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Pricing double-barrier options under a flexible jump diffusion model

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ABSTRACT

In this paper we present a Laplace transform-based analytical solution for pricing double-barrier options under a flexible hyper-exponential jump diffusion model (HEM). The major theoretical contribution is that we prove non-singularity of a related high-dimensional matrix, which guarantees the existence and uniqueness of the solution.

1. Introduction

Barrier options are among the most popular exotic options traded in financial markets. A barrier option offers the holder a payoff like that of a vanilla option, contingent on whether or not the underlying asset price process crosses some level(s)–called the barrier(s)–before or at the maturity date. In this paper we are going to study the pricing problem of double-barrier options under a flexible jump diffusion process for the underlying asset price.

The research of barrier options has been attracting a lot of attention in computational finance. Most studies on barrier option pricing are conducted under the Black-Scholes model (BSM) (see, e.g., [1–3]). Despite its simplicity, the BSM has obvious shortcomings but is a good description for the movements of the underlying asset prices. It assumes the asset returns are normally distributed and their variances remain constant. Empirical studies invalidate such assumptions by suggesting two observations for asset returns: the asymmetric leptokurtic feature, i.e., the actual return has much heavier tails than normal, and the volatility smile, i.e., the volatility implied from equity option prices is not a constant but presents a curve resembling a "smile".

In this paper we shall reinvestigate double-barrier option pricing problems under a new asset price model, the so called hyper-exponential jump diffusion model (HEM), proposed by Cai and Kou [4] recently. The model assumes the asset return follows a jump diffusion process with Poisson jump intensity and hyper-exponentially distributed jump sizes. It is appealing in two aspects. The asset returns of the HEM have heavier tails than normal distributions and then it is capable of generating smiles for equity options matching the empirical data. On the other hand, it retains some flexibility in modeling. As shown in Heyde and Kou [5], it may be very difficult to distinguish empirically the exponential-type tails from power-type tails even given a long period of daily data (e.g., 20 years daily prices). So, a sensible asset model should be more flexibility about the heaviness of the asset return tails. Thanks to the property of hypo-exponential distributions, it can approximate various distributions ranging from power tails to exponential tails by fine tuning parameters in the model (see, e.g., Feldmann and Whitt [6]).

Mathematically, the contributions of our work are twofold. First, we obtain analytical solutions to the prices of the standard double-barrier options in terms of Laplace transforms and then are able to invert them numerically via some efficient and accurate algorithms such as the Euler inversion algorithm proposed by Abate and Whitt [7] and Choudhury, Lucantoni and Whitt [8]. Second, we show the existence and uniqueness of the solutions. More precisely, our analytical pricing formulae involve solutions of some high-dimensional linear systems and thus their existence and uniqueness are reduced down to the non-singularity of the associated high-dimensional matrix. We manage to prove the matrix is invertible in this paper.

It is worth pointing out that similar technical issues also arise in some related work such as Cai and Kou [4] and Sepp [9]. In [4], Cai and Kou considered the single-barrier option pricing. They also showed the existence and uniqueness of their solution through

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non-singularity of a simpler matrix, which turns out to be a submatrix of ours in the double-barrier case. As a by-product of our work, we can duplicate their conclusion with a new proof. Sepp [9] priced standard double barrier options under the Kou's double exponential jump diffusion model (Kou [10], Kou and Wang [11]). The Kou's model assumes a double exponential distribution for jumps and therefore it is a special case of the HEM. In addition, Sepp [9] did not prove the existence and uniqueness of his solution.

The rest of the paper is organized as follows. In Section 2, we introduce the hyper-exponential jump diffusion model. Section 3 concentrates on deriving a general analytical formula relating to the joint distribution of the first passage time of the HEM to two flat barriers and the value of the HEM at the first passage time. Section 4 presents the analytical solution to the pricing problem of standard double-barrier options. Meanwhile, numerical results are also provided via the Euler inversion algorithm.

2. The model

We assume the asset price process $\{S_t : t \geq 0\}$ under the risk-neutral probability measure $\mathbb{P}$ is defined as $S_t := e^{X_t}$. The log-return process $\{X_t : t \geq 0\}$ follows the HEM given by

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $X_0 = \log(S_0)$, $\sigma > 0$, $\mu := r - \sigma^2/2 - \lambda\zeta$ with risk-free rate $r > 0$ and $\zeta = E[e^{Y_1}]$. $\{W_t : t \geq 0\}$ is a standard Brownian motion, $\{N_t : t \geq 0\}$ is a Poisson process with intensity λ. And $\{Y_i : i = 1, 2, \ldots\}$ is a sequence of independent identically distributed hyper-exponential random variables whose probability density function is given by

$$f_Y(y) = \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i y} \mathbf{1}_{\{y \geq 0\}} + \sum_{j=1}^{n} q_j \theta_j e^{\theta_j y} \mathbf{1}_{\{y<0\}}, \tag{2}$$

where $p_i \geq 0$, $\eta_i > 1$ for all $i = 1, \ldots, m$, $q_j \geq 0$, $\theta_j > 0$ for all $j = 1, \ldots, n$, and $\sum_{i=1}^m p_i + \sum_{j=1}^n q_j = 1$. From (2), we can see that there are m up-jumps and n down-jumps, among which the ith up-jump occurs with probability p_i and has an exponentially distributed jump size with mean $1/\eta_i$ for $i = 1, 2, \ldots, m$, and the jth down-jump occurs with probability q_j and has an exponential jump size with mean $1/\theta_j$ for any $j = 1, 2, \ldots, n$. We also assume $\{W_t\}$, $\{N_t\}$ and $\{Y_i\}$ are all independent.

It is easy to see that the infinitesimal generator of $\{X_t\}$ is given by

$$(Lu)(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y) \mathrm{d}y,$$

for any twice continuously differentiable function $u(x)$ and the the Lévy exponent of $\{X_t\}$ is given by

$$\begin{aligned} G(x) &:= \frac{1}{t} \log \mathbf{E}[\exp(xX_t)] \\ &= x\mu + \frac{1}{2}x^2\sigma^2 + \lambda \left(\sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right) \end{aligned}$$

for any $x \in (-\theta_1, \eta_1)$. By some elementary calculus, we can show for any given $a > 0$, the equation $G(x) = a$ has exactly $m + n + 2$ real roots $\beta_1, \ldots, \beta_{m+1}, -\gamma_1, \ldots, -\gamma_{n+1}$ satisfying

$$0 < \beta_1 < \eta_1 < \beta_2 < \cdots < \eta_m < \beta_{m+1} < \infty, \tag{3}$$

$$0 < \gamma_1 < \theta_1 < \gamma_2 < \cdots < \theta_n < \gamma_{n+1} < \infty. \tag{4}$$

We record this result for later references.

3. Distribution of the first passage time to two flat barriers

To price double barrier options, we need the joint distribution of first passage time of X to two barriers and the process value at the first passage time. In this section, we are going to derive the Laplace transform of the distribution. Define τ to be the first passage time of X_t to two flat barriers h and H ($h < H$), i.e.,

$$\tau := \inf\{t \geq 0 : X_t \geq H \text{ or } X_t \leq h\}. \tag{5}$$

From now on, use $\mathbf{E}^x$ and $\mathbf{P}^x$ to represent the expectation and the probability, respectively, when $\{X_t\}$ starts from $X_0 \equiv x$.

The joint distribution of τ and X_τ plays a crucial role when pricing double-barrier options. Our idea is to get it via the Laplace transform

$\mathbf{E}^x[e^{-a\tau + \theta X_\tau}]$.

The following theorem reaches a more general result for any expectations in the form of $\mathbf{E}^x[e^{-a\tau} f(X_\tau)]$, where f could be any nonnegative measurable function. The Laplace transform then becomes a direct corollary of the theorem.

Theorem 3.1. *Consider any nonnegative measurable function f such that $\int_0^{+\infty} f(y+H)e^{-\eta_i y}\mathrm{d}y$ and $\int_{-\infty}^0 f(y+h)e^{\theta_j y}\mathrm{d}y$ are integrable for all $1 \leq i \leq m$ and $1 \leq j \leq n$. For any $a > 0$ and $x \in (h, H)$, we have*

$$\mathbf{E}^x[e^{-a\tau} f(X_\tau)] = \boldsymbol{\varpi}(x)\mathbf{N}^{-1}\mathbf{f}, \tag{6}$$

where $\boldsymbol{\varpi}(x)$ is a row vector defined as

$$\begin{aligned} \boldsymbol{\varpi}(x) = \big(&e^{\beta_1(x-H)}, \ldots, e^{\beta_{m+1}(x-H)}, \\ &e^{-\gamma_1(x-h)}, \ldots, e^{-\gamma_{n+1}(x-h)}\big), \end{aligned} \tag{7}$$

$\mathbf{f}$ is a column vector such that $\mathbf{f} = (f_0^u, \ldots, f_m^u, f_0^d, \ldots, f_n^d)^T$,

$$\begin{aligned} f_0^u &= f(H), \qquad f_i^u = \int_0^{+\infty} f(y+H)e^{-\eta_i y}\mathrm{d}y, \quad 1 \leq i \leq m, \\ f_0^d &= f(h), \qquad f_j^d = \int_{-\infty}^0 f(y+h)e^{\theta_j y}\mathrm{d}y, \quad 1 \leq j \leq n; \end{aligned} \tag{8}$$

and $\mathbf{N}$ is an $(m+n+2) \times (m+n+2)$ non-singular matrix given by

$$\begin{bmatrix} 1 & \cdots & 1 & \bar{x}^{\gamma_1} & \cdots & \bar{x}^{\gamma_{n+1}} \\ \dfrac{1}{\eta_1 - \beta_1} & \cdots & \dfrac{1}{\eta_1 - \beta_{m+1}} & \dfrac{\bar{x}^{\gamma_1}}{\eta_1 + \gamma_1} & \cdots & \dfrac{\bar{x}^{\gamma_{n+1}}}{\eta_1 + \gamma_{n+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{\eta_m - \beta_1} & \cdots & \dfrac{1}{\eta_m - \beta_{m+1}} & \dfrac{\bar{x}^{\gamma_1}}{\eta_m + \gamma_1} & \cdots & \dfrac{\bar{x}^{\gamma_{n+1}}}{\eta_m + \gamma_{n+1}} \\ \bar{x}^{\beta_1} & \cdots & \bar{x}^{\beta_{m+1}} & 1 & \cdots & 1 \\ \dfrac{\bar{x}^{\beta_1}}{\theta_1 + \beta_1} & \cdots & \dfrac{\bar{x}^{\beta_{m+1}}}{\theta_1 + \beta_{m+1}} & \dfrac{1}{\theta_1 - \gamma_1} & \cdots & \dfrac{1}{\theta_1 - \gamma_{n+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\bar{x}^{\beta_1}}{\theta_n + \beta_1} & \cdots & \dfrac{\bar{x}^{\beta_{m+1}}}{\theta_n + \beta_{m+1}} & \dfrac{1}{\theta_n - \gamma_1} & \cdots & \dfrac{1}{\theta_n - \gamma_{n+1}} \end{bmatrix}$$

with $\bar{x} := e^{h-H}$.

To prove Theorem 3.1, the most difficult part is to show the non-singularity of the matrix $\mathbf{N}$. The proof is long and tedious. Due to the limitation of space, we just summarize the conclusion in the following proposition and include its proof to an online companion, which can be found in the second author's homepage (http://se.cuhk.edu.hk/people/nchen).

Proposition 3.1. *For any $\{\beta_i\}_{i=1}^{m+1}$ and $\{\gamma_j\}_{j=1}^{n+1}$ satisfying (3) and (4), the matrix $\mathbf{N}$ is non-singular.*

With the help of Proposition 3.1, we can show Theorem 3.1 now.

Proof of Theorem 3.1. Notice that τ is the first time the process X exits the band (h, H). It may leave the band from the boundaries, i.e., $X_\tau = H$ or $X_\tau = h$; or it may jump across the boundaries when leaving, i.e., $X_\tau > H$ or $X_\tau < h$. Introduce a sequence of events:

$$F_0 := \{\omega : X_\tau = H\}, \qquad G_0 := \{\omega : X_\tau = h\}$$

and

$$F_i := \{\omega : X_\tau - H > 0, Y_{N_\tau} \sim \mathrm{Exp}(\eta_i)\}$$

for $i = 1, 2, \ldots, m$ and

$$G_j := \{\omega : X_\tau - h < 0, -Y_{N_\tau} \sim \mathrm{Exp}(\theta_j)\}$$

for $j = 1, 2, \ldots, n$, indicating with which type of jump the process jumps across the boundaries when exiting (h, H). The events consist of a partition of the whole probability space and by the law of total probability, we have

$$\mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)] = \sum_{i=0}^{m} \mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)\mathbf{1}_{F_i}] + \sum_{j=0}^{n} \mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)\mathbf{1}_{G_j}]. \quad (9)$$

Emulating the proofs of Proposition 2.1 in [11], we can easily show that conditional on F_i, τ and X_τ are independent and moreover the overshoot $X_\tau - H$ is still exponentially distributed with mean $1/\eta_i$, thanks to the memoryless property of exponential distribution. Thus, for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, we have

$$\mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)\mathbf{1}_{F_i}] = \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}] \cdot \eta_i f_i^u. \quad (10)$$

$$\mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)\mathbf{1}_{G_j}] = \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}] \cdot \theta_j f_j^d. \quad (11)$$

Combining (9), (10) and (11),

$$\mathbf{E}^x[\mathrm{e}^{-a\tau} f(X_\tau)] = \sum_{i=0}^{m} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}] \cdot \eta_i f_i^u + \sum_{j=0}^{n} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}] \cdot \theta_j f_j^d, \quad (12)$$

with $\eta_0 = \theta_0 = 1$.

On the other hand, we are also able to obtain closed-form expressions for $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]$ and $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]$. Note that for any $a > 0$ and any pure imaginary number b,

$$M_t := \exp(-at + bX_t) - \exp(bX_0) - (G(b) - a)\int_0^t \exp(-as + bX_s)\mathrm{d}s$$

is a zero-mean martingale. By the optional sampling theorem, we know $\mathbf{E}^x[M_\tau] = 0$, i.e.,

$$0 = \mathbf{E}^x[\exp(-a\tau + bX_\tau)] - \mathrm{e}^{bx} - (G(b) - a)\mathbf{E}^x\left[\int_0^\tau \exp(-as + bX_s)\mathrm{d}s\right].$$

(12) provides us an expression for $\mathbf{E}^x[\exp(-a\tau + bX_\tau)]$. Substituting it in the right hand side of the above equality,

$$0 = \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_0}]\mathrm{e}^{bH} + \sum_{i=1}^{m} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]\mathrm{e}^{bH}\frac{\eta_i}{\eta_i - b} + \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_0}]\mathrm{e}^{bh} + \sum_{j=1}^{n} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]\mathrm{e}^{bh}\frac{\theta_j}{\theta_j + b} - \mathrm{e}^{bx} - (G(b) - a)\mathbf{E}^x\left[\int_0^\tau \exp(-as + bX_s)\mathrm{d}s\right]. \quad (13)$$

Denote the right hand side of (13) by $h(b)$. $h(b) \equiv 0$ for all pure imaginary b. Define $H(b) := \prod_{i=1}^{m}(\eta_i - b) \cdot \prod_{j=1}^{n}(\theta_j + b) \cdot h(b)$. Then $H(b)$ is well defined and analytic in the whole complex domain $\mathbb{C}$. $H(b)$ equals zero when b is a pure imaginary number. By the identity theorem of analytic functions in the complex domain ([12, Theorem 10.18]), we get $H(b) \equiv 0$ for all $b \in \mathbb{C}$. Accordingly, $h(b) = 0$ for all $b \in \mathbb{C} \setminus \{-\theta_n, \ldots, -\theta_1, \eta_1, \ldots, \eta_m\}$.

Replace b by β_i and $-\gamma_j$ in $h(b) = 0$, respectively. Note that β_i and $-\gamma_j$ are all the roots to $G(x) = a$. We have the following linear equations with respect to $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]$ and $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]$:

$$\mathrm{e}^{\beta_i x} = \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_0}]\mathrm{e}^{\beta_i H} + \sum_{i=1}^{m} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]\mathrm{e}^{\beta_i H}\frac{\eta_i}{\eta_i - \beta_i} + \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_0}]\mathrm{e}^{\beta_i h} + \sum_{j=1}^{n} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]\mathrm{e}^{\beta_i h}\frac{\theta_j}{\theta_j + \beta_i}$$

and

$$\mathrm{e}^{-\gamma_j x} = \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_0}]\mathrm{e}^{-\gamma_j H} + \sum_{i=1}^{m} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]\frac{\eta_i \mathrm{e}^{-\gamma_j H}}{\eta_i + \gamma_j} + \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_0}]\mathrm{e}^{-\gamma_j h} + \sum_{j=1}^{n} \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]\frac{\theta_j \mathrm{e}^{-\gamma_j h}}{\theta_j - \gamma_j}.$$

Proposition 3.1 has already shown the non-singularity of $\mathbf{N}$. It follows that the above linear system regarding $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_i}]$ and $\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_j}]$ is solvable and

$$\left(\mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_0}], \ldots, \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{F_m}], \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_0}], \ldots, \mathbf{E}^x[\mathrm{e}^{-a\tau}\mathbf{1}_{G_n}]\right) = \boldsymbol{\varpi}(x)\mathbf{N}^{-1}\mathrm{Diag}\left\{\frac{1}{\eta_0}, \ldots, \frac{1}{\eta_m}, \frac{1}{\theta_0}, \ldots, \frac{1}{\theta_n}\right\}, \quad (14)$$

where $\mathrm{Diag}\left\{\frac{1}{\eta_0}, \ldots, \frac{1}{\eta_m}, \frac{1}{\theta_0}, \ldots, \frac{1}{\theta_n}\right\}$ is a diagonal matrix. Plugging (14) into (12) yields (6) immediately. □

From Theorem 3.1, we can obtain closed-form expressions for the expectations of a variety of functions with respect to τ and X_τ. For instance, choosing $f(x)$ to be $\mathrm{e}^{\theta x}$ with $\theta \in (-\theta_1, \eta_1)$ in the above theorem, it is easy to derive the Laplace transform $\mathbf{E}^x[\mathrm{e}^{-a\tau + \theta X_\tau}]$, which is presented in the following corollary.

Corollary 3.1. *For any $\theta \in (-\theta_1, \eta_1)$, we have*

$$\mathbf{E}^x[\mathrm{e}^{-a\tau + \theta X_\tau}] = \mathrm{e}^{\theta H} \cdot \left(\sum_{i=1}^{m+1} \omega_i \mathrm{e}^{\beta_i (x - H)} + \sum_{j=1}^{n+1} \nu_j \mathrm{e}^{-\gamma_j (x - h)}\right) \quad (15)$$

where $(\omega_1, \ldots, \omega_{m+1}, \nu_1, \ldots, \nu_{n+1})^T = \mathbf{N}^{-1}\mathbf{J}(\theta)$ and $\mathbf{J}(\theta) = (1, \frac{1}{\eta_1 - \theta}, \ldots, \frac{1}{\eta_m - \theta}, \bar{x}^\theta, \frac{\bar{x}^\theta}{\theta_1 + \theta}, \ldots, \frac{\bar{x}^\theta}{\theta_n + \theta})^T$.

4. Pricing double-barrier options

In this section, we are going to derive pricing formulae for standard double-barrier options, based on the theoretical results obtained in the last section.

4.1. Standard double-barrier options

The payoff of a standard double-barrier option is activated (knocked in) or extinguished (knocked out) when the price of the underlying asset crosses barriers. For example, a knock-out call option will not give the holder the payoff of a European call option unless the underlying price remains within a pre-specified range before the option matures. More precisely, consider an interval (L, U) and the initial asset price S_0 is in it. The holder will receive $(S_T - K)^+ \mathbf{1}_{\{\tau > T\}}$ at the maturity T, where $\tau = \inf\{t \geq 0 : S_t \leq L \text{ or } S_t \geq U\}$. Under the risk-neutral measure $\mathbb{P}$ and the assumption

Table 1
The Laplace inversion (EI Price) vs. the Monte Carlo simulation (MC Value). For unvarying parameters, the default choices are $r = 0.05$, $m = n = 2$, $\eta_1 = 30$, $\eta_2 = 50$, $\theta_1 = 30$, $\theta_2 = 40$, $p_1 = p_2 = q_1 = q_2 = 0.25$, $S_0 = 100$, $U = 115$, $L = 80$, $T = 1$, and $\rho = 1$. Parameters for the Laplace inversion method are $A_1 = A_2 = 28.3$, $(n_1, n_2) = (11, 38)$, and the scaling factor $X = 1000$; while the MC values along with the associated 95% confidence intervals are obtained by using 60,000 time steps and simulating 100,000 sample paths.

Pricing knock-out call options

K	λ	$\sigma = 0.2$			$\sigma = 0.3$		
		EI price	MC value	95% CI	EI Price	MC Value	95% CI
	5	0.1052	0.1063	(0.1019, 0.1107)	0.01512	0.0158	(0.0141, 0.0175)
105	3	0.1156	0.1189	(0.1142, 0.1236)	0.01660	0.0175	(0.0157, 0.0193)
	1	0.1270	0.1300	(0.1252, 0.1348)	0.01822	0.0190	(0.0171, 0.0209)
	5	0.3456	0.3471	(0.3375, 0.3567)	0.05063	0.0522	(0.0484, 0.0560)
100	3	0.3804	0.3847	(0.3746, 0.3948)	0.05566	0.0575	(0.0535, 0.0615)
	1	0.4191	0.4210	(0.4105, 0.4315)	0.06120	0.0615	(0.0574, 0.0656)
	5	0.7812	0.7831	(0.7666, 0.7996)	0.1164	0.1182	(0.1116, 0.1248)
95	3	0.8606	0.8676	(0.8499, 0.8847)	0.1281	0.1306	(0.1236, 0.1376)
	1	0.9487	0.9478	(0.9298, 0.9658)	0.1410	0.1413	(0.1341, 0.1485)

that the underlying asset follows the HEM, the price of such option is given by $e^{-rT}\mathbf{E}[(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} | S_0]$.

We may use Corollary 3.1 to obtain a double Laplace transform for the above expectation. For this purpose, change some variables: let $h := \log(L/S_0)$, $H := \log(U/S_0)$ and $\kappa := -\log K$. Then, the expectation can be represented as

$$C(\kappa, T) := \mathbf{E}^x\left[(S_0 e^{X_T} - e^{-\kappa})\mathbf{1}_{\{\tau > T, S_0 e^{X_T} > e^{-\kappa}\}}\right].$$

Conduct a double Laplace transform on the new function $C(\kappa, T)$ with respect to κ and T. Note that the definition domains for κ and T are $(-\infty, \infty)$ and $(0, \infty)$, respectively. We have the following theorem:

Theorem 4.1. *For any* $0 < \varphi < \eta_1 - 1$ *and* $a > \max\{G(\varphi + 1), 0\}$, *let*

$$g(\varphi, a) = \int_0^\infty \int_{-\infty}^\infty e^{-\varphi\kappa - aT} C(\kappa, T) d\kappa dT. \quad (16)$$

Then,

$$g(\varphi, a) = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \frac{1}{a - G(\varphi+1)} \left(1 - e^{(\varphi+1)H}\left(\sum_{i=1}^{m+1} \omega_i e^{-\beta_i H} + \sum_{j=1}^{n+1} \nu_j e^{\gamma_j h}\right)\right), \quad (17)$$

where

$$(\omega_1, \omega_2, \ldots, \omega_{m+1}, \nu_1, \nu_2, \ldots, \nu_{n+1})^T = \mathbf{N}^{-1}\mathbf{J}(\varphi + 1).$$

Proof. For any fixed T, by the Fubini theorem, it is easy to show that

$$\int_{-\infty}^\infty e^{-\varphi\kappa} C(\kappa, T) d\kappa = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \cdot \frac{1}{a - G(\varphi+1)} (1 - \mathbf{E}^x[e^{-a\tau + (\varphi+1)X_\tau}]).$$

Applying Corollary 3.1 here, we can immediately obtain the conclusion. □

Once we have the double Laplace transform, we apply some numerical inversion algorithms to recover the value of the function $C(\kappa, T)$ at some specific κ and T we want to price. There are several other double-barrier options such as knock-out put, knock-in call or put traded in the market. The pricing formulae for them can be obtained through similar derivations.

4.2. Numerical examples

In this section, we intend to price the above standard knock-out call options by inverting the associated Laplace transforms (17) numerically via the Euler inversion algorithm. This algorithm was introduced by Abate and Whitt [7] and Choudhury, Lucantoni and Whitt [8]. Petrella [13] gave some improvements in inverting a two-sided Laplace transform. His method is faster and more stable numerically than the original Euler inversion when dealing with two-sided transforms, due to the introduction of a scaling factor. In (17) the Laplace transform with respect to κ is two-sided. Thus we use his algorithm in the following numerical examples.

In our numerical example, m and n are both 2 in the hyper-exponential distribution (2). The numerical results for the standard double-barrier options (denoted by EI Price) are given in Table 1, where we also show the Monte Carlo simulation result (denoted by MC Value) as a benchmark together with the associated 95% confidence interval (denoted by 95% CI). We can see that all the EI Prices stay within the 95% confidence intervals of the associated MC Values. Besides, based on a PC with Pentium(R) 4 CPU 2.80 GHz, 1 GB of RAM, the CPU time to produce one numerical result via Euler inversion algorithm is only around 6 s, while it takes about 20 min to generate one MC Value. Consequently, we draw the conclusion that the pricing method based on our analytical pricing formulae as well as the Euler inversion algorithm is accurate and efficient. It is worth mentioning that in Table 1, MC Values tend to be greater than EI Prices partly because the Monte Carlo simulation method overestimates the option prices due to the systematic discretization bias. Since our main purpose is to study the analytical solution rather than the Monte Carlo simulation method. We refer the interested readers to Metwally and Atiya [14] for more detailed discussions on the systematic discretization error reduction.

From the table, we can also see that the option price decreases as the strike K increases. This is intuitive because the payoff is a decreasing function in K. Meanwhile, when either σ or λ increases, the option price depreciates. That is because the option tends to be more likely knocked out when the underlying is more volatile.

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