\mathcal{A}$ is a subset of all Ψ-valued $\sigma(\xi)$-measurable random variables. Assume that for any $\theta \in \Theta$, there exists a solution $\psi^*(\theta) \in \mathcal{A}$ to the optimization problem (11). For a fixed $\theta_0 \in \Theta$, define

$$\alpha'(\theta_0) := \lim_{\Delta\theta \to 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{E[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta)] - E[L(\psi^*(\theta_0), \xi, \theta_0)]}{\Delta\theta}. \quad (12)$$

We are interesting in finding estimators for $\alpha'(\theta_0)$, which serves as an important measure to assess the impact of a mis-specification in the modeling parameter θ to the system performance.

It is straightforward to see that the problem of American option sensitivities is a special example of this setting, simply noting that L is now corresponding to the option payoff function and the random factor ξ consists of $\{R_1, \cdots, R_N\}$. The optimal τ^* is chosen from the set of stopping times $\mathcal{T}$, which is of course a subset of all $\sigma(R_1, \cdots, R_N)$-measurable random variables. To emphasize the difficulty we encounter in American options, we do not assume differentiability of the optimal $\psi^*(\theta)$ with respect to the parameter θ, excluding the possibility of using the vanilla IPA approach to obtain unbiased estimators directly.

The main result is summarized in the following theorem:

Theorem 3.1 *Suppose that*

(i)
$$E\left[\sup_{\psi \in \Psi} \sup_{\theta_1, \theta_2 \in \Theta} \frac{|L(\psi, \xi, \theta_1) - L(\psi, \xi, \theta_2)|}{|\theta_1 - \theta_2|}\right] < +\infty;$$

(ii) given any $\psi \in \Psi$ and $\theta \in \Theta$, $\partial L(\psi, \xi, \theta)/\partial\theta$ exists with probability 1;

(iii) there exists a constant δ such that

$$\lim_{\Delta\theta \to 0} E\left[\sup_{\tilde\theta \in (\theta_0 - \delta, \theta_0 + \delta)} \left|\frac{\partial L}{\partial\theta}(\psi^*(\theta_0 + \Delta\theta), \xi, \tilde\theta) - \frac{\partial L}{\partial\theta}(\psi^*(\theta_0), \xi, \tilde\theta)\right|\right] = 0.$$

Then, $\alpha'(\theta_0)$ exists and

$$\alpha'(\theta_0) = E\left[\frac{\partial L}{\partial\theta}(\psi, \xi, \theta_0)\bigg|_{\psi = \psi^*(\theta_0)}\right].$$

Theorem 3.1 sheds new insights into the problem of IPA sensitivity estimation in the presence of decision variables. To understand this result, given the existence of $\alpha'(\theta)$, let us conduct the following decomposition on it:

$$\begin{aligned}
\alpha'(\theta_0) &= \lim_{\Delta\theta \to 0} \frac{E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta)\right]}{\Delta\theta} \\
&+ \lim_{\Delta\theta \to 0} E\left[\frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta}\right] \\
&=: \text{Term 1} + \text{Term 2}, \quad\quad\quad (13)
\end{aligned}$$

by simultaneously adding and subtracting $E[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta)]$ on the right hand side of (12).

This decomposition reflects the discussion following Figure 2. In particular, Term 2 captures the essence of the scenario illustrated by the left plot: the parameter change does not affect the optimal decision, i.e., $\psi^*(\theta_0 + \Delta\theta) = \psi^*(\theta_0)$. Conditions (i) and (ii) in the theorem ensure that L is sufficiently smooth in θ with the optimal decision ψ^* being unchanged before and after perturbation. Therefore, we can invoke the traditional IPA method to take derivative under the expectation to get

$$\text{Term } 2 = E\left[\frac{\partial L}{\partial \theta}(\psi, \xi, \theta_0)\Big|_{\psi = \psi^*(\theta_0)}\right].$$

As noted at the beginning of this subsection, applying the IPA method directly on Term 1 is not appropriate, because the optimal decision ψ^* now changes discontinuously along such sample paths as shown in the right plot of Figure 2. However, we show in the proof that the magnitude of the expected change in the optimal function value, if we perturb the optimal decision from $\psi^*(\theta_0)$ to $\psi^*(\theta_0 + \Delta\theta)$, is actually a higher order infinitesimal than $\Delta\theta$. That is,

$$E[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta)] = o(\Delta\theta), \tag{14}$$

or in other words, Term 1 is actually zero. This can be viewed as a reminder of the celebrated first-order condition in mathematical optimization: due to the optimality of ψ^*, a small perturbation on it just leads to a negligible first-order impact on the expected value of L. We thereby can still obtain an IPA-like sensitivity estimator.

Eq. (14) and its relation to the IPA estimators point to an observation that may have broader implications beyond the context of American option sensitivities. To ensure that we can take differentiation under expectations to obtain IPA estimators, the existing simulation literature always emphasizes the importance of continuity of the payoff with respect to the underlying parameter. By Theorem 3.1, we complement this postulation with a new discovery that the conventional IPA estimator is still valid even in the absence of such continuity along each sample path. As long as, for the non-smooth part, the *expected value change* of the payoffs is in a higher order magnitude than the change of the parameter, the non-smoothness will have no contribution to the final sensitivity estimates.

It is worth stressing the indispensable role of the optimality condition in ψ^* in yielding a vanishing Term 1. If we drop this condition, the expected value change is generally in the same order of $\Delta\theta$ and the limit of Term 1 is therefore not zero. As shown by the following example, we may end up with a much more complicated estimator in absence of the optimality.

Example 3.1 *Consider a two-period time horizon $\{t_0, t_1, t_2\}$. Assume that the underlying S_t starts from $S_0 = s > K$ and it follows the discrete GBM (7), i.e.,*

$$S_{t_1} = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\xi_1\right) \quad and \quad S_{t_2} = S_{t_1} \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\xi_2\right)$$

for independent $\xi_1, \xi_2 \sim N(0, \Delta t)$. Let $\psi = \inf\{t \in \{t_1, t_2\} : S_t < K\} \wedge t_2$ (assume that $\inf \varnothing = \infty$). Define the performance measure function to be $L(\psi, \xi_1, \xi_2, s) = \max\{K - S_\psi, 0\}$. Appendix EC.4 shows

$$\frac{d}{ds}E[L(\psi, \xi_1, \xi_2, s)] = -E\left[\frac{dS_{t_1}}{ds}1_{\{\psi=t_1\}}\right] - E\left[\frac{dS_{t_2}}{ds}1_{\{\psi=t_2; S_{t_2}<K\}}\right]$$
$$+ E[(K - S_{t_2})^+ | S_{t_1} = K] \cdot \frac{K}{s} \cdot g_1(K; s), \tag{15}$$

where $g_1(K; s)$ is the probability density of S_{t_1} at $S_{t_1} = K$, given $S_0 = s$.

It is straightforward to see that the first two terms on the right hand side of (15) are obtained by fixing the value of ψ and taking differentiation on S_ψ with respect to s. The absence of the optimality of ψ introduces an additional term, the third summand in (15), to the final expression.

3.2. Unbiasedness of IPA Estimators: An Example from Dynamic Inventory Management

Theorem 3.1 suggests that our generalized IPA estimators will be unbiased under some stochastic optimization problems in which determining optimal decision policy is tractable. We provide one classical example from the literature of dynamic inventory management in this subsection to corroborate this observation.

Consider a dynamic version of the newsvendor problem over N periods (see, e.g., Veinott (1965a,b), Tsitsiklis (1984), and Porteus (2002) for details). The vendor manages a single-product inventory to meet random demands from her customers. As shown in Figure 3, she decides how much to order, if any, at the beginning of each period. A proportional ordering cost of c per unit is thus incurred, and orders placed are received immediately. For $0 \le i \le N - 1$, we denote x_i to be the initial inventory level at the beginning of period i, which connotes leftover stock when positive and the backlog when negative. Then the vendor makes an order of quantity u_i to increase the inventory level to y_i, i.e., $y_i = x_i + u_i$. Here we assume all admissible sizes of orders are in $\mathcal{U} \subseteq \mathbb{R}_+$. The demand of period i, D_i, is observed and the vendor will meet such demand based on the currently available inventory level y_i. The inventory level at the beginning of period $i + 1$ drops down to $x_{i+1} = y_i - D_i$. Demands in different periods are supposed to be independently and identically distributed with a common cumulative distribution function Φ_D. For simplicity, assume that backlogs must be met first before any future demands can be satisfied.
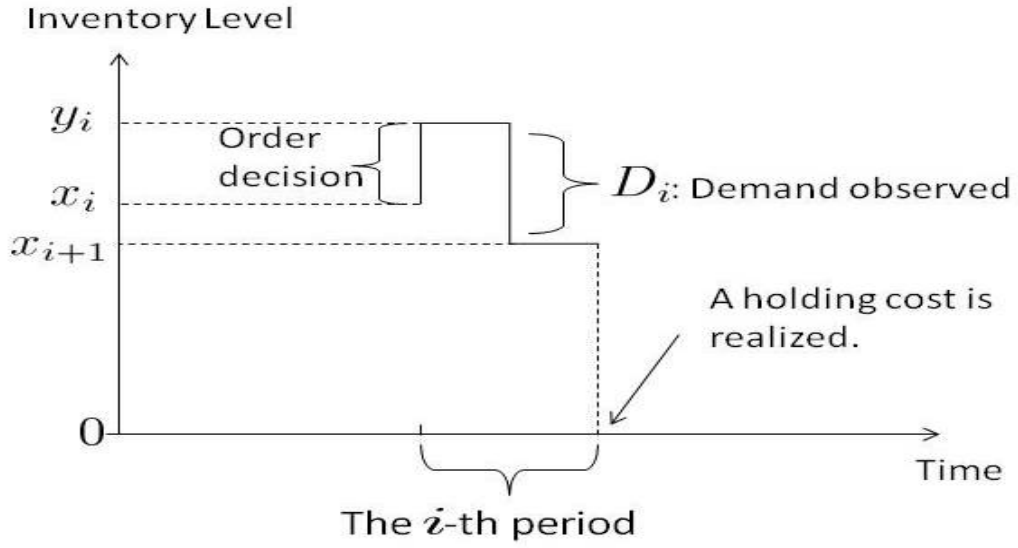
Figure 3   Timeline of the dynamic inventory management problem. At the beginning of the i-th period, the initial
                inventory is x_i. The vendor then increases the inventory level to y_i. A random demand of D_i is realized
                in the middle of the period. If $y_i \geq D_i$, all demands are satisfied and a proportional holding cost is
                incurred at the end of the period. If $y_i < D_i$, unsatisfied demands are backlogged and will cause a
                penalty cost. This figure shows a situation in which all demands are met and there is a positive leftover
                stock at the beginning of the next period.

The cost structure of the problem is as follows. Any unsold stock at the end of one period will
be kept in inventory and used for meeting future demands in the following periods. Each unit of
leftover stock causes a holding cost c_H per period, which is charged at the end of each period.
Excess demands, if any, will be backlogged in this period and satisfied at the beginning of the next
period when the vendor makes new orders. The backlogging penalty cost is c_P per unit. At the
end of the final period $N - 1$, the vendor can obtain a salvage value of c for each unit of leftover
inventory or must make an additional order immediately (at the usual unit cost c) to satisfy any
outstanding demands. That is, the terminal (salvage) value is

$$v_N(x) = -cx,$$

where $x = y_{N-1} - D_{N-1}$. Suppose that there is a discount factor per period α and assume $c_P >
(1 - \alpha)c$ and $c_H + (1 - \alpha)c > 0$ to exclude trivial solutions.

The objective of the vendor can be formulated as finding optimal inventory levels y_i for each
period to minimize the total expected cost

$$\min_{\{u_i \in \mathcal{U}: 0 \leq i \leq N-1\}} E\left[\sum_{i=0}^{N-1} \alpha^i \left(cu_i + c_H(x_i + u_i - D_i)^+ + c_P(D_i - x_i - u_i)^+\right) - \alpha^N cx_N\right], \qquad (16)$$

where x_0 is a known constant to represent the initial inventory at time 0. Veinott (1965a,b), Tsitsiklis (1984), and Porteus (2002) show that there exists a tractable optimal policy, known as the *base stock* policy, such that the vendor should ensure that the stock level after ordering is as close to the base stock level as possible.

From the discussion in Appendix EC.5, we can easily see that the optimal policy is highly dependent on the distribution function of random demands. In practice, people rely on statistical estimation to find the value of related parameters to determine Φ_D from observed demand samples. This process of course is subject to considerable statistical errors. We can apply the estimators developed in Theorem 3.1 here to assess how significant the influence of such error will be on the quality of our inventory decision.

Appendix EC.5 takes continuous demand with continuous or discrete ordering quantities ($\mathcal{U}$ is a continuum set or a lattice) as examples to demonstrate technical details. Note that the optimal ordering policy is not smooth with respect to the demand model parameter under the constraint that $\mathcal{U}$ is discrete. The numerical experiments show that our IPA sensitivity estimator is indeed unbiased. Moreover, an interesting sideline observation from our numerical experiments is that the costs under the base stock policy sometimes are very sensitive to the value of model parameter of random demands, meaning that users should make substantial efforts to obtain accurate parameter estimates before applying the policy.

4. First-Order Sensitivity Estimators for American Options

We will use the generalized IPA approach developed in the last section to construct first-order price sensitivity estimators for American options in this section. The optimal exercising rules of many American-type options are not known explicitly. Therefore, their numerical approximations introduce a new bias to sensitivity estimates using IPA. We discuss the error analysis issue in Section 4.2 and point out that in addition to increasing the number of simulation samples to reduce the estimation variance, we may also need a separate passage to the limit, in which the computational effort to obtain better approximate exercising rule increases, in order to achieve the asymptotic unbiasedness.

4.1. Option Sensitivity Estimators

Some regularity conditions are needed beforehand to ensure we can apply Theorem 3.1 here. First, assume that

Assumption 4.1 *The payoff function h is Lipschitz, i.e., there exists a constant $k \geq 0$ such that*

$$|h(i, x_1) - h(i, x_2)| \leq k\|x_1 - x_2\|$$

for any $x_1, x_2 \in \mathbb{R}_+^d$ and $1 \leq i \leq N$.

16

Chen and Liu: *Monte Carlo and American Option Sensitivities*
Article submitted to *Operations Research*; manuscript no. (Please, provide the manuscript number!)

Various American-style securities, such as the vanilla put and American max-call option, satisfy Assumption 4.1. The put's payoff is $h(i, x) = \max\{K - x, 0\}$. The max option is written on multiple assets and it entitles a payoff of $[\max\{X_i^1, \cdots, X_i^d\} - K]^+$ to the holder when she exercises it at t_i, where X_i^j is the price of the jth underlying, $1 \leq j \leq d$. It is easy to check that both functions are Lipschitz.

The previous two examples reveal that option payoffs often fail to be differentiable everywhere. But the points at which differentiability fails can often be ignored because the probability that the state process X hits them is usually 0. We need this technical condition as well. To make it more precise, let $D_i^h := \{x \in \mathbb{R}_+^d : h(i, \cdot) \text{ is differentiable at } x\}$ and we require

Assumption 4.2 $P(X_i^\theta \in D_i^h | X_0^\theta = x) = 1$, $1 \leq i \leq N$.

Clearly, the examples of standard put and American max-call both satisfy Assumption 4.2 too, when the underlying asset prices follow either GBM or VG. The put option's payoff is not differentiable only at the strike price K. The probability that the underlying asset prices exactly equals K is zero. The payoff of American max-call is differentiable in the sets

$$\{(x_1, \cdots, x_d) : \max_{1 \leq i \leq d} x_i < K\} \text{ and } \{(x_1, \cdots, x_d) : x_i > \max_{j \neq i} x_j, \ x_i > K\}, \ 1 \leq i \leq d.$$

With probability 1, the underlying asset prices will fall into one of these sets.

Under Assumptions 4.1 and 4.2, the payoff $h(i, X_i^\theta)$ is pathwise smooth. Using the chain rule of differentiation again, we have

$$\frac{\partial h}{\partial x_j}(i, X_i^\theta) = \sum_{k=1}^{d} \frac{\partial h}{\partial x_k}(i, X_i^\theta) \cdot \frac{\partial X_{i,k}^\theta}{\partial x_j} = \nabla^{\mathrm{T}} h(i, X_i^\theta) \cdot Y_i^j \tag{17}$$

and

$$\frac{\partial h}{\partial \theta}(i, X_i^\theta) = \sum_{k=1}^{d} \frac{\partial h}{\partial x_k}(i, X_i^\theta) \cdot \frac{\partial X_{i,k}^\theta}{\partial \theta} = \nabla^{\mathrm{T}} h(i, X_i^\theta) \cdot Z_i. \tag{18}$$

These two equations demonstrate the use of the derivative processes clearly: after Y and Z are simulated, we can easily obtain the derivatives of $h(X^\theta)$ along each path of X^θ from (17) and (18).

Based on these two derivatives, we present the main results of the paper in Theorem 4.1.

Theorem 4.1 *Fix $x \in \mathbb{R}^d$ and $\theta \in \Theta$. Suppose that Assumptions 2.1, 4.1, and 4.2 hold. Furthermore, we assume*

$$P[h(i, X_i^\theta) = C_i(X_i^\theta; \theta) | X_0^\theta = x] = 0, \ 1 \leq i \leq N - 1.$$

Then,

$$\frac{\partial Q_0}{\partial x_j}(x; \theta) = E\left[\nabla^{\mathrm{T}} h(\tau^*, X_{\tau^*}^\theta) \cdot Y_{\tau^*}^j \middle| X_0^\theta = x\right] \ and \ \frac{\partial Q_0}{\partial \theta}(x; \theta) = E\left[\nabla^{\mathrm{T}} h(\tau^*, X_{\tau^*}^\theta) \cdot Z_{\tau^*} \middle| X_0^\theta = x\right],$$

where τ^* is defined in (5).

This theorem is a direct application of Theorem 3.1. To see the connection between these two conclusions, be aware that τ^* is now the decision variable and $h(i, X_i^\theta)$ is the performance measure function. The theorem asserts that we can derive estimators simply by substituting τ^* into the derivatives of $h(i, X_i^\theta)$. A more rigorous proof is deferred to the Appendix.

At the end of this subsection, we need to point out that the additional assumption in Theorem 4.1 is not restrictive at all. Note that the set $\{x \in \mathbb{R}_+^d : h(i, x) = C_i(x; \theta)\}$ is typically a $(d - 1)$-dimensional manifold in $\mathbb{R}_+^d$. Therefore, as long as the probability density of X_i^θ exists, the probability of the event $h(i, X_i^\theta) = C_i(X_i^\theta; \theta)$ must be 0. The aforementioned vanilla put under the GBM or VG models exemplifies the cases that satisfy this assumption. In this example, the set of $\{x \in \mathbb{R}_+^d : h(i, x) = C_i(x; \theta)\}$ consists of a single point. Because the underlying price follows continuous distributions, it hits this set with probability zero.

On the basis of Theorem 4.1, we construct the following algorithm to generate the estimates of option prices and sensitivities simultaneously.

Sensitivity Estimation Algorithm

   1. Find an (approximate) optimal exercise policy $\tilde{\tau}$.

   2. Simulate L paths of $\{X_i^{\theta, l}, 0 \leq i \leq \tilde{\tau}^l\}$ from (6) for $1 \leq l \leq L$.

   3. Along with each sample path, generate the derivative processes $\{Y_i^{j, l} : 0 \leq i \leq \tilde{\tau}, \ 1 \leq j \leq d\}$ and $\{Z_{i, l} : 0 \leq i \leq \tilde{\tau}\}$ according to the recursions (9) and (10).

   4. Evaluate $A^l := \nabla^{\mathrm{T}} h(\tilde{\tau}^l, X_{\tilde{\tau}^l}^{\theta, l}) \cdot Y_{\tilde{\tau}^l}^{j, l}$ and $B^l := \nabla^{\mathrm{T}} h(\tilde{\tau}^l, X_{\tilde{\tau}^l}^{\theta, l}) \cdot Z_{\tilde{\tau}^l}^l$.

   5. Form the price estimator by $\sum_{l=1}^L h(\tilde{\tau}^l, X_{\tilde{\tau}^l}^{\theta, l})/L$ and the sensitivity estimators by $\sum_{l=1}^L A^l/L$ and $\sum_{l=1}^L B^l/L$, respectively.

Remark 4.1 *Note that this algorithm is designed only for a fixed initial position X_0^θ. In some applications, people may need to estimate sensitivities at another position X_s^θ with $0 < s \leq t_1$, where t_1 is the first exercise opportunity. We can of course re-simulate a new set of sample paths starting from X_s^θ and use the above algorithm again. However, Belomestny, Milstein, and Shoemakers (2010) propose an innovative method to infer the sensitivities from the old sample paths without re-simulation.*

4.2. Implementation Issues and Error Analysis

Unlike the example in Section 3.2, the exercising rules of American-type options are usually not tractable. As mentioned in Section 2, there are basically two ways to price American options numerically in the existing literature: through approximations of either continuation values or exercising rules. Our estimators are compatible to both of them and hence we can easily embed

the preceding algorithm to a variety of existing pricing methods to generate sensitivities as a by-product.

In particular, if the used pricing method can generate accurate approximations to the continuation functions $C_i(x; \theta)$, we can then forge an approximation to τ^* by letting

$$\tilde{\tau} = \min\{i \in \{1, \cdots, N\} : h(i, X_i^\theta) \geq \tilde{C}_i(X_i^\theta; \theta)\}.$$

Based on this exercising rule, we may proceed to use the above algorithm. For those methods in which an approximate exercising strategy $\tilde{\tau}$ is known in advance, substituting the obtained policy into the algorithm will directly lead to sensitivity estimators.

We should acknowledge that the discrepancy between $\tilde{\tau}$ and τ^* will introduce a new bias to the estimators in Theorem 4.1. For completeness, it is important to analyze the impact of such bias on the estimation quality of sensitivities. Let us consider the sensitivity with respect to x only in this section. The discussion about the other sensitivity with respect to θ can be done in a similar manner.

It is easy to see that the expectation of our delta estimator, $\sum_{l=1}^{L} A^l / L$, equals

$$\widetilde{\frac{\partial Q_0}{\partial x_j}} := E[\nabla^\mathsf{T} h(\tilde{\tau}, X_{\tilde{\tau}}^\theta) \cdot Y_{\tilde{\tau}}^j | X_0 = x]. \tag{19}$$

Intuitively, high-quality estimates of $\tilde{C}$ should lead to a good approximation $\tilde{\tau}$ and in turn produce a good estimate to $\partial Q_0 / \partial x_j$. But we find that this is not necessarily true when

$$P[h(i, X_i^\theta) = C_i(X_i^\theta; \theta)] > 0, \text{ for some } 1 \leq i \leq N - 1.$$

To see this, just consider the sample paths on which we have $h(i, X_i^\theta) = C_i(X_i^\theta; \theta)$. As long as there exists a discrepancy between C_i and $\tilde{C}_i$, no matter how small it is, it is always the case that $h(i, X_i^\theta) \neq \tilde{C}_i(X_i^\theta; \theta)$ on those paths. In other words, we will always misidentify the optimal stopping time for those paths, regardless of the estimation quality of $\tilde{C}$. To exclude this pathological case, we need an additional assumption:

Assumption 4.3 *There exists an $\alpha > 0$ such that*

$$\lim_{\delta \downarrow 0} \frac{P[|h(i, X_i^\theta) - C_i(X_i^\theta; \theta)| \leq \delta]}{\delta^\alpha} < +\infty$$

for all $1 \leq i \leq N$.

From the continuity property of probability measures, this assumption essentially requires that the probability of $h(i, X_i^\theta) = C_i(X_i^\theta; \theta)$ is zero, which is one assumption we used previously to establish the unbiased IPA estimators in Theorem 4.1. We note that a similar condition first appears

in Belomestny (2011) in which the author discusses the convergence rates of various nonparametric American-option pricing algorithms. Interestingly, that paper also points out possible values of α in different modeling contexts. For instance, one can show that $\alpha = 1$ when $h(i,x) - C_i(x;\theta)$ is smooth and has non-vanishing Jacobian in the vicinity of the exercise boundary, which are satisfied in many practical examples.

The following theorem relates the worst-case performance of our sensitivity estimates to the accuracy of the approximations $\tilde{C}$ or $\tilde{\tau}$.

Theorem 4.2 *Suppose that the conditions of Theorem 4.1 and Assumption 4.3 both hold. In addition, assume that $\sup_{1 \leq i \leq N} E[\|Y_i^j\|^2] < +\infty$. Then, there exists a constant K, independent of X_0 and θ, such that*

$$\left| \frac{\widetilde{\partial Q_0}}{\partial x_j} - \frac{\partial Q_0}{\partial x_j} \right| \leq K \sum_{i=1}^{N-1} \left(E[|\widetilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|] \right)^{\frac{\alpha}{2(1+\alpha)}}, \tag{20}$$

where $\partial Q_0 / \partial x_j$ is the true value of the delta.

Eq. (20) explicitly points out that the only source of bias of our algorithm comes from the approximation error between $\tilde{C}_i$ and C_i. Qualitatively speaking, any pricing methods that can produce accurate estimates of the continuation values are able to generate accurate estimates for the sensitivities. Furthermore, the above theorem also shows that in order to improve the overall performance of our estimators, say, the mean-squared error (MSE), in a problem where the optimal exercising rule is not tractable, we need to increase computational effort in obtaining better estimates to the continuation values to control the bias when we increase the number of simulation samples to reduce the variance.

Take the celebrated Longstaff and Schwartz method as an example. As we increase the number of basis functions, the spanned functional subspace will come closer to containing all true continuation value functions C_i for $1 \leq i \leq N - 1$. Hence, we can expect the bias generated by approximate continuation value functions to diminish to zero. Glasserman and Yu (2004) and Stentoft (2004) discuss in a greater detail the convergence of $\widetilde{C}_i(X_i^\theta;\theta)$ to $C_i(X_i^\theta;\theta)$ for all i. For instance, letting M and H be the respective total numbers of the basis functions and samples used to estimate C_i, Stentoft (2004) proves that, as long as M is increasing in H and M^3/H tends to 0, such convergence will be guaranteed when $H \to +\infty$ under some regular conditions on the underlying process and the used basis functions. Combining these known results in the existing literature with Eq. (20), we know that the sensitivity estimator presented in this paper can achieve asymptotic unbiasedness as both the number of basis functions and the number of simulated paths increase. The numerical experiment in Section 5.3 corroborates the aforementioned observation. It is still unclear

for us how to allocate computational effort between sample path simulation and continuation value function estimation to achieve the optimal convergence rate of the simulation bias. We leave a more comprehensive investigation of this issue for future research.

The requirement that $M^3/H \to 0$ is necessary to avoid potential overfitting problems. If we increase the number of basis functions too fast relative to the growth of the sample path number, the fitted $\tilde{C}_i$ using the Longstaff-Schwartz regression will excessively depend on simulated sample paths and this produces unacceptably high variability in the final outcomes. One may refer to Tompaidis and Yang (2010) for other alternatives to overcome the overfitting problem.

We stress here that Theorem 4.2 describes the worst-case performance of our algorithm. From its proof, we can easily see that, to achieve the upper bound given on the right-hand side of (20), it is necessary for the option to be mistakenly exercised or left unexercised at each time period. For this to happen, the underlying X_i^θ must be very close to the optimal exercising boundary $\{x : h(i,x) = C_i(x;\theta)\}$ at every time i. However, Assumption 4.3 stipulates that the probability of such an event is very small. Therefore, we expect that the bias magnitude of our sensitivity estimator should be far less than the right-hand side of (20).

By the discussion in the last paragraph, Theorem 4.2 also implicitly indicates the crucial role of the optimality of an exercising rule $\tilde{\tau}$ on the estimation bias. The more accurate $\tilde{\tau}$ is as an approximation to the optimal τ^*, the less the probability will be that the option is mistakenly exercised or left unexercised at each time period. This will lead to a small value of the expectation on the right-hand side of (20) and therefore accurate estimates of the sensitivities.

Finally, it is worth mentioning that similar error bounds like (20) also appear in the American-option pricing literature. For instance, Tsitsiklis and Van Roy (1999, 2001) and Haugh and Kogan (2004) establish that

$$|\widetilde{Q_0} - Q_0| \leq K' \sum_{i=1}^{N-1} E[|\widetilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|],$$

where $\widetilde{Q_0}$ is the expectation of $\sum_{l=1}^{L} h(\tilde{\tau}^l, X_{\tilde{\tau}^l}^{\theta,l})/L$, the price estimator in our algorithm. Belomestny (2011) significantly improves this bound.

5. Numerical Results

In this section, we undertake various numerical experiments to test the accuracy and efficiency of the obtained estimators. The section consists of four parts. First, we demonstrate the performance of the estimators in the contexts of different dimensionality. We find that they work robustly from single-dimensional to high-dimensional cases. Second, we test the effects of the types of underlying processes on their performance. The models under consideration include correlated pure diffusions, mixed jump diffusions, and pure jump processes. The efficiency of the estimators is not

influenced by such a wide modeling spectrum. Third, we use one example to show that our method is asymptotically unbiased if we simultaneously increase the number of simulated sample paths and the number of the basis functions used to estimate continuation values. Fourth, we compare our IPA estimators with several competing methods to point out their computational benefits.

In all these experiments, we used mainly the least square regression method proposed by Longstaff and Schwartz (2001) to obtain continuation value approximations and then to construct the optimal exercising policy approximately. We also implemented some other alternative pricing algorithms such as the stochastic mesh method by Broadie and Glasserman (2004) when we prepared this paper. The performance of the estimators was not sensitive to the underlying pricing methods we chose. We skip a detailed report about other methods in the interest of space. All numerical experiments were conducted on a PC equipped with an Intel Core 2 Quad 2.66 GHz CPU and 2.87 GB of RAM. All computing time is with respect to this configuration and measured in seconds. In particular, we obtain numerical results by repeating our experiments over multiple independent trials. In each trial, we re-estimate policies, option prices and sensitivities based on a newly generated set of sample paths. The reported computing time corresponds to the average time for one trial.

5.1. Effects of Dimensionality

Tables 1 and 2 record the numerical outcomes for a single asset and for 5 and 10 assets, respectively. Compared to the benchmarks in the existing literature, the numerical results indicate that our estimation method is fast and accurate, which is not influenced by the dimensionality of the underlying problem.

Table 1 summarizes the results in a single-asset vanilla put under the GBM model. Huang, Subrahmanyam, and Yu (1996) take an integral-equation-based approach to price the option value in this case and provide the corresponding sensitivities in the meantime. We use their method to generate the benchmark values, which are documented in the columns with a superscript $*$. The "$Bias_{price}$" column reports the bias of price estimates generated by the standard Longstaff-Schwartz method. The "$Bias_{Delta}$" and "$Bias_{Vega}$" columns report the sensitivities estimation biases using our proposed estimators. We can see that the computation is very efficient: the sensitivity estimates generated by our IPA estimators have very small biases and standard deviations.

For the continuation value approximation approach, we need to pay special attention to one subtle issue: independence between the sample paths used to train $\tilde{C}$ and the sample paths used to estimate the sensitivities. It is possible to reuse those paths generated in the training stage for the purpose of price and sensitivity estimation. In principle, doing so will save simulation efforts to a significant degree, but will expose us to the potential effect of "error maximization" because

Table 1   Sensitivities for single-asset American put options with the GBM model.

			In-sample					
K	T	$Price^*$	$Bias_{Price}$	$Delta^*$	$Bias_{Delta}$	$Vega^*$	$Bias_{Vega}$	$Time$
35	1/3	0.2003	-0.0003(0.00091)	-0.0904	-0.0001(0.00034)	3.7526	0.0053(0.0148)	26.3
35	7/12	0.4326	-0.0015(0.0017)	-0.1341	-0.0003(0.00035)	6.564	-0.017(0.0211)	32.5
40	1/3	1.5794	-0.0001(0.0023)	-0.4463	0.0010(0.00052)	8.9876	0.0086(0.0145)	59.8
40	7/12	1.9901	-0.0002(0.0029)	-0.4294	-0.0008(0.00063)	11.7303	-0.014(0.0167)	61.2
45	1/3	5.0837	-0.0018(0.0021)	-0.8857	0.0003(0.00039)	3.9311	-0.0136(0.0075)	83.4
45	7/12	5.2661	-0.0031(0.0041)	-0.7922	-0.0004(0.00041)	7.907	0.0129(0.0072)	86.7
			Out-of-sample					
K	T	$Price^*$	$Bias_{Price}$	$Delta^*$	$Bias_{Delta}$	$Vega^*$	$Bias_{Vega}$	$Time$
35	1/3	0.2003	-0.0002(0.00092)	-0.0904	-0.0001(0.00036)	3.7526	-0.0002(0.0153)	72.5
35	7/12	0.4326	-0.0003(0.0016)	-0.1341	-0.0002(0.00037)	6.564	0.0058 (0.0209)	80.6
40	1/3	1.5794	-0.0001(0.0022)	-0.4463	0.0002(0.00051)	8.9876	-0.01(0.0144)	102.9
40	7/12	1.9901	-0.0008(0.0029)	-0.4294	-0.0007(0.00062)	11.7303	-0.02(0.0161)	104.8
45	1/3	5.0837	-0.0037(0.0020)	-0.8857	-0.0006(0.00042)	3.9311	-0.0109(0.0059)	128.5
45	7/12	5.2661	-0.0012(0.0039)	-0.7922	-0.0004(0.00044)	7.907	-0.0119(0.0075)	132.7

Note. Sensitivities for single-asset American put options with the GBM model $dS_t = rS_t dt + \sigma S_t dW_t$. The defaulting parameters are $S_0=40$, $r = 0.0488$, and $\sigma = 0.2$. Set the number of exercising opportunities N to be 400 between $t = 0$ and T. The numbers in parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 0.5 million sample paths to train the exercising boundary and reuse them to obtain the in-sample estimation. We simulate an additional 0.5 million paths to obtain the out-of-sample estimation. The following six basis functions are used: a constant and the first five Hermite polynomials in the asset price.

of the high correlation of the samples used in the two stages. However, we do not find strong evidence in Table 1 that this should become a serious concern. In the out-of-sample panel, the set of paths used to obtain prices and sensitivities is different and independent of that used to construct exercising policies. In the in-sample panel, we use the same set of sample paths for both purposes. The numerical outcomes change very little across these two panels. This observation applies generally for the other experiments we have done. Hence, we only report the results for in-sample estimation in all the following experiments. This echoes a similar observation in the American option pricing literature by Raymar and Zwecher (1997), Longstaff and Schwartz (2001), and Broadie and Glasserman (2004), where they find that in-sample and out-of-sample estimates for prices are almost indistinguishable.

We use max-call options written on 5 and 10 uncorrelated assets in Table 2 to exhibit the performance of our estimators in a high-dimensional setting. The underlying dynamics is given by

$$\frac{dS_t^i}{S_t^i} = (r - \delta)dt + \sigma dW_t^i, \quad i = 1, \cdots, n,$$

where $n = 5$ or 10, and $\{W_t^i\}$ are independent standard Brownian motions. Kaniel, Tompaidis, and Zemlianov (2008) provide 90% confidence intervals for the estimates of prices and deltas. We use their results as the benchmark values in Table 2; see the "$Price^*$" and "$Delta^*$" columns. The results show that within a very small computational budget, all IPA estimates fall in the corresponding confidence intervals obtained by the likelihood ratio duality (LRD) estimators developed

in Kaniel, Tompaidis, and Zemlianov (2008). In contrast, it takes much longer for the LRD method to achieve these confidence intervals according to that paper.

Table 2 Sensitivities for American max-call options on high-dimensional uncorrelated assets.

S_0	$Price^*$	$Price^{LS}$	$Delta^*$	$Delta^{IPA}$	$Vega^*$	$Vega^{IPA}$	$Time$
			$n = 5$				
90	[16.6062,16.6568]	16.6220(0.0073)	[0.17104,0.17896]	0.1732(0.00005)	NA	32.6662(0.0123)	79
100	[26.0992,26.1788]	26.1228(0.0088)	[0.19608,0.20892]	0.2020(0.00004)	NA	37.6026(0.0119)	90
110	[36.7012,36.8148]	36.7289(0.0099)	[0.21408,0.22792]	0.2186(0.00003)	NA	41.1545(0.0124)	92
			$n = 10$				
S_0	$Price^*$	$Price^{LS}$	$Delta^*$	$Delta^{IPA}$	$Vega^*$	$Vega^{IPA}$	$Time$
90	[26.2102,26.2868]	26.2352(0.0085)	[0.11304,0.12296]	0.1162(0.00002)	NA	24.9140(0.0066)	178
100	[38.2732,38.3768]	38.3134(0.0097)	[0.11708,0.13292]	0.1226(0.00001)	NA	28.0862(0.0068)	183
110	[50.8052,50.9308]	50.8324(0.0107)	[0.11808,0.13392]	0.1232(0.00001)	NA	30.8147(0.0073)	190

Note. Sensitivities for American max-call options on 5-dimensional and 10-dimensional uncorrelated assets. The defaulting parameters are $K = 100$, $r = 5\%$, $\delta = 10\%$, $\sigma = 0.2$, and $T = 3$. Three exercise opportunities are evenly distributed in each year. For the case of $n=5$, the following 19 basis functions are used: a constant, the first five Hermite polynomials in the maximum of the values of five assets, the four values and squares of the values of the second through fifth highest asset prices, the product of the highest and second highest, second highest and third highest, etc., and finally, the product of all five asset values. For the case of $n = 10$, we use 32 basis functions in a similar specification. For simplicity, assume that all the assets start from the same initial value S_0. The numbers in parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 5 million sample paths to obtain the estimates. Delta is defined as the sensitivity with respect to the initial value of the first asset and vega is the sensitivity in the volatility of the first asset. All estimates are within their corresponding confidence intervals provided in Kaniel, Tompaidis, and Zemlianov (2008), which are used as our benchmark values here. We cannot find a corresponding vega estimator in that paper. According to Kaniel, Tompaidis, and Zemlianov (2008), all the calculations of benchmark confidence intervals in this table took 1 to 2 hours using between 16 and 32 1 GHz processors in parallel.

5.2. Performance under Various Underlying Processes

Tables 3, 4, and 5 show the results under different types of underlying models. The performance of the proposed sensitivity estimators is very stable across all the models. Whether the sample paths are continuous has no impact on the efficiency of our method. Table 3 considers a two-correlated GBM model such as

$$dS_t^1 = (r - \delta_1)S_t^1 dt + \sigma_1 dW_t^1$$
$$dS_t^2 = (r - \delta_2)S_t^1 dt + \sigma_2[\rho dW_t^1 + \sqrt{1 - \rho^2}dW_t^2],$$

where (W_t^1, W_t^2) are two independent standard Brownian motions. The same example is also used in Kaniel, Tompaidis, and Zemlianov (2008). We include their confidence interval estimates of prices and deltas in the table for reference. It is obvious that our estimators perform efficiently under a wide range of correlation values: the estimated values of price and delta fall into the benchmark confidence interval within a very small computational budget.

Table 3   Sensitivities for American max-call options on correlated assets.

ρ	$Price^*$	$Price^{LS}$	$Delta^*$	$Delta^{IPA}$	$Vega^*$	$Vega^{IPA}$	$Time$
-0.75	$[15.4341, 15.4759]$	$15.4482(0.0043)$	$[0.3742, 0.3998]$	$0.3844(0.00009)$	NA	$53.3521(0.0141)$	62
-0.5	$[15.0081, 15.0459]$	$15.0220(0.0045)$	$[0.3672, 0.3868]$	$0.3704(0.00009)$	NA	$52.1823(0.0148)$	60
-0.25	$[14.5021, 14.5379]$	$14.5172(0.0047)$	$[0.3481, 0.3639]$	$0.3513(0.00009)$	NA	$50.5327(0.0155)$	59
0	$[13.8841, 13.9189]$	$13.8985(0.0048)$	$[0.3271, 0.3379]$	$0.3301(0.00009)$	NA	$48.4686(0.0161)$	58
0.25	$[13.0821, 13.1159]$	$13.0925(0.0048)$	$[0.3041, 0.3199]$	$0.3114(0.00009)$	NA	$45.7715(0.0162)$	56
0.5	$[12.1351, 12.1679]$	$12.1422(0.0048)$	$[0.2821, 0.2979]$	$0.2896(0.00009)$	NA	$42.4195(0.0159)$	56
0.75	$[10.8791, 10.9149]$	$10.8941(0.0045)$	$[0.2592, 0.2798]$	$0.2647(0.00009)$	NA	$37.8083(0.0151)$	54

Note. The defaulting parameters are $S_0^1 = S_0^2 = 100$, $K = 100$, $r = 5\%$, $\delta_1 = \delta_2 = 10\%$, $\sigma_1 = \sigma_2 = 0.2$ and $T = 3$. Three exercise opportunities are evenly distributed in each year. The columns with superscript $*$ are the 90% confidence intervals given by Kaniel, Tompaidis, and Zemlianov (2008). The following nine basis functions are used: a constant, the first five Hermite polynomials in $\max(S_t^1, S_t^2)$, the value and square of the values of $\min(S_t^1, S_t^2)$, and $S_t^1 \cdot S_t^2$. The numbers in parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 10 million sample paths to obtain the estimates. Delta is defined as the sensitivity with respect to S_0^1 and vega is the sensitivity in σ_1. All estimates are within their corresponding confidence intervals provided in Kaniel, Tompaidis, and Zemlianov (2008), which are used as our benchmark values here. We cannot find a corresponding vega estimator in that paper. According to Kaniel, Tompaidis, and Zemlianov (2008), all the calculations of benchmark confidence intervals in this table took 1 to 2 hours using between 16 and 32 1 GHz processors in parallel.

In Tables 4 and 5, we investigate price processes with discontinuous sample paths. The models used in Table 4 include Merton's normal jump diffusion (Merton (1976)) and Kou's double-exponential jump diffusion (Kou (2002)). They represent the class of Levy processes with compound Poisson jumps. We can specify the models through the following log-price processes:

$$\log\left(\frac{S_t}{S_0}\right) = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i,$$

where N_t is a Poisson process with constant intensity λ and $\{Z_i\}$ are i.i.d. random variables. In Merton's model, the jump size Z follows a normal distribution $N(m, s)$ and Kou's model specifies a double exponential distribution for Z with density

$$p(z) = p\eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + (1-p)\eta_2 e^{-\eta_2 z} 1_{\{z < 0\}},$$

where $\eta_1, \eta_2 > 0$. One may refer to Section 3.5.1 of Glasserman (2004) for a detailed discussion on how to generate paths for both processes. The benchmark values, denoted with a superscript $*$, are computed by the implicit-explicit finite-difference solvers developed in Zhang (1997) and d'Halluin, Forsyth, and Labahn (2004). Table 5 displays numerical results for the VG model (8). We compare our estimates with the results from Hirsa and Madan (2003). For all these models, our method can generate quite accurate estimates for the sensitivities in a reasonable time horizon.

5.3. Effects of Exercising Policies

As noted in Section 4, the accuracy of our sensitivity estimators is subject to the influence of the quality of the approximating exercise policies we adopt. To achieve asymptotic unbiasedness, along with the increment of simulation trials, we need to increase the total number of basis functions as well to improve the estimation of continuation value functions.

Table 4 Sensitivities for single-asset American put options under jump diffusion processes.

Merton's Model

K	T	$Price^*$	$Bias_{Price}$	$Delta^*$	$Bias_{Delta}$	$Vega^*$	$Bias_{Vega}$	$Time$
35	1/3	0.5439	-0.0023(0.0021)	-0.139	0.0002(0.00035)	3.91	-0.071(0.0136)	62.34
35	7/12	0.9314	-0.0008(0.0027)	-0.1791	-0.0004(0.00041)	6.025	0.0222(0.0179)	66.56
40	1/3	2.1191	-0.0021(0.0036)	-0.4266	0.0003(0.00059)	8.3	0.0166(0.0154)	87.18
40	7/12	2.6876	-0.0047(0.0048)	-0.4154	0.0007 (0.00047)	9.57	0.0091(0.0201)	88.02
45	1/3	5.3834	-0.0027(0.0049)	-0.782	0.0004(0.00044)	7.805	0.0088(0.0121)	112.12
45	7/12	5.7662	-0.0049(0.0055)	-0.6941	-0.0004(0.00056)	9.11	0.0258(0.0177)	108.32

Kou's Model

K	T	$Price^*$	$Bias_{Price}$	$Delta^*$	$Bias_{Delta}$	$Vega^*$	$Bias_{Vega}$	$Time$
35	1/3	0.5718	-0.0002(0.0019)	-0.1293	0.0001(0.00041)	3.255	-0.014(0.0132)	93.34
35	7/12	0.979	-0.0043(0.0028)	-0.1717	0.0006(0.00044)	5.43	-0.0143(0.0175)	100.11
40	1/3	2.0676	-0.0014(0.0039)	-0.4107	-0.0001(0.00059)	7.26	0.0103 (0.0158)	121.43
40	7/12	2.6797	-0.0034(0.0052)	-0.4014	-0.0002(0.00062)	9.135	-0.0031(0.0201)	126.89
45	1/3	5.2936	-0.0039(0.0051)	-0.7866	-0.0004(0.00056)	5.685	-0.0035(0.0113)	156.21
45	7/12	5.7075	-0.0049(0.0058)	-0.689	0.0005(0.00058)	8.68	-0.0044(0.0148)	158.66

Note. Sensitivities for single-asset American put options under compound Poisson jump diffusion processes. For Merton's model, the defaulting parameters are $S_0 = 40$, $r = 0.0488$, $\sigma = 0.2$, $\lambda = 3$, $m = -0.05$, and $s = 0.086$. Set N to be 400 between $t = 0$ and T. For Kou's model, $S_0 = 40$, $r = 0.0488$, $\sigma = 0.2$, $\lambda = 3$, $p = 0.3$, $\eta_1 = 40$, and $\eta_2 = 12$. To let both processes be martingales, choose $\mu = r - \sigma^2/2 + \lambda(1 - e^{m+s^2/2})$ and $\mu = r - \sigma^2/2 + \lambda[(1-p)(\eta_2+1)^{-1} - p(\eta_1-1)^{-1}]$, respectively. The numbers in parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 0.5 million sample paths to obtain the estimates. Delta is defined as the sensitivity with respect to S_0 and vega is the sensitivity in σ. All benchmark values are within two standard deviations of the sensitivity estimates from our method.

Table 5 Sensitivities for single-asset American put options under the variance gamma process.

Variance Gamma Model

K	T	$Price^*$	$Bias_{Price}$	$Delta^*$	$Bias_{Delta}$	$Vega^*$	$Bias_{Vega}$	$Time$
1200	0.5616	35.5647	-0.0765(0.1221)	-0.1429	0.0002(0.00039)	168.77	-0.7821(0.5822)	107.61
1280	0.5616	54.102	-0.1831(0.1432)	-0.2158	0.0003(0.00043)	207.32	-1.0339(0.6991)	114.44
1360	0.5616	80.1335	-0.1434(0.1901)	-0.3181	-0.0004(0.00051)	237.985	0.341(0.6104)	136.32
1400	0.5616	96.6741	-0.1123(0.2001)	-0.3871	-0.0006(0.00048)	246.92	0.2802(0.6117)	145.51

Note. Sensitivities for single-asset American put options under the variance gamma process (8). The defaulting parameters are $S_0 = 1369.41$, $r = 0.0541$, $q = 0.012$, $\sigma = 0.20722$, $\theta = -0.22898$, and $\beta = 0.50215$. Set N to be 400 between $t = 0$ and T. To let the process be a martingale, choose $\mu = r - q + \log(1 - \theta\beta - \sigma^2\beta/2)/\beta$. The numbers in parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 0.5 million sample paths to obtain the estimates. Delta is defined as the sensitivity with respect to S_0 and vega is the sensitivity in σ. All benchmark values are within two standard deviations of the sensitivity estimates from our method.

In this subsection, we illustrate this effect numerically with a one-dimensional case. Consider the American put in Table 1 with $K = 40$ and $T = 7/12$. Table 6 displays the outcomes of price and sensitivity estimates when we simultaneously increase the numbers of basis functions and simulation trials. Let M be the number of basis functions we use for each numerical experiment. We change the number of sample paths H according to a rule of $H = CM^4$, where C is a pre-specified constant. The purpose of choosing this growth rate for the simulation effort is to avoid the overfitting issue mentioned in Section 4.2. An immediate observation from the table is that both prices and sensitivities become increasingly accurate, converging to the benchmark values, when the computational effort enlarges. Such improvement is due to the fact that we can obtain higher quality approximations for the continuation function with larger sets of basis functions and sample

26

Chen and Liu: *Monte Carlo and American Option Sensitivities*
Article submitted to *Operations Research*; manuscript no. (Please, provide the manuscript number!)

paths. Moreover, we find that the convergence rate of price estimates is faster than the convergence rates of the sensitivities. For example, with 9 basis functions, the price estimate is already very accurate, while there still exist small gaps between the true and the estimated sensitivity values.

Table 6 Sensitivity estimation under different sets of basis functions.

Basis Functions	Price	Delta	Vega
$LS-3$	1.9721(0.0136)	-0.4363(0.00248)	11.4532(0.0704)
$LS-4$	1.9876(0.0072)	-0.4326(0.00139)	11.5013(0.0405)
$LS-5$	1.9895(0.0048)	-0.4266(0.00089)	11.7069(0.0258)
$LS-6$	1.9899(0.0029)	-0.4302(0.00063)	11.7263(0.0167)
$LS-7$	1.9901(0.0024)	-0.4301(0.00046)	11.7281(0.0132)
$LS-8$	1.9901(0.0019)	-0.4298(0.00041)	11.7295(0.0101)
$LS-9$	1.9901(0.0015)	-0.4296(0.00034)	11.7306(0.0081)
Benchmark	1.9901	-0.4294	11.7303

Note. The model setting and parameter values are the same as the in-sample estimation for the case of $K = 40$ and $T = 7/12$ in Table 1. We perform the Longstaff-Schwartz regression method with the rule of $H = 386M^4$ in each row, where M is the number of basis functions and H is the number of sample paths. For instance, to compute the row of LS-9 we use a set of basis functions consisting of the constant and the first eight Hermite polynomials, and a set of 2532546 sample paths.

5.4. Comparison with Some Competing Methods

In this subsection, we report the numerical outcomes by comparing our IPA sensitivity estimators with a set of competing methods, including the finite-difference (FD) method, a direct differentiation (DD) method, the LRD estimator proposed in Kaniel, Tompaidis, and Zemlianov (2008), a heuristic point estimator (HPE) appearing in the same paper, and the modified Longstaff-Schwartz method (MLSM) in Wang and Caflisch (2010). In particular, the details of the above five alternative methods are as follows.

• **FD**. It builds up sensitivity estimators by taking value difference after a small perturbation on the parameter of interest. That is,

$$Delta^{FD} = \frac{\text{Price}(S_0 + \Delta S_0) - \text{Price}(S_0 - \Delta S_0)}{2\Delta S_0}$$

and

$$Vega^{FD} = \frac{\text{Price}(\sigma + \Delta\sigma) - \text{Price}(\sigma - \Delta\sigma)}{2\Delta\sigma}.$$

• **DD**. For a pre-specified class of basis functions $\{b_1(x), \cdots, b_n(x)\}$, use the Longstaff-Schwartz regression method to obtain an approximation to the continuation value function at time t_1, $C_1(x; \theta)$, such that

$$C_1(x; \theta) \approx \tilde{C}_1(x; \theta) := c_1(\theta)b_1(x) + \cdots + c_n(\theta)b_n(x),$$

where $c_i(\theta)$, $1 \leq i \leq n$, are all the regression coefficients. Then, we can form an estimator of delta by taking differentiation on $\tilde{C}_1(x; \theta)$:

$$
\begin{aligned}
\text{delta} &= \frac{d}{dx} E[Q_1(X_1; \theta)|X_0 = x] = \frac{d}{dx} E[\max\{h(X_1; \theta), C_1(X_1; \theta)\}|X_0 = x] \\
&\approx \frac{d}{dx} E[\max\{h(X_1; \theta), \tilde{C}_1(X_1; \theta)\}|X_0 = x] \\
&= E\left[\left(\tilde{C}_1'(X_1; \theta)1_{\{h(1,X_1)<C_1(X_1;\theta)\}} + h'(1, X_1)1_{\{h(1,X_1)\geq C_1(X_1;\theta)\}}\right) \cdot \frac{dX_1}{dx}|X_0 = x\right].
\end{aligned}
$$

- **LRD**. Assume that the one-step transition density of X_i^θ at time t_0 is given by

$$
P[X_1^\theta \in dy | X_0^\theta = x] = g_1(y; x, \theta)dy.
$$

Using the LR method in EC.3, we have

$$
\frac{\partial Q_0}{\partial x}(x; \theta) = E\left[Q_1(X_1^\theta; \theta) \cdot \eta\right],
$$

where the likelihood weight

$$
\eta = \frac{\partial \ln g_1}{\partial x}(X_1^\theta; x, \theta).
$$

The primal-dual pricing literature developed by Rogers (2002), Andersen and Broadie (2004), and Haugh and Kogan (2004) provides estimators to obtain upper and lower bounds of Q_1. Based on this observation, Kaniel, Tompaidis, and Zemlianov (2008) find that one set of upper and lower bounds for delta should be given by

$$
E[\mathbf{1}_{\{\eta<0\}}\eta U_1 + \mathbf{1}_{\{\eta\geq0\}}\eta L_1] \leq \text{delta} \leq E[\mathbf{1}_{\{\eta<0\}}\eta L_1 + \mathbf{1}_{\{\eta\geq0\}}\eta U_1], \tag{21}
$$

where we denote U_1 and L_1 to be the upper and lower bounds of Q_1 obtained from the primal-dual method.

- **HPE**. Kaniel, Tompaidis, and Zemlianov (2008) also propose HPE as a heuristic algorithm to speed up the calculation of sensitivities; see Algorithm 4 in that paper.
- **MLSM**. Wang and Caflisch (2010) extends the work of Longstaff and Schwartz (2001) by obtaining a regression equation for the option value function at the initial time. They propose to differentiate the regression equation analytically to derive estimates for sensitivities.

We use a single-asset American put option under the geometric Brownian motion model to assess the performance of all these methods. Table 7 shows that our IPA method demonstrates several benefits in terms of bias, standard deviation, and computation time. Although it is biased, many other competing methods, such as FD, DD, HPE, and MLSM, have the same problem. With comparable computational time budgets, the IPA estimators enjoy the lowest bias. Although there exist alternatives that provide confidence intervals for the sensitivity estimates, e.g., the LRD,

they appear to require significant additional computation effort. Furthermore, we find that the likelihood weight η in the LRD method, like other LR based sensitivity estimators in general, suffer from a large simulation variance. Under the GBM assumption,

$$\eta = \frac{W_{t_1}}{x\sigma\Delta t}.$$

Therefore, the variance of the weight tends to be high when the exercising frequency is larger, i.e., Δt is small; see also p.813 of Kaniel, Tompaidis, and Zemlianov (2008) for a related discussion on this issue. We also find the obtained confidence intervals are very wide in some experiments in Table 7. The reason is that both the upper and lower bounds in Eq. (21) involve the price upper bound U_1, which may suffer from a huge duality gap sometimes.

Table 7   Comparison among competing methods for single-asset American put options with the GBM model.

		IPA					FD				
K	T	$Bias_{Delta}$	SD_{Delta}	$Bias_{Vega}$	SD_{Vega}	$Time$	$Bias_{Delta}$	SD_{Delta}	$Bias_{Vega}$	SD_{Vega}	$Time$
40	1/3	0.0010	0.00052	0.0086	0.0145	59.8	-0.0023	0.0121	-0.0139	0.189	221.5
40	7/12	-0.0008	0.00063	-0.014	0.0167	61.2	-0.0014	0.0115	-0.026	0.201	224.6
45	1/3	0.0003	0.00039	-0.0163	0.0075	83.4	-0.0007	0.0089	0.007	0.029	331.2
45	7/12	-0.0004	0.00041	0.0129	0.0072	86.7	-0.0006	0.0096	0.0221	0.099	341.7
		DD					MLSM				
K	T	$Bias_{Delta}$	SD_{Delta}	$Bias_{Vega}$	SD_{Vega}	$Time$	$Bias_{Delta}$	SD_{Delta}	$Bias_{Vega}$	SD_{Vega}	$Time$
40	1/3	-0.0117	0.0365	NA	NA	58.1	0.0193	0.00094	NA	NA	66.1
40	7/12	0.0109	0.0379	NA	NA	58.6	0.0147	0.00089	NA	NA	63.6
45	1/3	0.0061	0.0137	NA	NA	86.5	0.0404	0.00087	NA	NA	74.6
45	7/12	0.0103	0.0117	NA	NA	83.9	0.0261	0.00086	NA	NA	72.4
		HPE					LRD				
K	T	$Bias_{Delta}$	SD_{Delta}	$Bias_{Vega}$	SD_{Vega}	$Time$	$Delta_{Lower}$	$SD_{Delta_{Lower}}$	$Delta_{Upper}$	$SD_{Delta_{Upper}}$	$Time$
40	1/3	-0.0193	0.0519	NA	NA	57.5	-2.9715	0.1343	2.9519	0.0159	29981
40	7/12	0.0196	0.0527	NA	NA	58.2	-4.9089	0.2107	5.4989	0.2702	30078
45	1/3	0.0134	0.0334	NA	NA	84.1	-0.9045	0.2218	-0.4296	0.2226	36085
45	7/12	-0.0112	0.0338	NA	NA	83.2	-2.0248	0.199	1.0917	0.2205	35471

Note. Comparison between our generalized IPA method with the five alternatives in a single-asset GBM model. The model and parameter settings are the same as in Table 1. The biases and standard deviations of every experiment are reported in the *Bias* and *SD* columns. The upper and lower limits of the confidence intervals obtained by the LRD method are recorded in $Delta_{Lower}$ and $Delta_{Upper}$, respectively. We use the same set of 6 basis functions for all methods when we conduct regression procedures. For the FD method, we use $\Delta S_0 = 0.001$ and $\Delta\sigma = 0.0001$. For the MLSM method, we use 0.5 million sample paths and specify the initial distribution strictly following the guidance in Section 3.3 of Wang and Caflisch (2010). In accordance with Kaniel, Tompaidis, and Zemlianov (2008), we specify for the LRD method the number of sample paths in different levels of simulation as follows: 10,000 outer paths, 10,000 paths for the lower bound of the option price, 1 intermediate path, and 10,000 inner paths.

6. Concluding Remarks

In this paper, we develop an efficient IPA Monte Carlo method for estimating American option sensitivities. As shown in the paper, this new approach demonstrates attractiveness theoretically and numerically. It also has potential applicable implications on stochastic optimization problems with model uncertainty. People often frame these problems to a min-max formulation to obtain a robust solution; that is, they are looking for a best response in the most adverse situation. If we

parameterize the set of all conceivable models relevant to the considered problem, then searching for the robust solution is equivalent to finding the worst modeling parameter and solving a stochastic optimization problem under it. The estimator obtained in the paper indicates a way to estimate the gradient of the optimal value of a stochastic optimization problem under a given parameter. In conjunction with some simulation-based optimization methods, we believe that our discovery will help to deal with such problems.

Acknowledgments

References

Anderson, L. 2000. A simple approach to the pricing of Bermudan swaptions in the multi-factor Libor market model. *J. Comput. Finance* **3** 5–32.

Andersen, L., J. Andreasen. 2001. Factor dependence of Bermudan swaptions: fact or fiction? *J. of Finan. Econ.* **62** 3–37.

Andersen, L., M. Broadie. 2004. Primal-dual simulation algorithm for pricing multidimensional American options. *Management Sci.* **50** 1222–1234.

Asmussen, S., P. W. Glynn. 2007. Stochastic Simulation: Algorithms and Analysis. Springer, New York.

Avramidis, A., H. Mitzinger. 2002. Convergence of the stochastic mesh estimator for pricing American options. *Proc. 2002 Winter Simulation Conf.*. Eds. by E. Yücesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes. 1560-1567.

Belomestny, D. 2011. Pricing Bermudan options using nonparametric regression: optimal rates of convergence for lower estimates. *Finance Stochastics* **15** 655–683.

Belomestny, D., G. N. Milstein, and J. Shoenmakers. 2010. Sensitivities for Bermudan options by regression methods. *Decisions Econ Finan* **33** 117–138.

Broadie, M., P. Glasserman. 1996. Estimating security price derivatives using simulation. *Management Sci.* **42** 269–285.

Broadie, M., P. Glasserman. 2004. A stochastic mesh method for pricing high-dimensional American options. *J. Comput. Finance* **7** 35–62.

Chen, N., P. Glasserman. 2007. Malliavin Greeks without Mailliavin calculus. *Stoch. Proc. Appl.* **117** 1689–1723.

Clément, E., D. Lambertion, P. Protter. 2002. An analysis of a least squares regression method for American option pricing. *Finance Stochastics* **6** 449–471.

Fournié, E., J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. 1999. Applications of Malliavin calculus to Monte Carlo methods in finance.*Finance Stochastics* **3** 391–412.

Fu, M. 2006. Gradient estimation. In *Handbook on Operations Research and Management Science: Simulation, S.G. Henderson and B.L. Nelson, editors*, 575–616. Elsevier.

Glasserman, P. 1991. Gradient Estimation via Perturbation Analysis. Kluwer.

Glasserman, P. 2004. Monte Carlo Methods in Financial Engineering. Springer, New York.

Glasserman, P., B. Yu. 2004. Number of paths versus number of basis functions in American option pricing. *Ann. Appl. Probab.* **14** 2090–2119.

Gobet, E. 2004. Revisiting the Greeks for European and American options. *Proceedings of the International Symposium on Stochastic Processes and Mathematical Finance at Ritsumeikan University, Kusatsu, Japan, March 2003.* Eds. by J. Akahori, S. Ogawa, S. Watanabe. World Scientific, 53-71.

d'Halluin, Y., P. A. Forsyth, and G. Labahn. 2004. A penalty method for American options with jump diffusion processes. *Numerische Mathematik* **97(2)** 321–352.

Haugh, M., L. Kogan. 2004. Pricing American options: a duality approach. *Oper. Res.* **52** 258–270.

Hirsa, A., D. B. Madan. 2003. Pricing American options under variance gamma. *J. Comp. Finance* **7(2)** 63-80.

Ho, Y. C., X. R. Cao. 1983. Optimization and perturbation analysis of queueing networks. *Journal of Optimization: Theory and Applications* **40** 559–582.

Huang, J., M. Subrahmanyam, and G. Yu. 1996. Pricing and hedging American options: a recursive integration method. *Rev. Financial Stud.* **9** 277–300.

Hull, J. C. 2009. Options, Futures, and Other Derivatives. Pearson Prentice Hall, Upper Saddle River, New Jersey.

Kaniel, R., S. Tompaidis, and A. Zemlianov. 2008. Efficient computation of hedging parameters for discretely exercisable options. *Oper. Res.* **56** 811–826.

Kou, S. G. 2002. A jump-diffusion model for option pricing. *Management Sci.* **48** 1086–1101.

L'Ecuyer, P. 1990. A unified view of the IPA, SF, and LR gradient estimation techniques. *Management Sci.* **36** 1364–1383.

L'Ecuyer, P. 1995. Note: On the interchange of derivative and expectation for likelihood ratio derivative estimators. *Management Sci.* **36** 738–748.

Liu, G., L. J. Hong. 2010. Kernel estimation of the Greeks for options with discontinuous payoffs. *Oper. Res.* Forthcoming.

Longstaff, F. A., E. S. Schwartz. 2001. Valuing American options by simulation: a simple least-squares approach. *Rev. Financial Stud.* **14** 113–147.

Madan, D. B., P. Carr, and E. C. Chang. 1998. The variance gamma process and option pricing. *Euro. Finan. Rev.* **2** 79–105.

Merton, R. 1976. Option pricing when underlying stock returns are discontinuous. *J. Finan. Econ.* **3** 125–144.

Piterbarg, V. 2003. Computing Deltas of callable Libor exotics in forward Libor models. *J. Comput. Finance* **7** 107–144.

Piterbarg, V. 2004. Risk sensitivities of Bermuda swaptions. *International Journal of Theoretical and Applied Finance* **7** 465–509.

Porteus, E. 2002. Stochastic Inventory Theory. Stanford University Press, Stanford, California.

Raymar, S., M. Zwecher. 1997, A Monte Carlo valuation of American call options on the maximum of several stocks. *J. Derivatives* **5** 7–23.

Rogers, L. C. G. 2002, Monte Carlo valuation of American options. *Math. Finance* **12** 271–286.

Stentoft, L. 2004. Convergence of the least square Monte Carlo approach to American option valuation. *Management Sci.* **50** 1193–1203.

Suri, R., M. Zazanis. 1988. Perturbation analysis gives strongly consistent estimates for the M/G/1 queue. *Management Sci.* **34** 39–64.

Svenstrup, M. 2005. On the suboptimality of single-factor exercise strategies for Bermudan swaptions. *J. Finan. Econ.* **78** 651–684.

Tsitsiklis, J. 1984. Periodic review inventory systems with continous demand and discrete order sizes. *Management Sci.* **30** 1250–1254.

Tsitsiklis, J., B. Van Roy. 1999. Optimal stopping of Markov processes: Hilbert space theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives. *IEEE Trans. Automatic Control* **44** 1840–1851.

Tsitsiklis, J., B. Van Roy. 2001. Regression methods for pricing complex American-style options. *IEEE Trans. Neural Networks* **12** 694–703.

Tompaidis, S., C. Yang. 2010. Pricing American-style options by Monte Carlo simulation: alternatives to ordinary least squares. Working paper of University of Texas at Austin.

Veinott, A., Jr. 1965a. Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Sci.* **12** 206–222.

Veinott, A., Jr. 1965b. The optimal inventory policy for batch ordering. *Oper. Res.* **13** 424–432.

Wang, Y., R. Caflisch. 2010. Pricing and hedging American-style options: a simple simulation-based approach. *J. Comput. Finance* **13(4)** 95–125.

Wang, Y., M. C. Fu, and S. I. Marcus. 2009. Sensitivity analysis for barrier options. *Proc. 2009 Winter Simulation Conf.* Eds. by M. D. Rossetti, R. R. Hill, B. Johansson, A. Dunkin, and R. G. Ingalls. 1272–1282.

Zhang, X. L. 1997. Numerical analysis of american option pricing in a jump-diffusion model. *Math. Oper. Res.,* **22** 668–690.

Nan Chen is an associate professor at Dept. of Systems Engineering and Engineering Management, Chinese University of Hong Kong. He obtained his PhD degree in Operations Research from Columbia University in 2006. His main research interests include quantitative methods in finance, Monte Carlo simulation, and applied probability.

Yanchu Liu is an assistant professor of finance at Lingnan (University) College, Sun Yat-sen University, Guangzhou, China. He received his Ph.D. degree in systems engineering and engineering management from the Chinese University of Hong Kong in 2012. His main research interests include financial engineering and related areas.

Electronic Companion

This online appendix consists of the technical proofs of all statements in the main text, details of the likelihood ratio estimators, derivation of Example 3.1, and supplementary details for Section 3.2.

EC.1. Technical Lemmas

Lemma EC.1.1 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}^n$ are both Lipschitz continuous. Then the composition of these two, $f(g(x))$, is also Lipschitz continuous.*

Proof. It is easy to see that, for any $x, y \in \mathbb{R}^d$,

$$|f(g(x)) - f(g(y))| \leq K\|g(x) - g(y)\| \leq KL\|x - y\|$$

for some constants K and L, thanks to the Lipschitz property of f and g. Therefore, $f(g(x))$ is also Lipschitz continuous. $\square$

Lemma EC.1.2 *Under Assumptions 2.1, 4.1, and 4.2 in the paper, all continuation value functions $C_i(x; \theta)$, $0 \leq i \leq N - 1$, are Lipschitz continuous with respect to both x and θ.*

Proof. Establish this lemma via a backward induction. First, notice that

$$C_{N-1}(x; \theta) = E[h(N, X_N^\theta)|X_{N-1}^\theta = x] = E[h(N, F_N(x, R_N; \theta))].$$

Using the Lipschitz continuity of F_N and h and Lemma EC.1.1, the function $h(N, F_N(x, R_N; \theta))$ should satisfy

$$|h(N, F_N(x, R_N; \theta)) - h(N, F_N(y, R_N; \theta))| \leq kK_N(R_N, \theta)\|x - y\|$$

for any $x, y \in \mathbb{R}^d$. Therefore,

$$|C_{N-1}(x; \theta) - C_{N-1}(y; \theta)| \leq E[|h(N, F_N(x, R_N; \theta)) - h(N, F_N(y, R_N; \theta))|] \leq kE[K_N(R_N, \theta)] \cdot \|x - y\|.$$

The integrability of K_N implies that C_{N-1} must be Lipschitz continuous in x. A similar argument will lead to the conclusion that C_{N-1} is Lipschitz continuous in θ as well.

Now assume that C_{i+1} is Lipschitz continuous. Consider the option value function $Q_{i+1}(x; \theta) = \max\{h(i + 1, x), C_{i+1}(x; \theta)\}$. It is a composition function of $h(i, x)$, $C_{i+1}(x; \theta)$, and a max function. All of them are Lipschitz continuous. By Lemma EC.1.1, we know that $Q_{i+1}(x; \theta)$ should be Lipschitz continuous too; that is, there exists a constant $c_{i+1}(\theta)$ such that

$$|Q_{i+1}(x; \theta) - Q_{i+1}(y; \theta)| \leq c_{i+1}(\theta) \cdot \|x - y\|.$$

This implies

$$|C_i(x;\theta) - C_i(y;\theta)| \leq E[|Q_{i+1}(F_{i+1}(x,R_{i+1};\theta);\theta) - Q_{i+1}(F_{i+1}(y,R_{i+1};\theta);\theta)|]$$
$$\leq c_{i+1}(\theta) \cdot E[\|F_{i+1}(x,R_{i+1};\theta) - F_{i+1}(y,R_{i+1};\theta)\|].$$

Making use of the Lipschitz continuity of F_{i+1}, we can easily show that C_i is Lipschitz continuous in x. Following similar arguments, we can also show that C_i is Lipschitz continuous with respect to θ. $\square$

Lemma EC.1.3 *Suppose that Assumption 2.1 holds. Then,*

$$E\left[\sup_{\theta,\theta'\in\Theta} \frac{\|X_i^\theta - X_i^{\theta'}\|}{|\theta - \theta'|}\right] < +\infty, \quad \textit{for all } 1 \leq i \leq N.$$

Proof. For any $1 \leq i \leq N$, by the recursive definition of X_i^θ, we have

$$\|X_i^\theta - X_i^{\theta'}\| = \|F_i(X_{i-1}^\theta;R_i,\theta) - F_i(X_{i-1}^{\theta'};R_i,\theta')\|$$
$$\leq \|F_i(X_{i-1}^\theta;R_i,\theta) - F_i(X_{i-1}^\theta;R_i,\theta')\| + \|F_i(X_{i-1}^\theta;R_i,\theta') - F_i(X_{i-1}^{\theta'};R_i,\theta')\|.$$

Using Assumption 2.1, the right hand side of the above inequality is bounded above by

$$G_i(X_{i-1}^\theta,R_i)|\theta - \theta'| + K_i(R_i,\theta')\|X_{i-1}^\theta - X_{i-1}^{\theta'}\|.$$

Therefore,

$$E\left[\sup_{\theta,\theta'\in\Theta} \frac{\|X_i^\theta - X_i^{\theta'}\|}{|\theta - \theta'|}\right] \leq E[\sup_{\theta\in\Theta} G_i(X_{i-1}^\theta,R_i)] + E\left[\sup_{\theta'\in\Theta} K_i(R_i,\theta') \cdot \sup_{\theta,\theta'\in\Theta} \frac{\|X_{i-1}^\theta - X_{i-1}^{\theta'}\|}{|\theta - \theta'|}\right]$$
$$\leq E[\sup_{\theta\in\Theta} G_i(X_{i-1}^\theta,R_i)] + E[\sup_{\theta'\in\Theta} K_i(R_i,\theta')] \cdot E\left[\sup_{\theta,\theta'\in\Theta} \frac{\|X_{i-1}^\theta - X_{i-1}^{\theta'}\|}{|\theta - \theta'|}\right], \tag{EC.1}$$

where the second inequality is due to the fact that R_i is independent of X_{i-1}^θ and $X_{i-1}^{\theta'}$.

Note that

$$E\left[\sup_{\theta,\theta'\in\Theta} \frac{\|X_1^\theta - X_1^{\theta'}\|}{|\theta - \theta'|}\right] \leq E\left[\sup_{\theta,\theta'\in\Theta} \frac{\|F_1(x;R_i,\theta) - F_1(x;R_i,\theta')\|}{|\theta - \theta'|}\right] \leq E[G_1(x,R_1)] < +\infty.$$

It is easy to prove the lemma by induction, combining with the recursive relationship (EC.1). $\square$

Lemma EC.1.4 *Suppose the conditions of Theorem 4.1 are satisfied. Then,*

$$\left|\frac{\partial Q_0}{\partial x_j} - \frac{\widetilde{\partial Q_0}}{\partial x_j}\right| \leq E\left[\sum_{i=1}^{N-1} \left|\nabla h(i,X_i^\theta) \cdot Y_i^j - \nabla C_i(X_i^\theta;\theta) \cdot Y_i^j\right| 1_{\{X_i^\theta \in I_i^\theta\}}\right],$$

where

$$I_i^\theta := \{x \in \mathbb{R}^d : \tilde{C}_i(x;\theta) \le h(i,x), C_i(x;\theta) > h(i,x)\} \cup \{x \in \mathbb{R}^d : \tilde{C}_i(x;\theta) > h(i,x), C_i(x;\theta) \le h(i,x)\},$$

for $1 \le i \le N-1$.

Proof. The proof of this lemma is inspired by Belomestny (2011). Use one-dimensional cases as an illustration for the purpose of notational simplicity. From the proof of Lemma EC.1.2, we have

$$C_i'(x;\theta) = E\left[\frac{dQ_{i+1}}{dx}(F_{i+1}(x, R_{i+1};\theta);\theta)\right]$$

$$= E\left[\left(h'(i+1, X_{i+1}^\theta)1_{\{C_{i+1}(X_{i+1}^\theta;\theta)\le h(i+1,X_{i+1}^\theta)\}} + C_{i+1}'(X_{i+1}^\theta;\theta)1_{\{C_{i+1}(X_{i+1}^\theta;\theta)>h(i+1,X_{i+1}^\theta)\}}\right)\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\right]$$

$$\text{(EC.2)}$$

for $1 \le i \le N$, where we use the chain rule of differentiation in the last equality. In particular,

$$\frac{dQ_0}{dx} = \frac{dC_0}{dx} = E[h'(1, X_1^\theta)\cdot Y_1 1_{\{h(1,X_1^\theta)\ge C_1(X_1^\theta;\theta)\}} + C_1'(X_1^\theta;\theta)\cdot Y_1 1_{\{h(1,X_1^\theta)<C_1(X_1^\theta;\theta)\}}].$$

Given $\tilde{C}_i(x;\theta)$, $1 \le i \le N$, define a new function recursively such that $\mathcal{C}_N(x;\theta) = 0$ and

$$\mathcal{C}_i(x;\theta)$$
$$= E\left[\left(h'(i+1, X_{i+1}^\theta)1_{\{\tilde{C}_{i+1}(X_{i+1}^\theta;\theta)\le h(i+1,X_{i+1}^\theta)\}} + \mathcal{C}_{i+1}(X_{i+1}^\theta;\theta)1_{\{\tilde{C}_{i+1}(X_{i+1}^\theta;\theta)>h(i+1,X_{i+1}^\theta)\}}\right)\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\right]$$

for $1 \le i \le N-1$. In words, $\mathcal{C}_i(x;\theta)$ is obtained when we replace C by $\tilde{C}$ in (EC.2). Following the proof of Theorem 4.1, it is straightforward to establish that

$$\frac{\widetilde{dQ_0}}{dx} = \mathcal{C}_0(x;\theta).$$

On the other hand, taking a difference between C_i' and $\mathcal{C}_i$, we have

$$C_i'(x;\theta) - \mathcal{C}_i(x;\theta)$$
$$= E\left[(h'(i+1, X_{i+1}^\theta) - C_{i+1}'(X_{i+1}^\theta;\theta))\cdot\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\cdot 1_{\{h(i+1,X_{i+1}^\theta)\ge C_{i+1}(X_{i+1}^\theta;\theta), h(i+1,X_{i+1}^\theta)<\tilde{C}_{i+1}(X_{i+1}^\theta;\theta)\}}\right]$$
$$+ E\left[(C_{i+1}'(X_{i+1}^\theta;\theta) - \mathcal{C}_{i+1}(X_{i+1}^\theta;\theta))\cdot\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\cdot 1_{\{h(i+1,X_{i+1}^\theta)<C_{i+1}(X_{i+1}^\theta;\theta), h(i+1,X_{i+1}^\theta)<\tilde{C}_{i+1}(X_{i+1}^\theta;\theta)\}}\right]$$
$$+ E\left[(C_{i+1}'(X_{i+1}^\theta;\theta) - h'(i+1, X_{i+1}^\theta))\cdot\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\cdot 1_{\{h(i+1,X_{i+1}^\theta)<C_{i+1}(X_{i+1}^\theta;\theta), h(i+1,X_{i+1}^\theta)\ge\tilde{C}_{i+1}(X_1^\theta;\theta)\}}\right].$$

This implies that

$$|C_i'(x;\theta) - \mathcal{C}_i(x;\theta)|$$
$$\le E\left[|h'(i+1, X_{i+1}^\theta) - C_{i+1}'(X_{i+1}^\theta;\theta)|\cdot\left|\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\right|\cdot 1_{\{X_{i+1}^\theta\in I_{i+1}^\theta\}}|X_i^\theta = x\right]$$
$$+ E\left[|C_{i+1}'(X_{i+1}^\theta;\theta) - \mathcal{C}_{i+1}(X_{i+1}^\theta;\theta)|\cdot\left|\frac{\partial F_{i+1}}{\partial x}(x, R_{i+1};\theta)\right|\cdot|X_i^\theta = x\right]. \qquad \text{(EC.3)}$$

Repeat using Eq. (EC.3) and exploiting the recursive definition of Y, we can prove that

$$\left| \frac{dQ_0}{dx} - \frac{\widetilde{dQ_0}}{dx} \right| \leq E\left[\sum_{i=1}^{N-1} |h'(i, X_i^\theta) \cdot Y_i - C_i'(X_i^\theta; \theta) \cdot Y_i| 1_{\{X_i^\theta \in I_i^\theta\}} \right]. \quad \square$$

EC.2. Proofs of Main Results

Proof of Theorem 3.1. By the definition, we have

$$\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0) = E[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)]$$

$$= E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) \right]$$

$$+ E\left[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0) \right],$$

where we add and subtract $E[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta)]$ simultaneously on the right-hand side of the above equality. The sub-optimality of $\psi^*(\theta_0)$ when the parameter value is $\theta_0 + \Delta\theta$ implies that

$$\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0) \geq E\left[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0) \right]. \tag{EC.4}$$

Therefore,

$$\liminf_{\Delta\theta \downarrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \geq \liminf_{\Delta\theta \downarrow 0} E\left[\frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta} \right] \tag{EC.5}$$

and

$$\limsup_{\Delta\theta \uparrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \leq \limsup_{\Delta\theta \uparrow 0} E\left[\frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta} \right]. \tag{EC.6}$$

Note that

$$\left| \frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta} \right| \leq \sup_{\psi \in \Psi} \sup_{\theta_1, \theta_2 \in \Theta} \frac{|L(\psi, \xi, \theta_1) - L(\psi, \xi, \theta_2)|}{|\theta_1 - \theta_2|}$$

for sufficiently small $\Delta\theta$ such that $\theta_0, \theta_0 + \Delta\theta \in \Theta$. The integrability of the term on the right-hand side by (i), together with the dominated convergence theorem, yields that

$$\lim_{\Delta\theta \to 0} E\left[\frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta} \right] = E\left[\lim_{\Delta\theta \to 0} \frac{L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)}{\Delta\theta} \right]$$

$$= E\left[\frac{\partial L}{\partial \theta}(\psi, \xi, \theta_0) \Big|_{\psi = \psi^*(\theta_0)} \right]. \tag{EC.7}$$

Therefore, we can take the limits inside the expectations on the right-hand sides of both (EC.5) and (EC.6) to obtain

$$\liminf_{\Delta\theta \downarrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \geq E\left[\frac{\partial L}{\partial \theta}(\psi, \xi, \theta_0) \Big|_{\psi = \psi^*(\theta_0)} \right], \tag{EC.8}$$

and

$$\limsup_{\Delta\theta\uparrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \leq E\left[\frac{\partial L}{\partial\theta}(\psi, \xi, \theta_0)\Big|_{\psi=\psi^*(\theta_0)}\right]. \tag{EC.9}$$

Another way to decompose $\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)$ is to simultaneously add and subtract $E[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)]$. We have

$$\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0) = E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)\right]$$
$$+ E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0) - L(\psi^*(\theta_0), \xi, \theta_0)\right]$$
$$\leq E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)\right],$$

where we use the fact that $\psi^*(\theta_0 + \Delta\theta)$ is a sub-optimal policy when the parameter value is θ_0. This inequality leads to

$$\limsup_{\Delta\theta\downarrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta}$$
$$\leq \limsup_{\Delta\theta\downarrow 0} E\left[\frac{L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)}{\Delta\theta}\right] \tag{EC.10}$$

and

$$\liminf_{\Delta\theta\uparrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta}$$
$$\geq \liminf_{\Delta\theta\uparrow 0} E\left[\frac{L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)}{\Delta\theta}\right]. \tag{EC.11}$$

Now we claim that the limits on the right-hand sides of both (EC.10) and (EC.11) equal

$$E\left[\frac{\partial}{\partial\theta}L(\psi, \xi, \theta_0)\Big|_{\psi=\psi^*(\theta_0)}\right].$$

In fact, for any ψ, by the fundamental theorem of calculus,

$$E\left[L(\psi, \xi, \theta_0 + \Delta\theta) - L(\psi, \xi, \theta_0)\right] = E\left[\int_{\theta_0}^{\theta_0+\Delta\theta} \frac{\partial}{\partial\theta}L(\psi, \xi, \tilde{\theta})d\tilde{\theta}\right] = \int_{\theta_0}^{\theta_0+\Delta\theta} E\left[\frac{\partial}{\partial\theta}L(\psi, \xi, \tilde{\theta})\right]d\tilde{\theta}.$$

Substituting $\psi^*(\theta_0 + \Delta\theta)$ and $\psi^*(\theta_0)$ respectively into the above equality, we have

$$\left|\frac{E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)\right]}{\Delta\theta} - \frac{E\left[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)\right]}{\Delta\theta}\right|$$
$$\leq \frac{1}{\Delta\theta}\int_{\theta_0}^{\theta_0+\Delta\theta} E\left[\left|\frac{\partial}{\partial\theta}L(\psi^*(\theta_0 + \Delta\theta), \xi, \tilde{\theta}) - \frac{\partial}{\partial\theta}L(\psi^*(\theta_0), \xi, \tilde{\theta})\right|\right]d\tilde{\theta}$$
$$\leq E\left[\sup_{\tilde{\theta}\in(\theta_0-\delta,\theta_0+\delta)}\left|\frac{\partial}{\partial\theta}L(\psi^*(\theta_0 + \Delta\theta), \xi, \tilde{\theta}) - \frac{\partial}{\partial\theta}L(\psi^*(\theta_0), \xi, \tilde{\theta})\right|\right] \rightarrow 0,$$

as $\Delta\theta \to 0$ according to condition (iii). This implies

$$\lim_{\Delta\theta\to 0} \frac{E\left[L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0 + \Delta\theta), \xi, \theta_0)\right]}{\Delta\theta}$$
$$= \lim_{\Delta\theta\to 0} \frac{E\left[L(\psi^*(\theta_0), \xi, \theta_0 + \Delta\theta) - L(\psi^*(\theta_0), \xi, \theta_0)\right]}{\Delta\theta}.$$

By (EC.7), we know the claim is true and

$$\limsup_{\Delta\theta\downarrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \le E\left[\left.\frac{\partial L}{\partial\theta}(\psi, \xi, \theta_0)\right|_{\psi=\psi^*(\theta_0)}\right], \tag{EC.12}$$

and

$$\liminf_{\Delta\theta\uparrow 0} \frac{\alpha(\theta_0 + \Delta\theta) - \alpha(\theta_0)}{\Delta\theta} \ge E\left[\left.\frac{\partial L}{\partial\theta}(\psi, \xi, \theta_0)\right|_{\psi=\psi^*(\theta_0)}\right]. \tag{EC.13}$$

Finally, the theorem is proved from (EC.8-EC.9) and (EC.12-EC.13). □

Proof of Theorem 4.1. We need to verify that all the technical conditions of Theorem 3.1 are satisfied in the case of American options. To see this, we can view the payoff $h(i, X_i^\theta)$ as a function mapping the initial value x, the parameter θ, and the random vector $\{R_i, 1 \le i \le N\}$ to the exercising value. The verification of conditions (i) and (ii) is straightforward. Take the analysis for θ as an illustration. First, for any $\theta, \theta' \in \Theta$,

$$\sup_{1\le i\le N} \sup_{\theta,\theta'\in\Theta} \frac{|h(i, X_i^\theta) - h(i, X_i^{\theta'})|}{|\theta - \theta'|} \le k \sum_{i=1}^N \sup_{\theta,\theta'\in\Theta} \frac{\|X_i^\theta - X_i^{\theta'}\|}{|\theta - \theta'|}$$

from the Lipschitz property of h. Lemma EC.1.3 shows that the right-hand side of the above is integrable. Therefore, condition (i) is satisfied. Second, Assumption 4.2 ensures $h(i, X_i^\theta)$ is differentiable with respect to θ with probability 1.

As for the condition (iii), we know that

$$\tau^*(\theta + \Delta\theta) := \min\{i \in \{1, 2, \ldots, N\} : h(i, X_i^{\theta+\Delta\theta}) \ge C_i(X_i^{\theta+\Delta\theta}; \theta + \Delta\theta)\}.$$

It is the optimal stopping time under the perturbed parameter $\theta + \Delta\theta$. For any $1 \le i \ne j \le N$, based on the fact that both $h(i, X_i^\theta)$ and $C_i(X_i^\theta; \theta)$ are (almost surely) continuous in θ, we can show easily that

$$1_{\{\tau^*(\theta)=i, \ \tau^*(\theta+\Delta\theta)=j\}} \to 0, \ \text{a.s.,} \tag{EC.14}$$

as $\Delta\theta \to 0$. Fix any $\delta > 0$ such that $B_\delta(\theta) = (\theta - \delta, \theta + \delta) \subseteq \Theta$. We have

$$\lim_{\Delta\theta\to 0} E\left[\sup_{\tilde\theta\in B_\delta(\theta)} \left|\frac{\partial h}{\partial x}(\tau^*(\theta+\Delta\theta), X_{\tau^*(\theta+\Delta\theta)}^{\tilde\theta})\frac{\partial X_{\tau^*(\theta+\Delta\theta)}^{\tilde\theta}}{\partial\theta} - \frac{\partial h}{\partial x}(\tau^*(\theta), X_{\tau^*(\theta)}^{\tilde\theta})\frac{\partial X_{\tau^*(\theta)}^{\tilde\theta}}{\partial\theta}\right|\right]$$
$$= \lim_{\Delta\theta\to 0} \sum_{i,j=1, i\ne j}^N E\left[1_{\{\tau^*(\theta)=i, \ \tau^*(\theta+\Delta\theta)=j\}} \sup_{\tilde\theta\in B_\delta(\theta)} \left|\left(\frac{\partial h}{\partial x}(j, X_j^{\tilde\theta})\frac{\partial X_j^{\tilde\theta}}{\partial\theta} - \frac{\partial h}{\partial x}(i, X_i^{\tilde\theta})\frac{\partial X_i^{\tilde\theta}}{\partial\theta}\right)\right|\right] \tag{EC.15}$$

Thanks to the Lipschitz continuity of h, $|\partial h/\partial x| \le k$. Therefore,

$$\sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \left(\frac{\partial h}{\partial x}(j, X_j^{\tilde{\theta}}) \frac{\partial X_j^{\tilde{\theta}}}{\partial \theta} - \frac{\partial h}{\partial x}(i, X_i^{\tilde{\theta}}) \frac{\partial X_i^{\tilde{\theta}}}{\partial \theta} \right) \right| \le k \left(\sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial X_j^{\tilde{\theta}}}{\partial \theta} \right| + \sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial X_i^{\tilde{\theta}}}{\partial \theta} \right| \right). \quad \text{(EC.16)}$$

Furthermore, for any $1 \le i \le N$,

$$\sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial X_i^{\tilde{\theta}}}{\partial \theta} \right| \le \sup_{\theta, \theta' \in \Theta} \frac{\|X_i^\theta - X_i^{\theta'}\|}{|\theta - \theta'|}.$$

By Lemma EC.2 and the above inequality, we know that the right-hand side of (EC.16) is bounded by an integrable random variable. Using the dominated convergence theorem, we take the limit inside the expectation on the right-hand side of (EC.15) to obtain

$$\lim_{\Delta\theta \to 0} E \left[\sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial h}{\partial x}(\tau^*(\theta + \Delta\theta), X_{\tau^*(\theta+\Delta\theta)}^{\tilde{\theta}}) \frac{\partial X_{\tau^*(\theta+\Delta\theta)}^{\tilde{\theta}}}{\partial \theta} - \frac{\partial h}{\partial x}(\tau^*(\theta), X_{\tau^*(\theta)}^{\tilde{\theta}}) \frac{\partial X_{\tau^*(\theta)}^{\tilde{\theta}}}{\partial \theta} \right| \right]$$

$$\le k \sum_{i,j=1, i\ne j}^N E \left[\lim_{\Delta\theta \to 0} 1_{\{\tau^*(\theta)=i,\ \tau^*(\theta+\Delta\theta)=j\}} \left(\sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial X_j^{\tilde{\theta}}}{\partial \theta} \right| + \sup_{\tilde{\theta} \in B_\delta(\theta)} \left| \frac{\partial X_i^{\tilde{\theta}}}{\partial \theta} \right| \right) \right] = 0,$$

implied by (EC.14). $\square$

Proof of Theorem 4.2. Still use the one-dimensional case to show the main idea of the proof. By Lemma EC.1.4, we know that

$$\left| \frac{dQ_0}{dx} - \frac{\widetilde{dQ_0}}{dx} \right| \le \sum_{i=1}^{N-1} E \left[|h'(i, X_i^\theta) - C_i'(X_i^\theta; \theta)| \cdot Y_i \cdot 1_{\{X_i^\theta \in I_i^\theta\}} \right], \quad \text{(EC.17)}$$

where I_i^θ is defined as follows: for $1 \le i \le N-1$,

$$I_i^\theta := \{x \in \mathbb{R}^d : \tilde{C}_i(x; \theta) \le h(i, x), C_i(x; \theta) > h(i, x)\} \cup \{x \in \mathbb{R}^d : \tilde{C}_i(x; \theta) > h(i, x), C_i(x; \theta) \le h(i, x)\}.$$

Applying the Cauchy-Schwartz inequality to the right-hand side of (EC.17) will yield

$$E \left[|h'(i, X_i^\theta) - C_i'(X_i^\theta; \theta)| \cdot |Y_i| \cdot 1_{\{X_i^\theta \in I_i^\theta\}} \right] \le E^{1/2}[|h'(i, X_i^\theta) - C_i'(X_i^\theta; \theta)|^2 \cdot |Y_i|^2] \cdot P^{1/2}[X_i^\theta \in I_i^\theta]. \quad \text{(EC.18)}$$

Assumption 4.1 and Lemma EC.1.2 imply that both h and C_i are Lipschitz continuous functions in x. Therefore, their first-order derivatives should be bounded. In other words, we can find a constant K_1 such that

$$|h'(i, X_i^\theta) - C_i'(X_i^\theta; \theta)|^2 \le |h'(i, X_i^\theta)|^2 + |C_i'(X_i^\theta; \theta)|^2 \le K_1.$$

By the finiteness of the second moment of Y_i, it is easy to conclude that there exists another constant K_2 such that

$$E[|h'(i, X_i^\theta) - C_i'(X_i^\theta; \theta)|^2 \cdot |Y_i|^2] \le K_2. \quad \text{(EC.19)}$$

Combining (EC.17), (EC.18), and (EC.19), we have

$$\left|\frac{\partial Q_0}{\partial x_j} - \frac{\widetilde{\partial Q_0}}{\partial x_j}\right| \le \sqrt{K_2} \sum_{i=1}^{N-1} (P[X_i^\theta \in I_i^\theta])^{1/2}. \tag{EC.20}$$

Then we prove (20). First, we point out that the theorem holds if $E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| = 0$ for all $1 \le i \le N-1$. From this assumption, $\tilde{C}_i(X_i^\theta;\theta) = C_i(X_i^\theta;\theta)$ almost surely. It implies that $P[X_i^\theta \in I_i^\theta] = 0$. The error bound in the theorem holds accordingly.

Turn to the case that there exist some i such that $E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| \ne 0$. Without loss of generality, we assume that inequality is true for all i. Note that

$$\{X_i^\theta \in I_i^\theta\} \subseteq \{|h(i, X_i^\theta) - C_i(X_i^\theta;\theta)| \le |\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|\} =: \Gamma.$$

For any $\varepsilon > 0$,

$$P[X_i^\theta \in I_i^\theta] \le P[\Gamma]$$
$$= P[\Gamma \cap \{|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| > \varepsilon\}] + P[\Gamma \cap \{|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| \le \varepsilon\}]. \tag{EC.21}$$

The first summand on the right hand side of (EC.21) is less than $P[|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| > \varepsilon]$. Using the Kolmogrov inequality on this probability, we have

$$P[\Gamma \cap \{|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| > \varepsilon\}] \le \frac{E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|}{\varepsilon}.$$

As for the second summand in (EC.21), note that the event $\Gamma \cap \{|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)| \le \varepsilon\}$ implies that $|h(i, X_i^\theta) - C_i(X_i^\theta;\theta)| \le \varepsilon$, whose probability should be less than $K_3 \varepsilon^\alpha$ for some positive constant K_3 according to Assumption 4.3. In summary,

$$P[X_i^\theta \in I_i^\theta] \le \frac{E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|}{\varepsilon} + K_3 \varepsilon^\alpha \tag{EC.22}$$

In particular, if we let

$$\varepsilon = \left(E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|\right)^{\frac{1}{1+\alpha}},$$

in (EC.22), then

$$P[X_i^\theta \in I_i^\theta] \le (1 + K_3) \left(E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|\right)^{\frac{\alpha}{1+\alpha}}.$$

Combining it with (EC.20), it is easy to see that

$$\left|\frac{\partial Q_0}{\partial x_j} - \frac{\widetilde{\partial Q_0}}{\partial x_j}\right| \le \sqrt{K_2(1 + K_3)} \sum_{i=1}^{N-1} \left(E|\tilde{C}_i(X_i^\theta;\theta) - C_i(X_i^\theta;\theta)|\right)^{\frac{\alpha}{2(1+\alpha)}}.$$

We have proved the theorem. $\square$

EC.3. Likelihood Ratio Estimators

In contrast to the IPA method developed in the text, the LR method differentiates the probability density of the underlying price to produce unbiased sensitivity estimators; see Glynn (1987), Reiman and Weiss (1989), Rubinstein (1989), and Rubinstein and Shapiro (1993) for the developments of the method in the discrete-event simulation literature and Broadie and Glasserman (1996) and Glasserman and Zhao (1999) for the applications in finance. It typically requires less on the smoothness of the payoff functions. This feature is very appealing when we intend to derive second-order sensitivities for American options. As one can see, when applying the IPA method again on the first-order IPA estimators in the text, we will encounter a technical difficulty that ∇h in almost all popularly traded options is not differentiable. For instance, for the vanilla American put, $\nabla h(x) = -1_{\{x<K\}}$. It contains a discontinuity at $x = K$, which prevents us from using the IPA method to construct second-order derivative estimators.

This section is devoted to developing LR estimators for American options, including first and second orders for the purpose of completeness. To save notations, we consider one-dimensional cases only. We omit the proofs of all theorems in this appendix. They are available upon request. One can easily carry out the idea to apply for other higher dimensional problems. Assume in this appendix that the one-step transition density of X_i^θ at y, given $X_{i-1}^\theta = x$, is known as $g_i(y; x, \theta)$ for any $1 \le i \le N$, i.e.,

$$P[X_i^\theta \in dy | X_{i-1}^\theta = x] = g_i(y; x, \theta)dy.$$

The densities satisfy

Assumption EC.3.1 *Fix $\theta \in \Theta$. The transition probability density $g_1(y; x, \theta)$ is a continuous function in both y and x. The partial derivative $\partial g_1(y; x, \theta)/\partial x$ exists everywhere and it is continuous with respect to y. In addition, there exists a $\delta > 0$ such that*

$$\int_{\mathbb{R}_+} \sup_{u \in [-\delta, \delta]} \left| \frac{\partial g_1}{\partial x}(y; x + u, \theta) \right| \cdot |Q_1(y; \theta)| \, dy < +\infty.$$

Note that we can represent the option price by

$$Q_0(x; \theta) = E[Q_1(X_1^\theta; \theta) | X_0^\theta = x] = \int_{\mathbb{R}_+} Q_1(x_1; \theta) \cdot g_1(x_1; x, \theta) dx_1. \tag{EC.23}$$

Assumption EC.3.1 ensures that interchanging differentiation and integration is feasible, combining the general results from L'Ecuyer (1990, 1995) such that

$$\frac{\partial Q_0}{\partial x}(x; \theta) = \int_0^\infty Q_1(x_1; \theta) \cdot \frac{\partial g_1}{\partial x}(x_1; x, \theta) dx_1.$$

From this, we achieve

Theorem EC.3.1 *Under Assumption EC.3.1,*

$$\frac{\partial Q_0}{\partial x}(x;\theta) = E\left[h(\tau^*, X_{\tau^*}^\theta) \cdot \frac{\partial \log g_1(X_1^\theta; x, \theta)}{\partial x}\Big| X_0 = x\right].$$

Consider the sensitivity with respect to θ now. It is easy to see from (EC.23) that the parameter θ affects not only the one-step transition law $g_1(x_1; x, \theta)$ but also Q_1, the option value in t_1. This structure causes a subtle difference from applying the LR method to European-type options, in which we can shift all the dependency of θ to the probability law. To take differentiation under the integral in (EC.23), some additional technical conditions are imposed:

Assumption EC.3.2 *(i) For any $1 \leq i \leq N$, the transition density function $g_i(y; x, \theta)$ is continuously differentiable with respect to y, x, and θ.*

(ii) There exists an open neighborhood of θ, $\Upsilon \subseteq \Theta$, the set $\{\theta : C_i(x; \theta) = h(i, x)\} \cap \Upsilon$ is at most countably infinite for any given $x \in \mathbb{R}_+$ and $1 \leq i \leq N$.

(iii) For all $x \in \mathbb{R}_+$,

$$f_N(x) := \int_{\mathbb{R}_+} \sup_{\theta \in \Upsilon} \left|\frac{\partial g_N}{\partial \theta}(y; x, \theta)\right| \cdot h(N, y) dy < +\infty.$$

Furthermore, define a sequence of functions backward from $i = N-1$ to $i = 1$ such that

$$f_i(x) := \int_{\mathbb{R}_+} \sup_{\theta \in \Upsilon} \left[\left|\frac{\partial g_i}{\partial \theta}(y; x, \theta)\right| \cdot Q_i(y; \theta)\right] dy + \int_{\mathbb{R}_+} \sup_{\theta \in \Upsilon} g_i(y; x, \theta) \cdot f_{i+1}(y) dy.$$

All these functions satisfy $f_i(x) < +\infty$ for all $x \in \mathbb{R}_+$.

We can prove

Theorem EC.3.2 *Under Assumption EC.3.2,*

$$\frac{\partial Q_0}{\partial \theta}(x;\theta) = E\left[h(\tau^*, X_{\tau^*}^\theta) \cdot \sum_{i=1}^{\tau^*} \frac{\partial \log g_i(X_i^\theta; X_{i-1}^\theta, \theta)}{\partial \theta}\Big| X_0 = x\right].$$

The LR method is helpful when we develop second-order sensitivity estimators. As noted at the beginning of this appendix, we cannot apply the IPA method again on the first-order estimators directly due to the absence of smoothness. However, the LR method can help us to find a mixed unbiased estimator. Take gamma under the GBM model (7) as an example. We know that

$$\text{Delta} = E\left[-e^{-r\tau^*}\frac{S_{\tau^*}}{S_0}1_{\{S_{\tau^*} < K\}}\right]$$

from Theorem 4.1. To avoid the difficulty caused by the non-continuity of the indicator function inside the above expectation, we present it as follows:

$$\text{Delta} = E\left[\frac{E\left[-e^{-r\tau^*}S_{\tau^*}1_{\{S_{\tau^*}<K\}}|S_{t_1}\right]}{S_0}\right] = \int_0^{+\infty} \frac{E\left[-e^{-r\tau^*}S_{\tau^*}1_{\{S_{\tau^*}<K\}}|S_{t_1}=y\right]}{S_0}g_1(y; S_0)dy,$$

Table EC.1 LR and MIX estimators for single-asset American put options.

K	T	$Delta^*$	$Delta^{IPA}$	$Delta^{LR}$
35	7/12	-0.1341	-0.1344(0.00035)	-0.1279(0.0053)
40	7/12	-0.4299	-0.4302(0.00063)	-0.4196(0.0146)
45	7/12	-0.7922	-0.7926(0.00041)	-0.8113(0.0204)
K	T	$Vega^*$	$Vega^{IPA}$	$Vega^{LR}$
35	7/12	6.564	6.547(0.0211)	6.4134(0.1831)
40	7/12	11.7303	11.7163(0.0167)	11.1349(0.4216)
45	7/12	7.907	7.9199(0.0072)	7.0986(0.5076)
K	T	$Gamma^*$	$Gamma^{MIX}$	$Time$
35	7/12	0.0364	0.0351(0.0013)	58.13
40	7/12	0.0723	0.0706(0.0022)	86.71
45	7/12	0.0799	0.0811(0.0036)	118.26

Note. We use the same modeling setting and parameters as the in-sample estimation in Table 1. The superscript *IPA* indicates the estimators derived by our IPA approach and the superscript *LR* indicates the likelihood ratio estimators. Results in the column $Gamma^{MIX}$ are given by (EC.24). The numbers in the parentheses are the standard deviations of Monte Carlo across 1000 independent trials. In each trial, we use 0.5 million sample paths to obtain the estimates.

where the transition probability density of S_{t_1}, given $S_0 = s$, is

$$g_1(y; s) = \frac{1}{s\sigma\sqrt{\Delta t}}\phi\left(\frac{\log(y/s) - (\mu - \sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$$

and ϕ is the probability density function of a standard normal distribution. Such representation pushes out the dependency of S_0 from the indicator function. Applying the LR method, we can easily obtain an estimator such as

$$\text{Gamma}^{MIX} = E\left[-e^{-r\tau^*}\frac{S_{\tau^*}}{S_0^2}1_{\{S_{\tau^*} < K\}} \cdot \left(\frac{W_{t_1}}{\sigma\Delta t} - 1\right)\right]. \tag{EC.24}$$

Table EC.1 presents numerical results for the LR estimators of various sensitivities for the one-dimensional American put option. In general, the IPA estimators of the first-order sensitivities have smaller variances than their LR counterparts.

EC.4. Derivation of Example 3.1

By the definition of random variable ψ, we have

$$E[\max\{K - S_\psi, 0\}] = E[(K - S_{t_1})1_{\{S_{t_1} < K\}}] + E[(K - S_{t_2})1_{\{S_{t_1} \geq K,\ S_{t_2} < K\}}]. \tag{EC.25}$$

The integrand in the first summand of (EC.25), $(K - S_{t_1})1_{\{S_{t_1} < K\}} = (K - S_{t_1})^+$, is Lipschitz in S_{t_1}. Therefore, it is easy to show that

$$\frac{d}{ds}E\left[(K - S_{t_1})1_{\{S_{t_1} < K\}}\right] = -E\left[\frac{dS_{t_1}}{ds}1_{\{S_{t_1} < K\}}\right].$$

Consider the derivative of the second summand of (EC.25). Conditioning the value of S_{t_1}, we have

$$E[(K - S_{t_2})1_{\{S_{t_1} \geq K; S_{t_2} < K\}}] = E[E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}]1_{\{S_{t_1} \geq K\}}]. \qquad (EC.26)$$

Treat $E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}]1_{\{S_{t_1} \geq K\}}$ as a function of S_{t_1} and note that S_{t_1} depends on s through (7). By the chain rule,

$$\frac{d}{ds}\left(E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}]1_{\{S_{t_1} \geq K\}}\right)$$
$$= \frac{d}{dS_{t_1}}E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}] \cdot \frac{dS_{t_1}}{ds} \cdot 1_{\{S_{t_1} \geq K\}} + E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}] \cdot \delta_K(S_{t_1}) \cdot \frac{dS_{t_1}}{ds},$$

where $\delta_K(\cdot)$ is the Dirac delta function such that

$$\delta_K(x) = \begin{cases} +\infty, & x = K; \\ 0, & x \neq K \end{cases}$$

and we use the fact that the derivative of $f(x) = 1_{\{x < K\}}$ with respect to x is $\delta_K(x)$.

Therefore, if we take the derivative with respect to s under the expectation in (EC.26), we will get

$$\frac{d}{ds}E\left[(K - S_{t_2})1_{\{S_{t_1} \geq K; S_{t_2} < K\}}\right]$$
$$= E\left[\frac{d}{dS_{t_1}}E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}]\frac{dS_{t_1}}{ds}1_{\{S_{t_1} \geq K\}}\right] + E\left[E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}]\delta_K(S_{t_1}) \cdot \frac{dS_{t_1}}{ds}\right].$$
$$(EC.27)$$

Using the fact that

$$\frac{d}{dS_{t_1}}E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1}] = \frac{d}{dS_{t_1}}E[(K - S_{t_2})^+|S_{t_1}] = -E\left[\frac{dS_{t_2}}{dS_{t_1}}1_{\{S_{t_2} < K\}}\right],$$

the first summand on the right-hand side of (EC.27) equals

$$-E\left[\frac{dS_{t_2}}{ds}1_{\{S_{t_1} \geq K; S_{t_2} < K\}}\right].$$

On the other hand, the second summand on the right hand of (EC.27) equals

$$\int_0^{+\infty} E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1} = y] \cdot \delta_K(y) \cdot \frac{dS_{t_1}}{ds}g_1(y; s)ds = E[(K - S_{t_2})1_{\{S_{t_2} < K\}}|S_{t_1} = K] \cdot \frac{K}{s} \cdot g_1(K; s),$$

where $g_1(\cdot; s)$ is the transition density function of S_{t_1}, given $S_{t_0} = s$.

In summary, we have

$$\frac{d}{ds}E[\max\{K - S_\psi, 0\}] = -E\left[\frac{dS_{t_1}}{ds}1_{\{\psi = t_1\}}\right] - E\left[\frac{dS_{t_2}}{ds}1_{\{\psi = t_2; S_{t_2} < K\}}\right] + E[(K - S_{t_2})^+|S_{t_1} = K] \cdot \frac{K}{s} \cdot g_1(K; s).$$

EC.5. Supplementary Details for Section 3.2

Under the settings of Section 3.2, we consider a continuous random demand with log-normal distributions; that is, $D_i = \exp(\mu + Z_i)$ for $0 \leq i \leq N-1$, where μ is the model parameter of interest and $Z_i's$ are *i.i.d.* standard normal random variables. Two cases of the admissible order-size set $\mathcal{U}$ are investigated for the sake of checking the unbiasedness of our estimator. The first case is that the vendor can order arbitrary sizes of the product, i.e., $\mathcal{U} = \mathbb{R}_+$. Veinott (1965a) and Porteus (2002) prove that the inventory management problem (16) admits the following optimal policy in this case. Let

$$S = \Phi_D^{-1}\left(\frac{c_P - (1-\alpha)c}{c_P + c_H}\right).$$

The vendor should order $u_i^* = (S - x_i)^+$ at the beginning of period i for all $0 \leq i \leq N-1$. The other case is that each order must be in some nonnegative integral of a standard quantity q, i.e., $\mathcal{U} = \{kq : k \text{ is an integer}, k \geq 0\}$. This case generally occurs when the wholesale supplier of a vendor can only deliver orders in full containers with specific sizes. Take $q = 1$ for instance. Veinott (1965b) and Tsitsiklis (1984) point out that the optimal ordering policy should then be

$$u_i^* = \max\{0, \lceil S - x_i \rceil\},$$

where $\lceil x \rceil$ denotes the unique integer n satisfying $x \leq n < x+1$.

No matter which case it is, we can easily verify that the conditions (i-iii) in Theorem 3.1 are satisfied. The generalized IPA estimator for the sensitivity of the optimal cost with respect to μ is given by

$$E\left[\sum_{i=0}^{N-1} \alpha^i \left(c_P \mathbf{1}_{\{x_i+u_i^*<D_i\}} - c_H \mathbf{1}_{\{x_i+u_i^*>D_i\}}\right)\left(\sum_{j=0}^{i} D_j\right) + \alpha^N c\left(\sum_{j=0}^{N-1} D_j\right)\right]. \qquad \text{(EC.28)}$$

The following tables illustrate the unbiasedness of the estimator presented in (EC.28). Notice that a closed-form expression of the optimal cost is available under the case of continuum $\mathcal{U}$ if $x_0 < S$. It is equal to

$$\begin{aligned}
\text{Cost}^* = {} & c(S - x_0) + \frac{\alpha - \alpha^{N+1}}{1-\alpha} ce^{\mu+\frac{1}{2}} - \alpha^N cS \\
& + \frac{1-\alpha^N}{1-\alpha}c_H\left(S\mathcal{N}(\log S - \mu) - \exp\left(\mu+\frac{1}{2}\right)\mathcal{N}(\log S - \mu - 1)\right) \\
& + \frac{1-\alpha^N}{1-\alpha}c_P\left(\exp\left(\mu+\frac{1}{2}\right)\mathcal{N}(\mu+1-\log S) - S\mathcal{N}(\mu - \log S)\right), \qquad \text{(EC.29)}
\end{aligned}$$

where $\mathcal{N}(\cdot)$ denotes the cdf of a standard normal random variable.

We can compare the simulation outcomes from our estimator with the derivative of (EC.29) in μ in this case to verify if the generalized IPA estimator is unbiased. As for the other cases, we use the finite-difference estimators with common random numbers under a very large simulation cost and very small parameter perturbation to produce comparison benchmarks.

Table EC.2   Unbiased estimation for $dCost/d\mu$ in the case of $\mathcal{U} = \mathbb{R}_+$.

x_0	$Cost^{*CF}$	$Cost^{*MC}$	$Cost^{MC}$	$\frac{dCost}{d\mu}^{*CF}$	$\frac{dCost}{d\mu}^{*FD}$	$\frac{dCost}{d\mu}^{IPA}$
0.32	207.8692	207.8582	207.8585(0.2201)	209.7877	209.7711	209.7757(0.3811)
1.59	200.195	200.1795	200.1782(0.2204)	209.7877	209.7786	209.7815(0.3753)
3.2	190.6023	190.5861	190.5854(0.2204)	209.7877	209.7753	209.7896(0.3753)
3.52	NA	188.7411	188.7409(0.2196)	NA	208.2365	208.2396(0.3756)
4.8	NA	182.7143	182.7165(0.2162)	NA	201.4594	201.4623(0.3764)
6.4	NA	178.0232	178.0221(0.2113)	NA	190.6179	190.6205(0.379)

Note. Illustration of the unbiasedness for our generalized IPA estimators in the case of $\mathcal{U} = \mathbb{R}_+$. We set the following parameter values: $N = 4$, $\alpha = 0.97$, $\mu = 1.2$, and $c_H = c_P = c = 6$. From these settings, we can solve out $S = 3.2$. The columns $Cost^{*CF}$ and $\frac{dCost}{d\mu}^{*CF}$ are benchmark values for the cost and the sensitivity, respectively, based on closed-form expressions (EC.29). The column $Cost^{*MC}$ is the benchmark value for the cost based on Monte Carlo simulation with 100 million independent sample paths, while the column $\frac{dCost}{d\mu}^{*FD}$ is the benchmark value for the sensitivity based on the finite-difference method with a parameter perturbation $\Delta\mu = 0.0001$ and the same simulation budget with $Cost^{*MC}$. Finally, the columns $Cost^{MC}$ and $\frac{dCost}{d\mu}^{IPA}$ are estimates based on (16) and (EC.28), respectively, where the numbers in parentheses are the standard error computed from 1000 independent trials each with 0.5 million independent sample paths.

Table EC.3   Unbiased estimation for $dCost/d\mu$ in the case of $\mathcal{U} = \{0, 1, 2, \cdots\}$.

x_0	$Cost^{*CF}$	$Cost^{*MC}$	$Cost^{MC}$	$\frac{dCost}{d\mu}^{*CF}$	$\frac{dCost}{d\mu}^{*FD}$	$\frac{dCost}{d\mu}^{IPA}$
0.32	NA	208.5244	208.5358(0.2182)	NA	181.7006	181.7110(0.3726)
1.59	NA	200.9523	200.9687(0.2174)	NA	180.3691	180.3604(0.37)
3.2	NA	191.2386	191.2379(0.2181)	NA	182.2775	182.2908(0.3643)
3.52	NA	189.3820	189.3977(0.2169)	NA	180.7329	180.7336(0.3706)
4.8	NA	183.3176	183.3256(0.2138)	NA	174.5211	174.5811(0.3718)
6.4	NA	178.5621	178.5555(0.2090)	NA	165.0538	165.0639(0.3684)

Note. Illustration of the unbiasedness for our generalized IPA estimators in the case of $\mathcal{U} = \{0, 1, 2, \cdots\}$. Remaining settings are the same as that in Table EC.2.

References

Broadie, M., P. Glasserman. 1996. Estimating security price derivatives using simulation. *Management Sci.* **42** 269–285.

Glasserman, P., X. Zhao. 1999. Fast Greeks by simulation in forward LIBOR models. *J. Comput. Finance* **3** 5–39.

Glynn, P. 1987. Likelihood ratio gradient estimation: an overview. *Proc. 1987 Winter Simulation Conf.* Eds. by A. Thesen, H. Grant, and W. David Kelton. 366–374.

Porteus, E. 2002. Stochastic Inventory Theory. Stanford University Press, Stanford, California.

Reiman, M., A. Weiss. 1989. Sensitivity analysis for simulation via likelihood ratios. *Oper. Res.* **37** 830–844.

Rubistein, R. 1989. Sensitivity analysis and performance extrapolation for computer simulation models. *Oper. Res.* **37** 72–81.

Rubistein, R., A. Shapiro. 1993. Discrete Event System: Sensitivity Analysis and Stochastic Optimization. Wiley, New York.

Tsitsiklis, J. 1984. Periodic review inventory systems with continous demand and discrete order sizes. *Management Sci.* **30** 1250–1254.

Veinott, A., Jr. 1965a. Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Sci.* **12** 206–222.

Veinott, A., Jr. 1965b. The optimal inventory policy for batch ordering. *Oper. Res.* **13** 424–432.