An Optimization View of Financial Systemic Risk Modeling: Network Effect and Market Liquidity Effect

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Abstract

Financial institutions are interconnected directly by holding debt claims against each other (the network channel), and they are also bound by the market when selling assets to raise cash in distressful circumstances (the liquidity channel). The goal of our study is to investigate how these two channels of risk interact to propagate individual defaults to a system-wide catastrophe. We formulate a constrained optimization problem that incorporates both channels of risk, and exploit the problem structure to generate the solution (to the clearing payment vector) via a partition algorithm. Through sensitivity analysis, we are able to identify two key contributors to financial systemic risk, the network multiplier and the liquidity amplifier, and to discern the qualitative difference between the two, confirming that the market liquidity effect has a great potential to cause systemwide contagion. We illustrate the network and market liquidity effects — in particular, the significance of the latter — in the formation of systemic risk with data from the European banking system. Our results contribute to a better understanding of the effectiveness of certain policy interventions. In addition, our algorithm can be used to pin down the changes of the net worth (marked to market) of each bank in the system as the spillover effect spreads, so as to estimate the extent of contagion, and to provide a metric of financial resilience as well. Our framework can also be easily extended to incorporate the effect of bankruptcy costs.

Keywords: Systemic risk, financial network, contagion, market liquidity.

1 Introduction

The complex interconnectedness of the modern financial system binds financial institutions tightly together to an unprecedented degree, such that failure at one or several institutions due to excessive idiosyncratic risk taking can quickly propagate through it to set off cascading disasters. While the 2007-2009 US credit crisis and the European sovereignty debt crisis have triggered much debate as to the causes, culprits, and lessons learned, a growing body of literature points to the prominent roles played by two risk-transmission channels in amplifying the severity of the crises. The first channel is the direct debt exposures among financial institutions. They hold heavy liabilities against each other and therefore a loss caused by one default will be easily transmitted to the others. We refer to this as the *network channel* later in this paper. The second channel for contagion, referred to below as the *liquidity channel*, is that institutions are also interconnected indirectly through the market. A fire sale initiated by one distressed institution for the purpose of fund raising, in particular under difficult aggregate economic conditions, will drive down the asset price sharply. As the institutions across the system accumulate large positions in the assets of similar nature during normal periods, such price decline will create a serious negative externality to the systemic stability in a crisis.

Our investigation reported here is primarily motivated by two issues. How should we quantitatively characterize the systemic impacts of the two channels of risk, in particular through their interplay? How should we discern the difference in strengths between the two channels in causing cascading contagion in the financial system? Any insights we obtain in studying these issues will certainly contribute to our understanding of financial systemic risk, help the development of risk and resilience metrics for financial contagion, and inform regulatory practices and policies.

1.1 Contribution of the Current Paper

We use the same modeling framework as the one in [3], where the two channels of risk are clearly brought to bear. A relative liability matrix captures the network effect — how the banks interact with each other via interbank exposures; and an inverse demand function $Q(\cdot)$ captures the market liquidity effect — selling of illiquid assets at one bank will negatively impact all other banks holding such assets, as the value of the assets will be "discounted" by Q. Our model is, however, more general, in that we require minimal assumptions on Q.

As one main contribution of the current paper, we reformulate the model to an optimization problem with equilibrium constraints and develop a partition algorithm to solve for the maximal clearing payment vector and asset price. The key idea underpinning the algorithm is that, we identify some "obvious" defaults first, then adjust repayment and asset sales of each institution accordingly, and mark the asset values of the institutions to the market-clearing security price to identify more defaults for the next round. This procedure is iterated until no new default institutions are found. In this way, it reveals a *hierarchy*, or sequential order relation, among the defaults. Through this lens, we can better examine the interplay of the two effects, in particular, how the market liquidity is depleted by successive fire sales as defaults cascade and how the depressed asset price in turn reinforces financial contagion.

The hierarchy structure of our partition algorithm enables us to develop estimates of the probability that default at a given bank in the system causes defaults at other banks. From such estimation, we find that two factors determine the systemic impact of each institution on the rest of the financial system. One is its position in the topology of a liability network, or more specifically, how close it is to other banks in terms of liability exposure. The other is its illiquidity concentration, how large portfolio of illiquid assets it is holding.

We synthesizes these two factors into a Katz centrality-like metric to measure the system's resilience against external shocks. The metric is computed from the banks' net worths that are marked to the security price after the liquidation of the failed banks. Hence, it features the global and first-hand impact the failed banks yield to the system through the liquidity channel: low price depressed by their liquidation will impair the capital base of all the others, making them more susceptible to further contagion. In this sense, it is different from other centrality measures discussed intensively in the network literature that focuses more on diffusion-type contagion through the local neighborhood structure. Moreover, the metric takes a product form, showing that the network effect will magnify the price impact on equity value from the market in a *multiplicative* manner. Although it is a common sense that the market value, as opposed to the book value, of the net worth of a banking system better reflects its ability to weather adverse shocks, our finding concretizes this qualitative idea into an economic metric and explicitly relates it to contagion probability estimates.

To further discern the two effects, we apply sensitivity analysis, a standard tool in optimization area, to investigate how the optimal equilibrium repayments and security price change in response to external value shocks. It delineates these two channels clearly. The network effect takes the form of a linear factor (i.e., a *multiplier*) in determining sensitivities of the equilibrium with respect to the shocks. In contrast, the market liquidity effect takes the form of a denominator; and we call this an *amplifier*. In a given financial system (i.e., the relative liability matrix P is fixed), the network multiplier has a finite value because the impact through this channel decays exponentially in the course of transmission. However, as its structure suggests, the liquidity amplifier has a potential to become dominantly large when the total sales of illiquid asset approach what the market can absorb. The relationship between the liquidity amplifier and network multiplier as they appear in the sensitivity measures is also a product, indicating once again that the two effects are multiplicative in propagating the systemic risk.

The above comparison results lead to policy implications. Using the liquidity amplifier and network multiplier, we assess the effectiveness of several intervention policies used in the crisis management, including direct asset purchase and capital injection. We find that different policies have different regulatory focuses. One interesting observation (see Theorem 9 in Section 4) is that the asset purchase program should have a greater impact in improving the market liquidity than capital injection, but it may be less effective than the latter to preempt massive failures in a highly leveraged system. In Appendix C.2, we also discuss an undesirable pro-cyclical effect caused by the regulation of capital adequacy requirement. It is long observed in the literature that this regulation exacerbates fire sales during market turbulence and has perverse effects on stability. Our optimization framework provides an appropriate tool to characterize such adverse effects: this requirement introduces an additional constraint to the original optimization problem and therefore makes the solution more sensitive to external shocks.

Bankruptcy costs magnify the severity of systemic risk, through costs like legal and administrative fees associated with restructuring failed banks and delays in payments and service disruptions to their creditors and customers. We can easily extend our model to incorporate the effects of such non-market bankruptcy costs by introducing a fixed recovery rate on asset values when a bank defaults. The numerical result shows that even a small bankruptcy cost will lead to an appreciable contagion effect in the presence of market liquidity. This finding underscores the importance of orderly resolution of failed banks during a financial crisis when the market liquidity condition is adverse.

The optimization-based method can be used to develop counterfactual simulation schemes to test the resilience of financial systems against systemic shocks. Inclusion of the liquidity channel will contribute a new aspect to such schemes, as we note that a majority of the existing simulation methodologies, such as those reviewed in [50], account for only the network effect. In this paper, we perform numerical experiments on the data released by the European Banking Authority after its 2011 EU-wide stress test to illustrate the basic notions and methodologies of our formulation. The evidence from the experiments suggest that the liquidity channel has a greater potential to trigger massive contagion than the network.

Note that the reason why we cannot observe a significant contagion effect in the numerical experiment is that the interbank liabilities are just a very small fraction of the aggregate assets of the European banking sector in the dataset we examine. A more proper interpretation of the experiment outcomes should be that simple spillover effects through interbank lending will have only limited impact *if* the leverage that financial institutions are allowed to take is controlled. In other words, it still will pose a considerable threat to the systemic stability when the expansion of interbank claims goes unchecked as what the entire financial sector experienced prior to the US credit crisis.

In addition, counterparty credit risk in over-the-counter (OTC) derivatives contributes to another form of network interconnectedness. Due to the lack of information about the size of OTC derivatives taken by the European banks in the dataset, one limitation of the current paper is that we do not incorporate the impact of this particular interconnectedness, which will apparently affect our assessments on the systemic risk in a financial system.

1.2 Related Literature

The contagion effect in a financial system has been well investigated in the literature. Some early papers, such as [46], [40], [3], [28], and [41], study the economic origin of interbank networks. They demonstrate via some highly stylized models how the network channel, a risk-sharing mechanism of a financial system under normal conditions, will help to transmit the systemic crisis when there is a global shortage for liquidity. An influential work of [23] quantifies the network effect in a fixed point model, showing that in theory the original shortfall in the payment of a single institution can cascade through the interbank liability system. It triggers a substantial body of investigation on the relationship between systemic risk and network topology. We refer the reader to [30], [42], [31], [37], [43], [1], [6], and [24] for more details about the recent developments in this direction. Another line of research, represented by [48], [36], [4], [33], and [14], primarily focuses on the contagious effect of asset price. Their research indicates that defensive asset liquidation triggers a large scaled fire sale, generating adverse welfare consequences for the entire system such as high price volatility, more bank defaults, and market inefficiency. [49] provides an excellent survey on models of counterpart contagion and their application in systemic risk management. Most of the above literature concentrates on one of the two effects. The current paper aims to establish a general framework to integrate both effects.

As noted in Section 1.1, our analysis builds up on the basis of a variation of the model presented in [3]. Their paper proposes a procedure to solve for equilibrium in two steps. First, they rely on a fixed-point based mechanism to solve a market-clearing price. Then, with the price being fixed, they simplify the problem down to computing repayments to clear the liability network. Clearly, the idea behind their approach is to separate the two effects in order to solve them one by one. This separation idea focuses much on the ultimate equilibrium and ignores how the interplay between the network and market liquidity shapes up its formation. As a complement to their work, our optimization formulation and the related partition algorithm derive many new insights in this regard.

Several other papers in the literature share similar modeling vehicles as the current one. [5] incorporate the price impact through a model with exogenous default costs. With it, they execute simulation exercises on random graphs to examine the impact of the degree of interconnectedness to the system stability in the presence of liquidity risk. The model captures the cascade effect well, but does not suffice to reflect the severity of contagion compared with our model in which defaults are endogenously determined. The research of [7] explores the risk mitigation effect of a central clearing counterparty in an interbank market. Our work focuses more on the methodology developments. All these studies complement each other from different angles without much overlapping.

It is long known in the literature that only the factor of interbank liabilities itself cannot generate significant contagion effect; see [29], [26], [19] for example for numerical experiments conducted on the domestic interbank datasets from a variety of nations. [34] develop a general framework to derive estimates for contagion probability using observable aggregate liability information. Their numerical experiments based on a dataset of the European banking system also indicate that the network effect has only a very limited impact. In addition, [47] show that the banks have a positive incentive to form a consortium to rescue distressed financial institutions in order to avoid the costs caused by the liquidation of bankrupt banks.

All the above empirical and theoretical works call for a full investigation on other transmission channels of the systemic risk than the network effect, especially the channels involving the price effect. One nice contribution along this research line comes from [17]. They quantify the impact of loss-triggered fire sales on the portfolios of financial institutions across the market. The forensic analysis in their paper on the Quant Crash of August 2007 and the Great Deleveraging following the default of Lehman Brothers in September 2008 documents strong destabilizing effects caused by the market liquidity. [35] and [21] use detailed balance sheet data for the European and US banking industry to study the influence of fire-sale spillovers on systemic vulnerability, respectively. These literatures provide concrete examples of liquidity-based contagion. Our paper aims to develop new analytical tools in this direction.

1.3 Paper Organization

The rest of the paper is organized as follows. In Section 2, we present our optimization formulation and develop the partition algorithm to solve for the maximal equilibrium. We then examine in Section 3 how to derive estimates of contagion probability based on the net worth of the financial institutions, making use of the insights gained from the algorithm. Section 4 presents the sensitivity analysis and discusses the effectiveness of different intervention policies. Section 5 collects numerical experiments on the data from the 2011 EU-wide stress test. All the technical proofs are provided in the E-Companion.

2 The Optimization Approach of Systemic Risk Modeling

2.1 Notation

Throughout the paper all vectors are row vectors unless otherwise specified. In particular, any vector that pre-multiplies a matrix is, naturally, a row vector, and this is the majority case below. Only occasionally will we have cases in which a column vector post multiplies a matrix. The inner product of two vectors, u and v, will be simply written as uv, with the understanding that u is a row vector and v a column vector. We use I to denote the identity matrix; $\mathbf{e}_i$, the i-th standard basis vector of the Euclidean space; and $\mathbf{1}$ and $\mathbf{0}$, vectors of all 1's and all 0's, respectively. For a matrix M and two index subsets $\mathcal{I}$ and $\mathcal{J}$, $M_{\mathcal{I},\mathcal{J}}$ represents the submatrix of M consisting of the rows and columns indexed by $\mathcal{I}$ and $\mathcal{J}$, respectively; and write $M_{\mathcal{I}}$ if $\mathcal{I} = \mathcal{J}$. The partial ordering between two vectors u and v, $u \geq v$, is in the component-wise sense, and so are their minimum and maximum, $u \wedge v$ and $u \vee v$. For a (non-negative) vector u, denote the sum of its components (L_1 norm) as $|u|$. In addition, we use $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ to indicate a probability measure and its related expectation.

2.2 Model Formulation

We follow the framework of [3] to consider a financial system with n banks, indexed by $i = 1, \cdots, n$. Assume that each bank i invests at time 0 in three categories of assets: external projects such as loans to households and non-financial sectors, marketable securities, and interbank debts. Table 1 illustrates a detailed breakdown of the balance sheet of a representative bank in the system.

Assets	Liabilities and owner's equity
External investments: β_i	External debt claim: b_i
Interbank loans: L_{ki} for $k \neq i$	Interbank liabilities: L_{ij} for $j \neq i$
Liquid securities: $\bar{y}_i$	Equity: e_i
Illiquid securities: $\bar{s}_i$	

Table 1: The Balance Sheet of a Representative Bank in the Financial System

The system features two layers of interconnectedness discussed in the introduction. One is on its liability side: every bank owes some amount of money to creditors inside and outside the network. From now on, we will use $\ell := (\ell_i)$ and $P := (p_{ij})$ to indicate the liability vector and the relative liability matrix of this network, respectively, where

$$\ell_i := b_i + \sum_{j \neq i} L_{ij} \qquad \text{and} \qquad p_{ij} := L_{ij}/\ell_i, \qquad i,j = 1,\dots,n. \tag{1}$$

Note, P is a *substochastic* matrix (i.e., non-negative, and each row sum is ≤ 1). Assume throughout below, the spectral radius of P is < 1; thus, $I - P$ is invertible. In fact, $I - P$ is an M-matrix; hence, all of its principal sub-matrices are invertible and non-negative. Refer to [15], Chapter 5, for more details on M-matrix.

The other layer of interconnectedness resides in the asset side of every bank. As shown in Table 1, the assets that bank i initially owns are divided into liquid and illiquid securities, of amounts $\bar{y}_i$ and $\bar{s}_i$, respectively; refer to the discussion in Section 2.4 for a further clarification about this division. Assume that the liquid security can be converted into cash at its face value. In contrast, should the bank sell an amount $s_i (\in [0, \bar{s}_i])$ of its illiquid asset, the corresponding proceeds it receives will be $s_i q$, where

$$q = Q\left(|s|\right) := Q\left(\sum_i s_i\right). \tag{2}$$

with $s = (s_1, \cdots, s_n)$ recording the sale amounts of the illiquid asset from individual banks in the system. Here we use the function Q to model the degree of illiquidity in the market; as such we assume it is continuous and satisfies

Assumption 1. *(i)* $Q(0) = 1$*; (ii)* $Q(\cdot) \geq 0$ *and* $Q(\cdot)$ *is non-increasing.*

The non-increasing property of function Q captures the "discount for immediacy" in asset liquidation: larger amounts of the illiquid security are sold in the market concurrently, lower level its price will be pushed down to. Note the above conditions on Q is minimal. A special case of the Q function satisfying the above assumption, $Q(s) = \exp(-\gamma s)$ (with the constant parameter $\gamma > 0$), has been used by [3] and others.

Given a realization of $\beta := (\beta_1, \cdots, \beta_n)$, the value vector of the external investments from the system, our objective is to find the clearing repayment $x := (x_1, \cdots, x_n)$, where x_i denotes the amount bank i actually pays to its creditors. Alongside, we also need to determine vectors of sales of liquid and illiquid securities $y = (y_i)$ and $s = (s_i)$, and the market-clearing price of the illiquid security q. First, the repayment of each bank should comply with the principle of limited liability. Namely, it needs to pay all of its liabilities in full if it can; or, if short of that, it should declare default and liquidate its available assets to repay debts. Assume that all the debt claims are of equal seniority so that the banks' repayments are proportional to the amounts of their notional liabilities. Then, the total repayment received by bank i from all other banks is $\sum_{j \neq i} x_j p_{ji}$. In addition, bank i may sell amounts y_i and s_i of liquid and illiquid securities to meet its liability repayment, adding the total amount of cash available to this bank up to

$$\beta_i + \sum_{j \neq i} x_j p_{ji} + y_i + s_i q.$$

Limited liability of the bank thus requires that

$$x_i = \ell_i \wedge \big(\beta_i + y_i + \sum_{j\neq i} x_j p_{ji} + s_i q\big). \tag{3}$$

Second, the security markets should be cleared in the equilibrium. Assume that no short sales are allowed in our model. Note that this assumption may limit us from modeling derivative positions. Future work is needed to address this limitation. Assume that the banks will attempt to sell liquid assets first to raise cash when they have shortfalls between the due liability ℓ_i and the incomes they received from the external investments and the other banks' repayments. Therefore, the total amount of the liquid security sale is given by

$$y_i = \bar{y}_i \wedge d_i^1, \quad \text{with} \quad d_i^1 := [\ell_i - (\beta_i + \sum_{j\neq i} x_j p_{ji})]^+, \tag{4}$$

noting that such liquidation will be capped by $\bar{y}_i$ under the no-short-sale rule. After a bank exhausts all of its liquid holding, it will start to tap into illiquid security sale. The amount of the illiquid asset needed to be sold is then

$$s_i = \bar{s}_i \wedge d_i^2, \quad \text{with} \quad d_i^2 := \left\{ \frac{[\ell_i - (\beta_i + \sum_{j\neq i} x_j p_{ji}) - y_i]^+}{q} \right\}. \tag{5}$$

Given everything else the same, s_i tends to be larger when q is small in Eq. (5). Hence, a low price of the illiquid security, depressed for instance by massive fire sales initiated by some major financial institutions, will trigger the others to liquidate more to raise funds. In this sense, the market price q can yield a significant negative externality to the banks in the system. To summarize the preceding model setting up, we define

Definition 2. A quadruple (x, y, s, q) is called a *market-clearing repayment equilibrium* if it satisfies the market-clearing condition (2), the limited liability condition (3), and the asset sale equations (4) and (5), for $i = 1, \ldots, n$.

As shown in the following example, limited liquidity may cause multiple equilibria for the problem.

Example 3. *Consider two banks 1 and 2 whose balance sheets are given in Table 2. Both of them are holding 1 and 2 dollars (face value) of illiquid securities, i.e.,* $\bar{s}_1 = 1$ *and* $\bar{s}_2 = 2$*. Their businesses are partially financed by some borrowings from the creditors outside of the system with the notional values being* $b_1 = b_2 = 1$*. Note that these two banks are interlinked only through the market channel. We assume that the inverse demand function in this two-bank market is given by* $Q(s) = \exp(-s)$*.*

Bank 1	
External investments β_1	External debt b_1
Iliquid asset $\bar{s}_1$	Equity e_1

Bank 2	
External investments β_2	External debt b_2
Iliquid asset $\bar{s}_2$	Equity e_2

Table 2: Balance sheets of the two-bank system in Example 3.

Suppose that $\beta_1 = 0.1$ *and* $\beta_2 = 0.9$*. Apparently, both banks have to liquidate part (or all) of the assets to raise cash to pay off the debts in the equilibrium, i.e.,* $s_1, s_2 > 0$*. According to Definition 2, we need to* *solve the following equations to search for the repayment equilibria:*

$$s_1 = 1 \wedge \frac{0.9}{q}, \qquad s_2 = 2 \wedge \frac{0.1}{q}, \qquad \textit{and} \qquad q = \exp(-(s_1+s_2)), \tag{6}$$

where \$0.9 and \$0.1 are the respective shortfalls for both banks. No interior solution to (6) exists. In fact, suppose that it is not the case and we have $s_1 < 1$ *and* $s_2 < 2$ *satisfying the above equations. That implies*

$$s_1 q = 0.9, \quad s_2 q = 0.1 \quad \Rightarrow \quad (s_1+s_2)q = (s_1+s_2)\exp(-(s_1+s_2)) = 1,$$

leading to a contradiction because

$$\max_{0\le s_1\le 1, 0\le s_2\le 2} (s_1+s_2)\exp(-(s_1+s_2)) = \exp(-1) < 1. \tag{7}$$

Therefore, either $s_1 = 1$ *or* $s_2 = 2$ *must hold in any equilibrium. At least two possibilities will then arise:* $(s_1, s_2) = (1, 0.4092)$ *and* $(s_1, s_2) = (1, 2)$. *In the first equilibrium,* $q = 0.2443$, $x_1 = 0.3443 < 1$ *and* $x_2 = 1$, *meaning that bank 1 defaults but bank 2 does not. However, neither banks will survive in the second equilibrium because* $q = 0.0498$, $x_1 = 0.1498 < 1$ *and* $x_2 = 0.9996 < 1$ *under it.* □

Example 3 reveals that limited liquidity constitutes an important source of equilibrium multiplicity. The total shortfall of the two banks amounts to \$1, exceeding the maximal liquidity the market can provide as shown by (7). In this situation, different liquidation order will lead to different equilibria. Suppose that we force bank 1 to be liquidated first, i.e., letting $s_1 = 1$. This reduces the equations in (6) down to solving $s_2\exp(-(1+s_2)) = 0.1$. Clearly, $s_2 = 0.4092$ is the solution in $[0, 2]$ to the above equation, which leads to the first equilibrium. However, if we specify that the default and liquidation of bank 2 occur first, we cannot find any $s_1 \in [0, 1]$ satisfying $s_1\exp(-(s_1+2)) = 0.9$. The banking system will end up at the second equilibrium. No direct debt exposure exists between the two banks in this example. Market illiquidity should be the sole factor in triggering equilibrium multiplicity here.

[1] developed some sufficient conditions on Q for uniqueness of the clearing asset price and liability payments in the model of [3]. One strong assumption needed there is that $sQ(s)$ should be increasing in $s \in [0, \sum_i \bar{s}_i]$, which is not satisfied by the aforementioned exponential Q except for sufficiently small constant γ. Our primary interest in this research is on how systemic risk develops under extreme market environments, such as massive evaporation of market liquidity to absorb asset fire sales. In this sense, a large value of γ in the exponential demand function Q should be more relevant for our purpose. Thus, we omit this condition to maintain more flexibility in calibrating the model to a variety of liquidity situations.

2.3 The Optimization Approach

Observe that the failure of bank 1, the one with the largest shortfall in Example 3, is necessary, because it will default no matter which equilibrium ultimately materializes. However, the default of bank 2, together with the adverse social welfare outcome such as an extremely low market-clearing price for the illiquid securities in the second equilibrium, can be avoided if we handle the resolution of failed banks properly. This motivates us to use an optimization formulation to study the maximum equilibrium. It will help to identify sources of necessary defaults in a financial system so that we may adopt various intervention policies to preempt contagion from them to the remaining part of the system. To this end, consider the following problem to find an equilibrium with the greatest repayment vector:

$$\max_{x,s,y} |x| \qquad \text{s.t.} \quad x = \ell \wedge (\beta + y + sq + xP), \quad y = \bar{y} \wedge d^1, \quad s = \bar{s} \wedge (d^2/q), \quad q = Q(|s|). \tag{8}$$

Here we rewrite the four conditions of Definition 2 into their respective matrix forms to simplify the notations. In particular, the second and third constraints in the optimization problem (8) are corresponding to the asset sale equations (4) and (5), where $d^j = (d^j_1, \cdots, d^j_n)$, $j = 1, 2$.

Consider any feasible solution (x, y, s, q) of (8). We can partition the n banks into two subsets: $\mathcal{D} = \{i : x_i < \ell_i\}$ and $\mathcal{N} = \{i : x_i = \ell_i\}$, i.e., default and non-default banks. Furthermore, $y_i = \bar{y}_i$ and $s_i = \bar{s}_i$ if $i \in \mathcal{D}$. In fact, for such i,

$$x_i = \beta_i + y_i + s_i q + \sum_{j \neq i} x_j p_{ji} < \ell_i,$$

which implies that $y_i < d^1_i$ and $s_i q < d^2_i$. In view of the definitions of d^1_i and d^2_i in (4) and (5), we know that $y_i = \bar{y}_i$ and $s_i = \bar{s}_i$.

Thus, once a partition $(\mathcal{D}, \mathcal{N})$ is identified, the corresponding solution is partially determined by $x_{\mathcal{N}} = \ell_{\mathcal{N}}$, $y_{\mathcal{D}} = \bar{y}_{\mathcal{D}}$, and $s_{\mathcal{D}} = \bar{s}_{\mathcal{D}}$. Regroup the columns and rows of matrix P such that

$$P = \begin{pmatrix} P_{\mathcal{D}} & P_{\mathcal{D},\mathcal{N}} \\ P_{\mathcal{N},\mathcal{D}} & P_{\mathcal{N}} \end{pmatrix}.$$

Then, the maximization problem in (8) is reduced to the following:

$$\max_{\mathcal{D},\mathcal{N},\S_{\mathcal{D}},\dagger_{\mathcal{N}},\int_{\mathcal{N}}} |x|,$$

$$\text{s.t.} \quad x_{\mathcal{D}} = \beta_{\mathcal{D}} + y_{\mathcal{D}} + s_{\mathcal{D}} q + x_{\mathcal{D}} P_{\mathcal{D}} + x_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}, \quad x_{\mathcal{N}} \leq \beta_{\mathcal{N}} + y_{\mathcal{N}} + s_{\mathcal{N}} q + x_{\mathcal{D}} P_{\mathcal{D},\mathcal{N}} + x_{\mathcal{N}} P_{\mathcal{N}}, \tag{9}$$

$$x_{\mathcal{N}} = \ell_{\mathcal{N}}, \qquad x_{\mathcal{D}} < \ell_{\mathcal{D}}, \tag{10}$$

$$y_{\mathcal{D}} = \bar{y}_{\mathcal{D}}, \qquad s_{\mathcal{D}} = \bar{s}_{\mathcal{D}}, \tag{11}$$

$$y_{\mathcal{N}} = \bar{y}_{\mathcal{N}} \wedge d^1_{\mathcal{N}}, \quad s_{\mathcal{N}} = \bar{s}_{\mathcal{N}} \wedge (d^2_{\mathcal{N}}/q), \quad q = Q(|\bar{s}_{\mathcal{D}}| + |s_{\mathcal{N}}|). \tag{12}$$

The first equality constraint in (9) is an *accounting identity* the defaulting banks should comply with: the total repayments they make on its left hand side equal the total incomes they receive on the right hand side. The second inequality constraint in (9) will be referred to as *surplus constraint* below, which states that the non-default banks should have sufficient funds to meet their liabilities. The second constraint in (10) is automatically from the definition of $\mathcal{D}$.

Now we develop a partition algorithm to solve the optimization problem (9-12). The idea is to generate a sequence of tentative partitions, starting with $\mathcal{D} = \emptyset$ and $\mathcal{N} = \{1, 2, \cdots, n\}$ initially, until the optimal one is finally reached. For any partition generated during this course, we have $x_{\mathcal{N}} = \ell_{\mathcal{N}}$, $y_{\mathcal{D}} = \bar{y}_{\mathcal{D}}$, and $s_{\mathcal{D}} = \bar{s}_{\mathcal{D}}$ according to the first constraint in (10) and the two equality constraints in (11). Meanwhile, we use the iterative routine developed below to solve for the remaining $(x_{\mathcal{D}}, y_{\mathcal{N}}, s_{\mathcal{N}}, q)$ from the accounting identity regarding $x_{\mathcal{D}}$ in (9) and the market clearing constraint (12).

Check the feasibility of such obtained solution (x, y, s, q) against the constraints (9-12). Theorem 4 below shows that $x_{\mathcal{D}} < \ell_{\mathcal{D}}$, suggesting that the tentative solution must satisfy the second constraint in (10). But, some banks may violate the surplus constraint in (9). Thus, they must be moved into $\mathcal{D}$. Repeat the above procedure with the updated partition until no more defaults are identified. Obviously, the algorithm will terminate in at most n steps. In the proof of Theorem 4, we also prove that the L_1 norm of the repayment vector $x = (x_{\mathcal{D}}, x_{\mathcal{N}})$ obtained in each intermediate step is larger than the optimal value of problem (9-12). The algorithm keeps reducing the L_1-norm of these infeasible solutions by identifying more and more default banks. When it stops, i.e., when the surplus constraint is satisfied, it will output a feasible (thus, optimal) partition.

We rely on an iterative routine to determine $(x_{\mathcal{D}}, s_{\mathcal{N}}, q)$ associated with a given partition $(\mathcal{D}, \mathcal{N})$. (Note, $y_{\mathcal{N}}$ can be determined by $x_{\mathcal{D}}$ via the first equality in (12).) Define $H(\cdot)$ to be a mapping from the space $\mathcal{R} := \prod_{i\in\mathcal{D}}[0, \ell_i] \otimes \prod_{i\in\mathcal{N}}[0, \bar{s}_i] \otimes [0, 1]$ to itself. For any $z \in \prod_{i\in\mathcal{D}}[0, \ell_i]$, $t \in \prod_{i\in\mathcal{N}}[0, \bar{s}_i]$, and $p \in [0, 1]$, we have $H : (z, t, p) \mapsto (z', t', p')$, in which

$$z' = (\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \bar{s}_{\mathcal{D}}p + zP_{\mathcal{D}} + \ell_{\mathcal{N}}P_{\mathcal{N},\mathcal{D}}) \wedge \ell_{\mathcal{D}}, \quad t' = \bar{s}_{\mathcal{N}} \wedge \frac{[\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}}P_{\mathcal{N}} + zP_{\mathcal{D},\mathcal{N}}) - w]^+}{p}$$

and $p' = Q(|\bar{s}_{\mathcal{D}}| + |t|)$ with

$$w = \bar{y}_{\mathcal{N}} \wedge [\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}}P_{\mathcal{N}} + zP_{\mathcal{D},\mathcal{N}})]^+.$$

Starting with $(z^0, t^0, p^0) = (\ell_{\mathcal{D}}, \mathbf{0}_{\mathcal{N}}, 1)$, we generate a sequence of vectors $\{(z^i, t^i, p^i) : i \geq 1\}$ by repeatedly applying H to obtain $(z^i, t^i, p^i) := H(z^{i-1}, t^{i-1}, p^{i-1})$ for $i \geq 1$. The appendix provides more details about the properties of the mapping H. Utilizing these properties, we can show that the vector sequence must converge. Let $x_{\mathcal{D}} = \lim_{i\to+\infty} z^i$, $s_{\mathcal{N}} = \lim_{i\to+\infty} t^i$, and $q = \lim_{i\to+\infty} p^i$. From Lemma 11 in the appendix, we know that these limits constitute a maximal solution to the following system of equations:

$$x_{\mathcal{D}} = \beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \bar{s}_{\mathcal{D}}q + x_{\mathcal{D}}P_{\mathcal{D}} + \ell_{\mathcal{N}}P_{\mathcal{N},\mathcal{D}}; \tag{13}$$

$$y_{\mathcal{N}} = \bar{y}_{\mathcal{N}} \wedge d^1_{\mathcal{N}} = \bar{y}_{\mathcal{N}} \wedge [\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}}P_{\mathcal{N}} + x_{\mathcal{D}}P_{\mathcal{D},\mathcal{N}})]^+; \tag{14}$$

$$s_{\mathcal{N}} = \bar{s}_{\mathcal{N}} \wedge (d^2_{\mathcal{N}}/q) = \bar{s}_{\mathcal{N}} \wedge \left(\frac{[\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}}P_{\mathcal{N}} + x_{\mathcal{D}}P_{\mathcal{D},\mathcal{N}}) - y_{\mathcal{N}}]^+}{q}\right); \tag{15}$$

$$q = Q(|\bar{s}_{\mathcal{D}}| + |s_{\mathcal{N}}|). \tag{16}$$

Evidently, the above equations ensure that the solution $(x_{\mathcal{D}}, y_{\mathcal{N}}, s_{\mathcal{N}}, q)$ satisfy the accounting identity in (9) and the market clearing constraint (12).

Below is a summary of the algorithm:

Partition Algorithm in the Presence of Asset Liquidation

- Step 0. Set $\mathcal{D} = \emptyset$ and $\mathcal{N} = \{1, \ldots, n\}$.
- Step 1. Set $x_{\mathcal{N}} = \ell_{\mathcal{N}}$, $y_{\mathcal{D}} = \bar{y}_{\mathcal{D}}$, and $s_{\mathcal{D}} = \bar{s}_{\mathcal{D}}$. Use the preceding routine to solve $(x_{\mathcal{D}}, y_{\mathcal{N}}, s_{\mathcal{N}}, q)$ from the equation system (13-16). Let $x = (x_{\mathcal{N}}, x_{\mathcal{D}})$, $y = (y_{\mathcal{N}}, y_{\mathcal{D}})$, and $s = (s_{\mathcal{N}}, s_{\mathcal{D}})$.
- Step 2. Check the feasibility of the surplus constraint in (9) under (x, y, s, q). If it is satisfied, stop; otherwise, identify the violating banks, move them into $\mathcal{D}$ (from $\mathcal{N}$), and go to Step 1.

Theorem 4. *The above algorithm terminates in at most n iterations of Step 1. When it stops, it yields an optimal partition $(\mathcal{D}^*, \mathcal{N}^*)$, along with the optimal solution $x^* = (\ell_{\mathcal{N}^*}, x^*_{\mathcal{D}^*})$, $y^* = (y^*_{\mathcal{N}^*}, \bar{y}_{\mathcal{D}^*})$, and $s^* = (s^*_{\mathcal{N}^*}, \bar{s}_{\mathcal{D}^*})$ for the problem (8).*

As noted in the introduction, an important feature of the partition algorithm is that it generates a sequential order relation among the defaults. It highlights the effect of market liquidity in propagating contagion. To see this, consider the feasibility checking at the end of every step. The surplus constraint in (9) can be restated as

$$[\beta_{\mathcal{N}} + y_{\mathcal{N}} + s_{\mathcal{N}} q + x_{\mathcal{D}} P_{\mathcal{D},\mathcal{N}} + x_{\mathcal{N}} P_{\mathcal{N}}] - x_{\mathcal{N}} \geq 0.$$

Note that the sum in the brackets is the asset value of these banks, marked to the market price $q = Q(|\bar{s}_{\mathcal{D}}| + |s_{\mathcal{N}}|)$, and $x_{\mathcal{N}}$ is their liability payments. Thus, it is equivalent to checking whether the market values of these banks' net worth remain nonnegative. The price impact that the defaults identified in the previous steps yield is folded into q through their asset sale $\bar{s}_{\mathcal{D}}$ and will affect the contagion magnitude in the subsequent steps.

We present a simple numerical example below to illustrate the algorithm.

Example 5. *Figure 1 shows a system of two banks with a tandem structure. Bank 2 raises money from the external investors and lends it to bank 1, then the credit flows to ultimate external debtors. Both two banks*

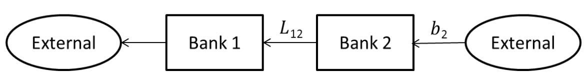

Figure 1: A system of two banks.

own equity capital and illiquid assets. Table 3 displays more granular information about the composition of the balance sheets of these banks at time 0. Still use $Q(s) = \exp(-\gamma s)$ as the price impact model in this

Bank 1			
External investments	$\beta_1 = \$50$	Interbank liability	$L_{12} = \$50$
Illiquid assets	$\bar{s}_1 = \$150$	Equity capital	$e_1 = \$150$

Bank 2			
Interbank lending	$L_{12} = \$50$	External debts	$b_2 = \$50$
Illiquid assets	$\bar{s}_2 = \$50$	Equity capital	$e_2 = \$50$

Table 3: Balance sheets information of the 2-bank system. The numbers shown here are the notional values of the banks' assets and liabilities at time 0.

market, in which $\gamma = 0.02$.

Suppose that bank 1 now suffers from a 40% loss in its external investment, i.e., the realized value of β_1 *is* \$30. *Starting with a partition* $\mathcal{D} = \emptyset$ *and* $\mathcal{N} = \{1, 2\}$*, we apply the preceding algorithm to search for a market-clearing equilibrium for this system. In the first round, after solving the equation system (13-16), we have* $s = (150, 0)$*,* $q = 0.0498$*, and* $x = (50, 50)$ *corresponding to this initial partition. The surplus constraint in (9) is violated at bank 1 under such* (x, s, q) *because*

$$x_1 = 50 > 30 + 150 \cdot 0.0498 = \beta_1 + s_1 q. \tag{17}$$

We should include bank 1 into the default set and update the partition to $\mathcal{D} = \{1\}$ *and* $\mathcal{N} = \{2\}$.

In the second round of algorithm execution, given bank 1 is already bankrupt, solving the equations (13-16) again leads to $s = (150, 50)$*,* $q = 0.0183$*, and* $x = (32.745, 50)$. *Checking the surplus constraint again on bank 2, we will find that its equity value is already negative because* $32.745 + 50 \cdot 0.0183 - 50 < 0$. *We have to move it into* $\mathcal{D}$. *Finally, the equilibrium is reached at* $s = (150, 50)$*,* $q = 0.0183$*, and* $x = (32.745, 33.66)$. □

In this example, the default and the accompanying liquidation of bank 1 exerts a decisive influence to the stability of this two-bank system, because of its large amount of illiquid asset holdings. Bank 1's default is fundamental in the sense that its failure is not caused by the interconnectedness of the system. The algorithm identifies it in the first round of execution. The liquidation amount from this bank, according to the computation result at the end of the first round, has already reached $s_1 = 150$, which will significantly drive the price of the illiquid asset from face value \$1 down to only \$0.0498. The depressed price then feeds back to cause bank 2 to default in the next round. In fact, we can foresee the bankruptcy of bank 2 without waiting until the second round. On one hand, marked to the price at the end of the first round, the equity value of bank 2 is $50 + 0.0498 \cdot 50 - 50 = \$2.49$; on the other hand, the total repayment from bank 1 to bank 2 is at most \$37.47 by the right hand side of (17), inflicting a loss of \$12.53 to the latter one. Therefore, the net worth of bank 2 is too thin to sustain the shock transmitted from its direct debt exposure to bank 1. It will fail in the subsequent round as a result of contagion.

From this example, we can see that the ultimate equilibrium is not the only goal achieved by the partition algorithm. More importantly, it can reveal the hierarchy structure of the forming process of the equilibrium. Note that in the previous example, the price q changes its value from \$1 at the beginning, to \$0.0498 at the end of the first round, and finally to \$0.0183. We actually use different prices in each step of the algorithm execution to identify defaults. In this way, the algorithm delineates clearly how the market liquidity is depleted by successive fire sales as defaults cascade and how the depressed asset price in turn impairs the net worth of a banking system to reinforce financial contagion.

2.4 Discussions

To simplify the analysis, we roughly group the banks' assets to three subcategories, besides the interbank loans, in this stylized model: external investments and liquid and illiquid securities. Further clarification is definitely needed to make our abstract classification more concrete, relating the model to realistic banking balance sheet data.

We refer to liquid securities as the assets whose liquidation will not generate much price impacts. Typical examples include cash and Treasury bills. The illiquid securities are also marketable, but their prices will be subject to significant changes when fire-sold. Possible examples are sovereign, municipal, and corporate bonds, asset and mortgage-backed securities, equities, and so on. Here we model the liquidity effect in reduced form by suppressing price impacts on different classes of assets into a universal demand function Q. This assumption is admittedly too strong to capture liquidity differentials across distinct asset classes. For instance, liquidating large amounts of equities generally results in much smaller price impacts than liquidating comparable quantities of non-agency asset backed securities (ABS).

In light of this limitation, we explore how different liquidity conditions about Q will change our numerical results in the experiments below as a robust check. An assumption used in the benchmark case therein is that all the illiquid securities in the system are roughly liquid as equities. Our estimate is somehow conservative, given that most of the assets owned by banks in reality should be much less liquid than stocks. Meanwhile, we vary the price-impact coefficients over a wide range of values to approximate recent empirical estimates of price impacts in a variety of markets such as agency collateralized mortgage obligation (CMO) and mortgage-backed securities (MBS), municipal and corporate bonds, and ABSs. One may refer to Section 5.1 for detailed discussion.

Furthermore, we are aware that the contagion effect caused by fire sale should greatly impact financial institutions that hold mainly assets that would be marked to market, including hedge funds, investment banks, and insurance companies. Recognizing that the holding ratio of these types of assets in a firm's portfolio may vary to a great extent across the system, we use some values consistent with the balance sheet data of the European banking system released after the 2011 stress test to perform the numerical experiments.

A large portion of assets owned by commercial banks is corporate or retail loans. They are typically not marked to market. When one bank unwinds its loan portfolio, other banks would not have to change the value they record for their own portfolios. We count such kind of assets as external investments. For bank i, the quantity β_i in the model therefore should be interpreted to be the realized value of these investments, such as repayments received from the external loan borrowers and the proceeds from loan sales.

Resolving failed banks is costly. As argued in [52], liquidation of the assets owned by banks in default, legal and administrative fees associated with restructuring or liquidating, delays in payments to creditors, and service disruptions to the failed bank's customers, all contribute to the costs associated with default and magnify the severity of systemic risk. Our model mainly captures the first type of bankruptcy cost, but it can be easily extended to incorporate the effects of other kinds of non-market bankruptcy costs. Introduce a factor λ, $0 \leq \lambda < 1$ and replace Eq. (3) in Definition 2 by

$$x_i = \begin{cases} \ell_i, & \text{if } \ell_i \leq \beta_i + y_i + \sum_{j \neq i} x_j p_{ji} + s_i q, \\ \lambda[\beta_i + y_i + \sum_{j \neq i} x_j p_{ji} + s_i q], & \text{otherwise.} \end{cases} \tag{18}$$

In words, we assume that a fraction of $(1-\lambda)$ of bank i's asset value will be destroyed when it defaults, due to, say, costly reorganization procedure or the loss of firm-specific knowledge and reputation. Note that a similar form of recovery rate is also used in [47]. We can show that the partition algorithm still applies in this case, with a minor modification that in (13) $x_{\mathcal{D}}$ now should be determined by

$$x_{\mathcal{D}} = \lambda[\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \bar{s}_{\mathcal{D}} q + x_{\mathcal{D}} P_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}].$$

3 Net Worth, Systemic Resilience, and Market Liquidity

In this section we intend to estimate the probability of contagion caused by the failure of a single bank, so that we can quantify the systemic influence of each bank on the resilience of a banking system. [34] developed a similar estimate in the presence of the network effect only. We extend their results to integrate the effect of the liquidity channel here.

To start, consider a banking system satisfying the following initial condition:

$$\beta_i + \bar{y}_i + \sum_{j=1}^{n} \ell_j p_{ji} > \ell_i, \quad \text{for all } i. \tag{19}$$

In words, every bank in this system at the beginning has sufficient liquid funds to meet its liabilities. No defaults will occur in this circumstance. For all i, define the *book value* of bank i's net worth as

$$e_i^{(0)} := \beta_i + \bar{y}_i + \bar{s}_i + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i, \tag{20}$$

by taking a difference between the bank's total asset value and its notational liability as shown in Table 1. Under (19), $e_i^{(0)} > 0$ for all i.

Then, suppose that a shock hits a representative bank, say bank 1, so that β_1 changes to $\beta_1 - Y_1$, where $Y_1 \in [0, \beta_1]$ is a random variable. We kick off the partition algorithm with $\mathcal{D} = \emptyset$ and $\mathcal{N} = \{1, 2, \cdots, n\}$. Solve Eqs. (13-16) to calculate how much illiquid security should be sold for each bank under this tentative partition. Denote $s_1^{(1)}$ and $q^{(1)}$ to be the calculation outcomes for the amount of illiquid asset sales of bank 1 and the market price of the illiquid asset, respectively. Observe that, if

$$\beta_1 - Y_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{ji} + s_1^{(1)} q^{(1)} < \ell_1, \tag{21}$$

bank 1 will violate the surplus constraint in (9); it thus will be partitioned into the default set $\mathcal{D}$.

Use the initial net worth $e_1^{(0)}$ to derive a sufficient condition for (21). Note that $s_1^{(1)} \le \bar{s}_1$ and $q^{(1)} \le Q(0) = 1$. If $Y_1 > e_1^{(0)}$, it implies that

$$Y_1 > \beta_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{j1} + \bar{s}_1 - \ell_1 \ge \beta_1 + \bar{y}_1 + \sum_{j=1}^{n} \ell_j p_{ji} + s_1^{(1)} q^{(1)} - \ell_1;$$

the inequality (21) follows. The preceding simple analysis yields

Proposition 6.

$$\mathbb{P}(\textit{Bank 1 defaults}) \geq \mathbb{P}(Y_1 > e_1^{(0)}).$$

This proposition has a clear economic interpretation. When Y_1 is overwhelmingly large, the shock will consume all the net worth the bank possesses and cause it to become insolvent. In this sense, the failure of bank 1 is fundamental, not resulted by the interconnectedness of the system.

The default of bank 1 may be contagious to a subset of other banks. Hence, one more interesting question is how likely a contagion will be caused by this default. The following theorem establishes some probabilistic estimates about the depth that cascading defaults can transmit in a given system. Mimicking what we did in Example 5, define

$$e_i^{(1)} := \big(\beta_i + \bar{y}_i + \bar{s}_i Q(\bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i\big) \vee 0, \quad \text{for all } i. \tag{22}$$

In contrast to $e_i^{(0)}$, $e_i^{(1)}$ represents the *market value* of bank i's net worth, in which the illiquid asset is marked not to its face value, but to the market price after bank 1 has sold out all its illiquid holdings. Thus, it emphasizes the negative price pressure coming from the liquidation of bank 1. We have

Theorem 7. *The probability that the shock on bank 1 causes bank j, $j \neq 1$, to default satisfies*

$$\mathbb{P}\left(\textit{Bank } j \textit{ defaults in the equilibrium} \,|\, \textit{Bank 1 defaults}\right) \geq \mathbb{P}\left(Y_1 > e_1^{(1)} + \frac{1}{z_{1j}} \sum_{i=2}^{n} e_i^{(1)} z_{ij}\right),$$

where $Z = (z_{ij})_{i,j \in \{1,\cdots,n\}} = (I-P)^{-1}$. Moreover,

$$\mathbb{E}\left(\textit{number of default banks} \,|\, \textit{Bank 1 defaults}\right) \geq \sum_{j \neq 1} \mathbb{P}\left(Y_1 > e_1^{(1)} + \frac{1}{z_{1j}} \sum_{i=2}^{n} e_i^{(1)} z_{ij}\right).$$

We call

$$e_1^{(1)} + \frac{1}{z_{1j}} \sum_{i=2}^{n} e_i^{(1)} z_{ij} = \frac{\sum_{i=1}^{n} e_i^{(1)} z_{ij}}{z_{1j}} \tag{23}$$

the *resilience index* of bank j against the shock on bank 1. Theorem 7 states that contagion from bank 1 has a high probability to cause bank j to default if its resilience index is low. The index integrates the effects of the two crucial risk transmission channels presented in this paper. We view the denominator as a measure of *exposure distance* between banks 1 and j on the liability network. Note, $I - P$ is invertible and we have the following expansion

$$Z = (I-P)^{-1} = I + P + P^2 + \cdots. \tag{24}$$

When bank 1 receives one dollar loss, it will pass $\mathbf{e}_1 P$ of the loss to its first-order neighbors, where $\mathbf{e}_1 = (1, 0, \cdots, 0)$. From them, the loss will be distributed to affect yet more banks: the creditors of these first-order neighboring banks will receive a fraction of this $\mathbf{e}_1 P$ loss, namely $\mathbf{e}_1 P^2$, and so on. Hence, z_{1j}, the

$(1, j)$-entry of matrix Z, reflects an aggregate impact of this one-dollar loss on bank j, taking into account all the possible risk-transmission paths from 1 to j. The matrix of $(I-P)^{-1}$ will be referred to as the network multiplier below, because it captures the above amplification mechanism through the liability network. Such notion is well known in the network literature (see, e.g., [43]). We will compare it with the liquidity effect developed below in the next section.

A more important part of the index lies in its numerator, which characterizes *financial strength* of bank j in light of the interconnected balance sheets of the banking system. We may explain the intuition behind it as follows. There are two factors defining the capability of a bank to absorb external shocks. One is the net worth of this bank. The other is the net worths of its neighboring banks. Given the interconnectedness of this financial system, a bank should be more able to weather negative shocks if all the banks that it has exposures to have high-valued net worths: these strong neighbors will effectively prevent the contagion originated by defaults occurring somewhere else in the system from affecting bank j. Bearing this intuition in mind, we define an n-dimensional vector $k = (k_1, \cdots, k_n)$ such that

$$k = e^{(1)} + kP. \tag{25}$$

The entry k_j, as a measure of the financial strength of bank j, combines both the bank's net worth $e_j^{(1)}$ and the strength of its neighbors. Eq. (25) admits a closed-form solution: $k = e^{(1)}(I-P)^{-1} = e^{(1)}Z$. That is, for each j, $k_j = \sum_{i=1}^n e_i^{(1)} z_{ij}$, the numerator of our resilience index.

The vector k has a flavor of the centrality measure introduced in the literature of social network by [39]; see also [38] and [44] for references. [20] also presents a centrality-based threat index to measure the spillover effect in a banking system described by [23]. We need to stress that these works are more interested in diffusion-type contagion through the local neighborhood structure of a system. In addition to the network effect, our measure k also features a global and direct impact through the liquidity channel. Recall that the definition of $e^{(1)}$ is marked to $Q(\bar{s}_1)$. Therefore, if a considerable fraction in the asset side of the banking system is illiquid, the asset liquidation of bank 1 will directly generate a serious negative shock to $e^{(1)}$ via the depressed price. The product form of k implies that, the loss on $e^{(1)}$ will be amplified by the network multiplier $(I-P)^{-1}$ to impair the financial strength of the system in a multiplicative manner.

The previous discussion shows that two factors determine the potential impact of a bank's failure on the rest of the financial system. The first is the relative position of the bank in the entire network, captured by the multiplier $(I-P)^{-1}$. The second is the concentration of illiquidity on the bank, whose liquidation will yield a price impact of size $Q(\bar{s}_1)$. Our analysis in this section synthesizes these two factors into one resilience index, and more importantly, explicitly relates it to probability estimates of contagion. It is worth mentioning that the index relies only on the balance sheet information of financial institutions, independent of the distribution assumptions of external shocks. Therefore, our index allows us to maintain a high flexibility in designing hypothetical adverse scenarios about external shocks to assess the systemic influence of one specific failure.

4 Sensitivity Analysis: Liquidity Amplifier

With the optimal partition $(\mathcal{D}^*, \mathcal{N}^*)$ and its related market-clearing equilibrium (x^*, y^*, s^*, q^*) obtained from Theorem 4, we proceed in this section to characterize the sensitivity of x^* and q^* with respect to small changes in the model parameters. (Here "small" means the partition set will not be affected.) One important consequence of such analysis is that we identify a *liquidity amplifier* to capture the effect of asset price to the systemic risk. Moreover, the analysis explicitly demonstrates the interplay of two amplification mechanisms of liability network and market liquidity, whereby we are able to examine the effectiveness of several intervention policies in preventing spillover of the systemic risk.

4.1 Liquidity Amplifier

Denote $\mathcal{L}^* = \{i \in \mathcal{N}^* : s_i^* > 0\}$. By it, we split the non-default banks further into two subgroups, $\mathcal{N}^* = \mathcal{L}^* \cup (\mathcal{N}^* \backslash \mathcal{L}^*)$. The former subgroup of banks relies partly on illiquid asset sales to meet their liabilities, whereas no illiquid asset liquidation is needed for the banks in the latter subgroup in the equilibrium. As we can expect, the banks in $\mathcal{L}^*$ should play a crucial role in the liquidity amplifier, because the competition for the limited market liquidity among them will drive down the price of the illiquid security, yielding a negative externality for the systemic resilience. In contrast, the banks in $\mathcal{N}^* \backslash \mathcal{L}^*$ should have no effect on the equilibrium because they fully repay their liabilities and do not participate in asset sales. Therefore they will not appear in the expression of the liquidity multipliers. More precisely, we have

Theorem 8. *Assume that $Q(\cdot)$ is differentiable. Define $\gamma = -Q'(|s^*|)/Q(|s^*|)$. Then, the sensitivity of the market price q^* with respect to the external investment value β is given by*

$$\frac{\partial q^*}{\partial \beta_i} = \frac{\gamma}{1 - \gamma\big(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*, \mathcal{L}^*}\mathbf{1}\big)}, \qquad \textit{for } i \in \mathcal{L}^*,$$

and

$$\frac{\partial q^*}{\partial \beta_i} = \frac{\gamma \mathbf{e}_i (I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*, \mathcal{L}^*}\mathbf{1}}{1 - \gamma\big(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*, \mathcal{L}^*}\mathbf{1}\big)}, \qquad \textit{for } i \in \mathcal{D}^*.$$

The sensitivity of the equilibrium repayment x^* with respect to β is given by

$$\frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} = \frac{\partial q^*}{\partial \beta_i} \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}, \quad \textit{for } i \in \mathcal{L}^*.$$

and

$$\frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} = \Big(\mathbf{e}_i + \frac{\partial q^*}{\partial \beta_i} \bar{s}_{\mathcal{D}^*}\Big)(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}, \quad \textit{for } i \in \mathcal{D}^*,$$

Moreover, all the sensitivities are nonnegative.

Theorem 8 presents complex knock-on effects due to the liquidity. To see that, consider the impact of a reduction of \$1 in β_i to the market price of the illiquid security. If this reduction does not change the equilibrium partition (namely, bank i is still in $\mathcal{L}^*$ and paying fully its liabilities), the bank will have to sell an additional amount of $1/q^*$ of illiquid security, at a price $q^* = Q(|s^*|)$ per unit, to compensate for this reduction. This extra sale subsequently will lower the price of the illiquid asset by

$$Q(|s^*|) - Q\left(|s^*| + \frac{1}{q^*}\right) \approx \frac{-Q'(|s^*|)}{q^*} = \frac{-Q'(|s^*|)}{Q(|s^*|)} = \gamma.$$

This is the first-order liquidity effect.

Such a price decline will feed back into the incomes of the banks in $\mathcal{L}^*$ through two channels. First, it immediately causes a shrinkage in the sale proceeds of asset liquidation of those banks. Note that the amounts of the illiquid security sold from the banks in $\mathcal{L}^*$ are given by $s^*_{\mathcal{L}^*}$. Hence the impact via this channel is that these banks will receive $\gamma s^*_{\mathcal{L}^*}$ dollars less, as the price declines by γ.

The second channel is from the network effect, more specifically, the repayments of the banks in $\mathcal{D}^*$. All the banks in this subgroup, defaulting in the equilibrium, are forced to liquidate all the asset holdings; hence, $s^*_i = \bar{s}_i$ for $i \in \mathcal{D}^*$. When the market price declines by γ, the values of these banks after liquidation will decrease by $\gamma \bar{s}_{\mathcal{D}^*}$. This initial loss caused exogenously by the market price decline will be amplified endogenously through the interconnectedness of the liabilities among the banks in $\mathcal{D}^*$. Denote $\lambda_{\mathcal{D}^*}$ to be an effective loss vector for the banks in $\mathcal{D}^*$, whose jth entry is the sum of exogenous and endogenous losses to bank j, $j \in \mathcal{D}^*$. We have

$$\lambda_{\mathcal{D}^*} = \gamma \bar{s}_{\mathcal{D}^*} + \lambda_{\mathcal{D}^*} P_{\mathcal{D}^*};$$

hence, $\lambda_{\mathcal{D}^*} = \gamma \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$. Through the liability exposures between the subsets $\mathcal{D}^*$ and $\mathcal{L}^*$, this $\lambda_{\mathcal{D}^*}$ will ultimately result in a loss to the repayments to $\mathcal{L}^*$ by $\gamma \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}$. Hence, the total income loss of the bank subgroup $\mathcal{L}^*$ caused from the above two channels amounts to

$$(\gamma s^*_{\mathcal{L}^*} + \gamma \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*})\mathbf{1} = \gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}).$$

They have to sell more to offset the impact of this loss, which further lowers the asset price by

$$\gamma \cdot [\gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})],$$

which is the second-order price effect. Continuing to taking all orders of ripple effects into account, the ultimate price decline should be

$$\gamma + \gamma\left[\gamma(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})\right] + \gamma\left[\gamma^2(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1})^2\right] + \cdots.$$

The sum of this geometric series is exactly the expression of $\partial q/\partial \beta_i$ for $i \in \mathcal{L}^*$ in Theorem 8. We can interpret the other sensitivities in the theorem in a similar manner. The related discussion is skipped in the interest of space.

Define a *liquidity amplifier* as

$$LA \;:=\; \frac{\gamma}{1 - \gamma\big(|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}\big)}. \tag{26}$$

As explained before, it characterizes how the market amplifies an initial price decline. This amplifier takes the form of a denominator. It will become infinitely large when

$$|s^*_{\mathcal{L}^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}$$

approaches $1/\gamma$, a measure of the market depth. In contrast, the value of the network multiplier $(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$ is finite if we fix the topology of our banking system. Therefore, the liquidity channel has a potential to play a dominant role in affecting the equilibrium x^* and q^*, especially when the market liquidity largely evaporates. Many studies, including [13], [14], and [2], observe empirically that a liquidity-induced loss spiral, visualized in Figure 2, significantly contributes to the severity of the 2007-2009 US crisis. [17] construct a quantitative framework to analyze the impact of loss-triggered fire sales on systemic risk.

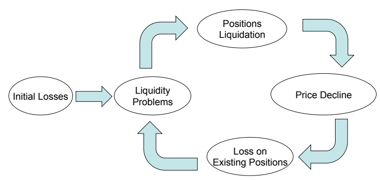

Figure 2: A loss spiral due to the liquidity effect.

4.2 Intervention Policies

We now use the above sensitivity analysis as a tool to examine the effectiveness of policy intervention. Two policies are studied for the purpose of idea illustration: (1) direct purchase of the illiquid asset by an external player, e.g., the government, and (2) capital injection. In practice, central banks undertook both to alleviate the negative impacts of financial crises. For instance, the US Treasury started in October 2008, shortly after the collapse of Lehman Brothers, to inject \$205 billion in the form of preferred stock to the financial industry as a part of the Troubled Asset Relief Program (TARP). In 2009, US Treasury, in conjunction with the Federal Reserve and FDIC, also launched the Public-Private Investment Program for Legacy Assets (PPIP), designed to create partnerships with private investors to buy so called "toxic" assets such as legacy commercial MBS and non-agency residential MBS.

The consequences of the above two policies can be modeled in our setting as modifications on the original structure of the banks' balance sheets (cf. Table 1). Suppose that the government injects \$$\Delta$ to one of the banks to mitigate its systemic impact. If it uses a direct asset purchase program, it pays cash in exchange for some amounts of illiquid securities held by bank i. Two possibilities arise under this category: the government may pay either the face value or the market price of the asset. As Figure 3 demonstrates, this policy will result in an increment for the amount of liquid holding of bank i from $\bar{y}_i$ to $\bar{y}_i + \Delta$ and meanwhile decrease its illiquid assets from $\bar{s}_i$ to $\bar{s}_i - \Delta$ or $\bar{s}_i - \Delta/q^*$, depending on whether the government pays the face value or market price. As for the capital injection, we assume that the bank uses \$$\Delta$, infused by the government in the form of equity capital, to scale up its liquid holding from $\bar{y}_i$ to $\bar{y}_i + \Delta$.

Figure 3: Changes on a bank's balance sheet caused by two intervention policies. The left plot shows the direct asset purchase leads to an increase of \$$\Delta$ in the part of liquid holdings of the bank and a decrease of \$$\Delta$ in its illiquid assets when the transaction is done under the face value. The policy of capital injection in the right plot enhances the equity base of a bank by \$$\Delta$.

Holding β_i unchanged, we compute relative value changes in equilibrium, in particular, in terms of the clearing price $q^*_{\text{Policy}}(\Delta)$ and repayment $x^*_{\text{Policy}}(\Delta)$, caused by the aforementioned balance sheet modifications. Let

$$PE^I_{\text{Policy}} := \lim_{\Delta \to 0} \frac{q^*_{\text{Policy}}(\Delta) - q^*}{\Delta} \quad \text{and} \quad PE^{II}_{\text{Policy}} := \lim_{\Delta \to 0} \frac{x^*_{\text{Policy}}(\Delta) - x^*}{\Delta}$$

be two gauges of policy effectiveness. We compare them for different intervention schemes in Theorem 9.

Theorem 9. *Fix $i \in \mathcal{D}^*$. The policy effectiveness under direct asset purchase (DAP) on the market price and capital injection is given by*

$$PE^I_{DAP,\ Market} = LA, \quad PE^{II}_{DAP,\ Market} = LA \cdot \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1},$$

and

$$PE^I_{Capital} = LA \cdot \mathbf{e}_i (I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} P_{\mathcal{D}^*, \mathcal{L}^*} \mathbf{1},$$
$$PE^{II}_{Capital} = \mathbf{e}_i (I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} + PE^I_{Capital} \cdot \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1},$$

respectively, where LA refers to the liquidity amplifier defined in (26). The effectiveness of the face-value purchase is a weighted average of the above two, namely,

$$PE^{I/II}_{DAP,\ Face} = q^* PE^{I/II}_{DAP,\ Market} + (1 - q^*) PE^{I/II}_{Capital}.$$

All the policies produce positive effectiveness, indicating that they can indeed influence the market equilibrium in a desirable direction such as stabilizing the market price of the illiquid asset and reducing the spillover of the systemic risk. However, the theorem also reveals that different policies may have different focuses. Note that we can show $\mathbf{e}_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1} \leq 1$. Hence,

$$PE^{I}_{\text{DAP, Market}} \geq PE^{I}_{\text{Capital}}.$$

In other words, the direct asset purchase program should have more influence on the market price than capital injection.

As for the repayment improvement, we find that PE^{II}_{Capital} is larger than $PE^{II}_{\text{DAP, Market}}$ when $\mathbf{e}_i(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}$, the network multiplier, is sufficiently large. Recall that the number of defaults in the ultimate equilibrium depends on the value of x^* (i.e., how many $x_i^* < \ell_i$). Thus, capital injection should have a greater effect in reducing the scale of contagious defaults in a highly leveraged banking system. From this discussion, we can see that our analysis provides theoretical supports for the two-pronged approach taken by the US Treasury, which aimed on two different, but related frontiers to alleviate the credit crisis.

We propose the following simple intuition to explain the differences between the policies. The asset purchase program mainly utilizes the liquidity channel to propagate its impact, whereas the capital injection program focuses more on the network channel. Calculating the value change in the bank's total assets under the policy of asset purchase, we find that

$$\begin{aligned}\text{Total Asset Value (TAV) After Purchase} &= \text{TAV Before Purchase} + \Delta - q^* \cdot \frac{\Delta}{q^*} \\ &= \text{TAV Before Purchase.}\end{aligned}$$

Hence, the policy does not change the bank's asset value and thereby will not result in a reduction in the default probability for the recipient bank. In contrast, the capital injection program increases the total asset value of the recipient bank by Δ, making it less likely to default. Therefore, the improvement effect of capital injection on x^* will be more significant.

The price impact is more related to the relative composition of liquid and illiquid assets for the banks. To capture it, consider the liquidity ratio of a bank, which is defined as the ratio of a bank's liquid holdings over its total asset value. The direct asset purchase program increases the liquidity ratio of bank i from $\bar{y}_i/TAV$ to $(\bar{y}_i + \Delta)/TAV$, noting that the policy does not change the asset value as previously argued. The improvement on this ratio under capital injection is only from $\bar{y}_i/TAV$ to

$$(\bar{y}_i + \Delta)/(TAV + \Delta) < (\bar{y}_i + \Delta)/TAV.$$

That explains why the liquidity improvement of capital injection should be less obvious than the former.

Of course, the previous discussion concerns only the benefits of these policies. We do not intend to make any claims here regarding the optimality of policy selection. To provide a more comprehensive assessment for the purpose of policy recommendation, one should count the costs of these interventions, which is absent so far in our sensitivity analysis. However, it still shed insights about effectiveness comparison between policies with different regulatory focuses.

5 Numerical Experiments

We undertake some numerical experiments on the data of the 11 Germany banks that participated in the 2011 EU-wide stress test. In light of incomplete information disclosed from our dataset, we see these experiments more like illustrative of the aforementioned methodologies and notions. Nevertheless, we do find that the market liquidity has the potential to become a prevailing force to trigger a massive contagion under the current market environment.

5.1 The Data and Network Reconstruction

Ninety banks in 21 countries were involved in the exercise of the 2011 stress test organized by the European Banking Authority (see [27]). For each, the authority reports the total assets and core tier 1 capital after the effects of mandatory restructuring plans publicly announced and fully committed before 31 December 2010. In addition, the EBA reports each bank's total claims (exposure at default, EAD) on domestic and foreign institutions, corporations, retail customers, and commercial real estate. Table 4 contains some relevant information extracted from the EBA's report.

Bank Code	Bank Name	Total Asset (A)	Capital	Domestic Interbank EAD (E)	Interbank EAD/ Total Assets (E/A%)
DE017	DEUTSCHE BANK AG	1,905,630	30,361	47,102	2.47%
DE018	COMMERZBANK AG	771,201	26,728	49,871	6.47%
DE019	LANDESBANK B-W	374,413	9,838	91,201	24.36%
DE020	DZ BANK AG DT.Z-G	323,578	7,299	100,099	30.94%
DE021	BAYERISCHE LANDESBANK	316,354	11,501	66,535	21.03%
DE022	NORDDEUTSCHE LANDESBANK -GZ	228,586	3,974	54,921	24.03%
DE023	HYPO REAL ESTATE HOLDING AG	328,119	5,539	7,956	2.42%
DE024	WESTLB AG DUSSELDORF	191,523	4,218	24,007	12.53%
DE025	HSH NORDBANK AG HAMBURG	150,930	4,434	4,645	3.08%
DE027	LANDESBANK BERLIN AG	133,861	5,162	27,707	20.70%
DE028	DEKABANK DEUTSCHE GIROZENTRALE	130,304	3,359	30,937	23.74%

Table 4: Data of German banks from the 2011 EBA Stress Test Report. All the quantities are in million euros.

The EBA data contains only aggregate information about the banks' assets and capital. The detailed breakup about bilateral interbank exposures for each participant bank is not available. Hence, we need to reconstruct the banking system model from the data before performing numerical experiments. To this end, assume that each bank's interbank liabilities equal its interbank assets, and the domestic interbank EAD of each one is held by some other banks in the table. These assumptions concentrate the interbank liabilities within these 11 banks, leaving us a closed system to be constructed. In so doing, we actually bias the experiments in favor of the network-caused contagion. To ensure that the resulted models are consistent with the above aggregate-level data, we search for an appropriate liability matrix $L = (L_{ij})$, $L_{ii} = 0$, such that

$$l_i = \sum_{j:j\neq i} L_{ij} \quad \text{and} \quad a_j = \sum_{i:i\neq j} L_{ij}, \tag{27}$$

where l_i and a_j are the interbank liability of bank i and the interbank asset of bank j, respectively. Both values can be obtained from the column of interbank EAD in Table 4.

Of course, an infinite number of matrix candidates can satisfy the requirement (27). In order to further fix the network configuration, we consider the following three stereotypes of structures:

A. Complete: Every bank has bilateral exposures with every other banks in the system.

B. Ring-like: Every bank concentrates its exposure to its neighboring banks.

C. Core-periphery: The 11 banks are divided into two groups: core and periphery. The core banks connect widely with all the other banks in the system, whereas the banks in the periphery have exposure to the core banks only.

As reported by some empirical studies, such as [8] and [18], interbank markets are typically tiered in the sense that most banks do not lend to each other directly but through money center banks acting as intermediaries. Hence, Structural type C may resemble more closely the reality of the banking industry.

However, two other types of network structure are also considered in the experiments. The conventional wisdom in the literature is that incomplete networks are more prone to large-scale contagion than complete networks, as the latter structure helps diversify away the loss caused by failed banks. For instance, [3] use exactly a complete graph and a ring-like graph as the representatives of the two opposite extremes in the spectrum of graph completeness to assess the influence of network diversification on financial contagion. In light of this given background, the purpose of including types A and B here is definitely not to claim that they are realistic reflections of the true German financial system, but to facilitate numerical comparisons between the impacts of different network topologies and the market liquidity to systemic stability. Figure 4 shows the reconstruction results under all the three structure types.

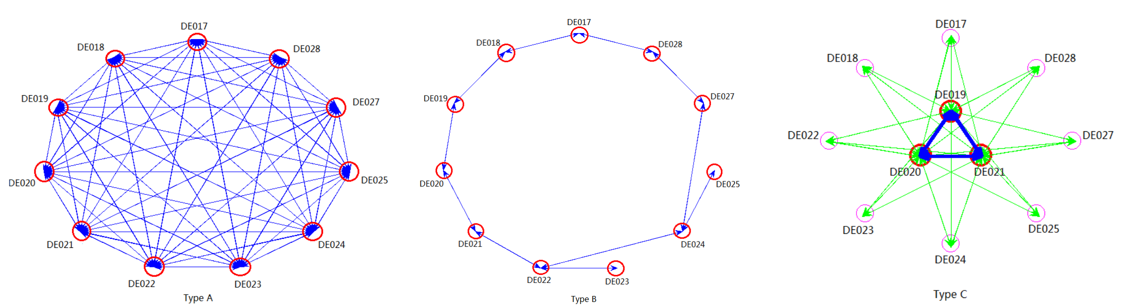

Figure 4: The recovered interbank liability networks from the EBA stress test data. The nodes in the graphs represent individual banks. The arrow from bank i to j represents a claim of bank i on bank j.

Take type A as an example to show how we recover the network by using an entropy-minimizing estimation method developed in [8] from the EBA data. The recovery details under the other two are similar and thus deferred to Appendix C.1 in the interest of space. Define a matrix $Y = (y_{ij})$ such that $y_{ij} = l_i a_j$, the product of bank i's interbank liability and bank j's interbank asset, for all i and j. Such Y should be corresponding to a complete-graph structure in which interbank liabilities and assets are independently distributed among the banks. Noting that Y may violate the consistency constraint (27), we then attempt to find a matrix L to solve the following minimization problem:

$$\min_{L} \sum_{i,j} L_{ij} \ln(L_{ij}/y_{ij}) \tag{28}$$

subject to the constraint (27), $L_{ii} = 0$, and $L_{ij} \geq 0$ for all i, j. In this way, we ensure that the obtained L is as close as possible to the complete structure specified by Y and meanwhile it is also consistent with the data we observe from the EU report.

We need to put the amounts of illiquid holdings and the price impact function Q in the recovered networks to incorporate the liquidity effect. The EBA data shows that the total exposure of major European banks to sovereign bonds is about 2.3 trillion euros (approx. 13% of aggregate banking sector assets), mortgages 4.7 trillion euros (approx. 20%), and corporate loans 6.7 trillion euros (approx. 29%). Correspondingly, we assume that $\theta = 10\%, 30\%, 60\%$ of the total assets of each bank (i.e., the first column of Table 4) are illiquid in the following experiments, by progressively adding the above three classes to the list of assets that can be sold (thus subject to the price impact). Note that, based on the same set of EBA stress test data, [35] use similar approximations (see Table 6 therein) to assess the impact of sizes of sellable assets on their estimates of the bank vulnerability to fire sales.

In addition, suppose that the inverse demand function for the illiquid asset takes a linear form $Q(s) = 1 - \nu s$. In Appendix C.2, we take another form, an exponential Q, to assess the impact of functional form to the results. Use $\nu = 1 \times 10^{-13}$ as a benchmark case. It means that 10 billion euro asset sales results in a price change of 10 basis points. [5] reports that this number is close to the liquidity of a broad spectrum of stocks. Therefore, by taking such ν, we implicitly assume that all the assets owned by the banks are roughly as liquid as equities. Given that most of the banking assets in reality are much less liquid, we are likely biasing low the liquidity effect for the system. To examine how the value of ν changes our numerical results, we also perform the experiments under $\nu = 0.5 \times 10^{-13}$ and 3×10^{-13}, which are corresponding to 5 and 30 basis-point price changes per 10 billion euro sale, respectively. The former is in a neighborhood of price impact for agency CMO and MBS and the latter is close to the liquidity of average corporate bonds and some ABSs used by some empirical research (see [21]). Finally, we estimate β_i, the total value of external investments for each bank, by subtracting the interbank EAD and the illiquid asset from the total asset of a bank.

5.2 Contagion via Market Liquidity

In this subsection, we compare the effects of network and liquidity as two important transmission channels for the systemic risk. Our numerical experiments point out that the market liquidity has the potential to trigger a massive contagion. Given the fact that interbank lending accounts for a relatively small fraction in the total assets of each bank as disclosed by the EBA data, we find that, absent the liquidity channel, the failure of one bank hardly affects the others. But, a significant contagion effect can be observed once we introduce sufficient illiquidity into the system.

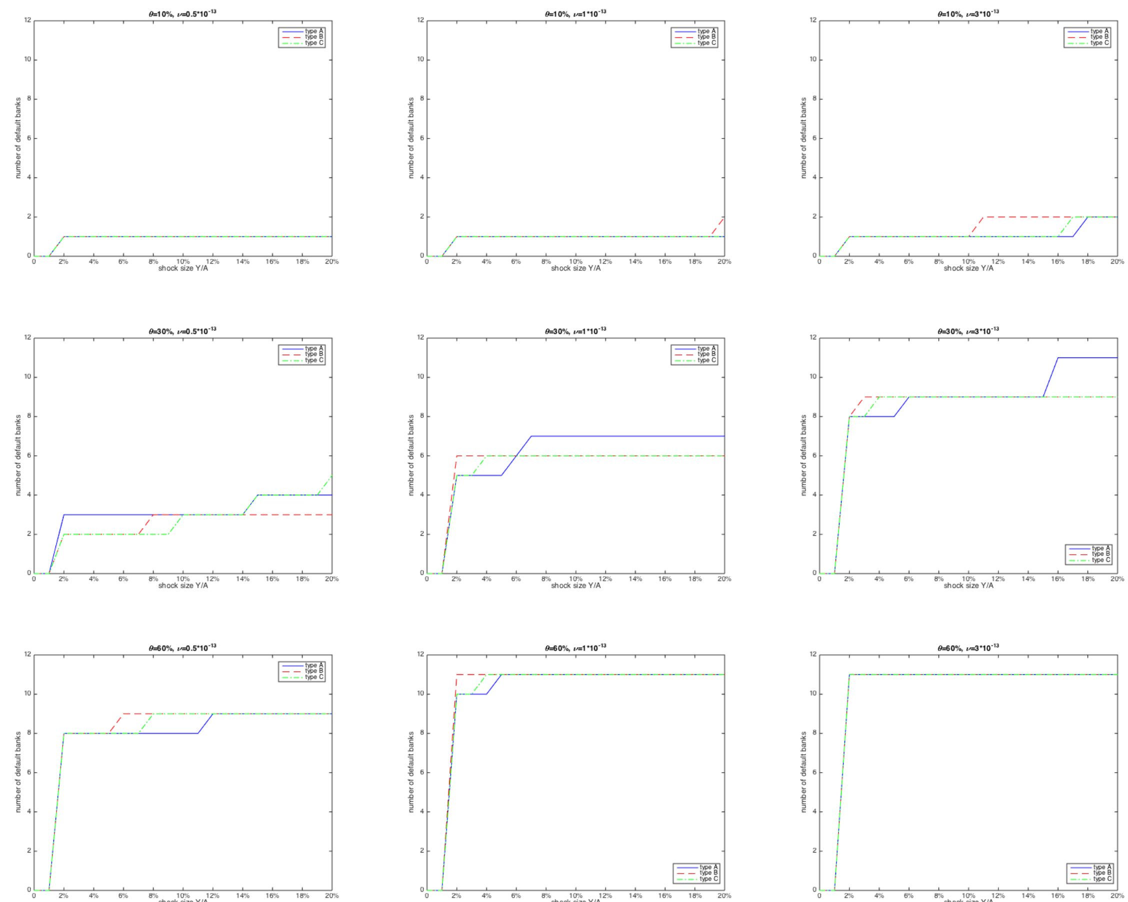

Figure 5: The number of defaults under different shock sizes. The vertical axis is the default number in the repayment equilibrium. The percentages in the horizontal axis are the relative size of an external shock Y to the total asset of Bank DE017. In the first, second, and third rows, we specify the illiquid asset ratios as $\theta = 10\%$, 30%, and 60%, respectively, while in the first, second, and third columns, we use the market depth as $\nu = 0.5 \times 10^{-13}$, 1×10^{-13}, and 3×10^{-13}.

Figure 5 illustrates the number of defaults in equilibrium when different sizes of external shocks are applied on the external projects of a bank. In this set of experiments, we assume that Bank DE017 loses the value of its external investment β by an amount of Y. As the shock size of Y increases, the bank fails and its failure will spill over to affect other banks. When both θ and ν are small (the plot in the northwest corner), no notable contagion effect occurs in any of the three structured networks even under a large external shock: only the recipient bank of shocks, DE017, defaults in the equilibria and no other defaults are caused by its failure. From this, our experiments corroborate the findings in the recent empirical and simulation studies on network stress testing that interbank liabilities alone can barely generate contagion. As we increase the values of θ or ν, i.e., the market gets more illiquid, the plots indicate that the severity of contagion becomes acute. In most of the plots, even a relatively mild external shock to the asset of DE017 can trigger a large number of banks to fail.

As revealed by the last column of Table 4, interbank exposures contribute a very small fraction of the total asset values of these 11 banks. When one bank fails, the loss it causes to its interbank debt holders will be at most the total amount of its bilateral debt exposures, no matter what kind of network structures are used in our simulation. Therefore, the impact from this channel to the system will be not significant for this given dataset. [34] derive some estimates of network-triggered contagion probability only based on the information of the banks' aggregate liability amounts. Their results also point out that the network effect will be very limited for a system with only small amounts of interbank liability exposures. Compared with the network effect, the liquidity can produce a global influence on every participant through the asset market price, not necessarily confined to those banks with direct exposure to the defaulting one. As a complement to their work, our numerical experiments show that fire sales can generate substantial losses from contagion.

Once a contagion is triggered, the networks in the experiments also demonstrate a well-documented robust-yet-fragile property, although its impact is somehow mild compared with the liquidity effect. Take the benchmark case (the central panel in Figure 5) as an illustration. We find a "phase transition" phenomenon; that is, the number of defaults in the system with type-A structure is smallest under small shocks, whereas it changes to the least stable network among the three structures as the shock size becomes large. Our explanation to this observation is that the well-interconnectedness in complete networks indeed has a double-edged effect. The banks may utilize its diversification effect to divert small external shocks; and on the other hand, it will serve as an efficient conduit to transmit losses when the shock size is sufficiently large. [1] theoretically identify two shock regimes in which the complete network is the most and least stable. Their analysis is mainly based on symmetric networks. Our numerical results show that such property of highly interconnected financial networks may still exist even in the presence of fire sale and asymmetric network structure. Nevertheless, we need to stress that this phenomenon occurs only for a market with intermediate degree of liquidity. As shown by the plot in the southeast corner, this difference caused by the network will be dominated by the liquidity effect for a highly illiquid market.

At the end of this subsection, we perform more experiments by changing the shock recipient to another bank DE020 to further highlight the role of liquidity contagion. In contrast to DE017, DE020 owns the largest amount of interbank EAD among the 11 banks. However, as indicated by Figure 6, the interbank EAD is apparently not as contagious as the market liquidity. The number of defaults in this set of experiments are significantly less than the second row of Figure 5, which uses the same combination of θ and ν. Obviously,

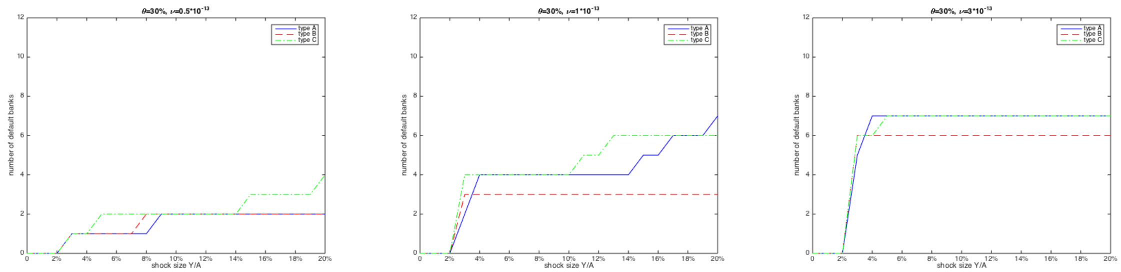

Figure 6: The number of defaults when DE020 is shocked. The vertical axis is the default number in the repayment equilibrium. The percentages in the horizontal axis are the relative size of an external shock Y to the total asset of the bank. The liquidity parameters in the three plots are $(\theta, \nu) = (30\%, 0.5 \times 10^{-13})$, $(30\%, 1 \times 10^{-13})$, and $(30\%, 3 \times 10^{-13})$, respectively.

this follows from the assumption that the banks universally hold 30% of illiquid assets in the system. Under it, the amount of illiquid holdings of DE017 is larger than that of DE020 merely due to its larger size of total assets. The figure suggests that, in a highly illiquid market, the failure of an institution holding a large amount of illiquid assets may pose a greater threat to the system stability than the failure of an institution with large interbank exposures.

5.3 Net Worth as an Indicator of Systemic Resilience

We now consider the importance of net worth, especially its market value, as a gauge of systemic resilience in the presence of the liquidity channel. Use the benchmark case with the type-A structured network as an example. Assume that an external shock hits on DE017 such that 20% of its total assets are lost. Running the partition algorithm, we obtain an equilibrium in which seven banks ultimately default. As the sole shock recipient, the failure of Bank DE017 is fundamental: the size of external shock on it is given by $Y_1 = 20\% \times 1,905,630 = 381,126$, exceeding its net worth $e_1^{(0)} = 30,361$. According to the analysis in Section 3, the bank will be identified by the partition algorithm into the default set $\mathcal{D}$ in the first iteration. The bankruptcy of DE017 will cause three more rounds of cascade (i.e, three more rounds of augmentations in the algorithm execution) through the system. Table 5 illustrates the hierarchy of contagion under the shock.

Default order	Banks Failing in Each Round	Cumulative Failures up to the Round
0-order	DE017	DE017
1st-order	DE022, 023	DE017, DE022, DE023
2nd-order	DE020, 024	DE017, DE022, DE023, DE020, DE024
3rd-order	DE019, 028	DE017, DE022, DE023, DE020, DE024, DE019, DE028

Table 5: Hierarchy of cascades under a 20% shock to the asset of DE017. The liability network is type A. The liquidity parameters are assumed to be $\theta = 30\%$ and $\nu = 1 \times 10^{-13}$.

One interesting observation is that DE023 defaults at so early a stage, although its interbank exposure to DE017 is relatively small compared with the other banks; refer to the technical appendix for the detailed liability matrix. To understand this, we compare in Table 6 the net worths of every bank before and after the failure of bank 1. Under $\nu = 1 \times 10^{-13}$, an illiquid market environment, the impact that the failure of DE017 exerts on the net worths of the other banks is prominent. In particular, DE022 and 023 would lose 98.7% and 100% of their net worth values, respectively. Note that the resilience indices for these two banks decline respectively to 204,299 and 190,684, far less than the magnitude of the shock Y_1. Using Theorem 7, we can infer that DE022 and 023 will default immediately after DE017 fails.

Banks	$e_i^{(0)}$	$e_i^{(1)}$	Loss ratio	Resilience
DE017	30,361	-	-	-
DE018	26,728	13,494	49.51%	4,167,518
DE019	9,838	3,413	65.31%	680,653
DE020	7,299	1,746	76.07%	411,792
DE021	11,501	6,072	47.20%	1,477,794
DE022	3,974	51	98.70%	204,299
DE023	5,539	0	100%	190,684
DE024	4,218	931	77.92%	793,171
DE025	4,434	1,844	58.41%	6,581,028
DE027	5,162	2,865	44.50%	1,784,500
DE028	3,359	1,123	66.57%	746,698

Table 6: Net worth changes caused by the default of DE017. The parameters used are the same as those in Table 5. The net worths $e^{(0)}$ and $e^{(1)}$ for each bank are computed according to (20) and (22), respectively. The loss ratio is defined as $(e_i^{(0)} - e_i^{(1)})/e_i^{(0)}$. We use (23) to compute the resilience for the banks.

This experiment serves as another strong supportive evidence that the liquidity contagion should not be neglected in the study of financial systemic risk. It shows that the price decline due to the asset liquidation of DE017 weakens the capital bases of the two banks to such an extent that they will not have sufficient cushion to sustain even a moderate shock propagated from DE017. Meanwhile, it also demonstrates that the market value of equity capital should be more accurate to reflect systemic resilience of a banking system. To see this, we compute the resilience indices of DE022 and 023 again, replacing $e^{(1)}$ with $e^{(0)}$. The values are 1,569,848 and 11,658,341, respectively, which are much stronger than the size of the external shock.

In general, if we taking the difference between $e^{(0)}$ and $e^{(1)}$, we have

$$e_j^{(0)} - e_j^{(1)} = \bar{s}_j(1 - Q(\bar{s}_1)) \geq 0, \quad \text{for all } j \neq 1. \tag{29}$$

By the non-negativeness of P,

$$\frac{\sum_{i=1}^n e_i^{(1)} z_{ij}}{z_{1j}} = \frac{[e^{(1)} + e^{(1)}P + e^{(1)}P^2 + \cdots]_j}{[I + P + P^2 + \cdots]_{1j}} \leq \frac{[e^{(0)} + e^{(0)}P + e^{(0)}P^2 + \cdots]_j}{[I + P + P^2 + \cdots]_{1j}} = \frac{\sum_{i=1}^n e_i^{(0)} z_{ij}}{z_{1j}}.$$

Therefore, the resilience of a bank would be seriously inflated if we used $e^{(0)}$ to calculate it. As shown in the experiment, using the book value of net worth will underestimates the true systemic danger and can be misleading in the presence of the liquidity channel.

5.4 Bankruptcy Cost

Figure 7 displays the impacts of bankruptcy costs discussed in Section 2.4. If we assume that, in addition to the loss caused by asset liquidation, a fixed fraction of the failed banks' value will be destroyed during the procedure of bankruptcy, the risk amplification effect in the banking system is more significant. For given external shock Y and market liquidity environment (θ, ν), smaller λ (i.e., less recovery rate) will lead to higher number of failures.

An interesting finding from the figure is that, without the liquidity channel, bankruptcy costs must be very large in order to have an appreciable impact on the equilibrium. For instance, the number of default banks in the type-C structure (the green dot-dash line) exceeds 4 when λ is less than 0.61 in the left plot in which $\nu = 0$. In contrast, when $\nu = 0.5 \times 10^{-13}$ (the central plot), more than 4 banks will default in the equilibrium even for $\lambda = 0.93$. In other words, a relatively small bankruptcy cost can be easily magnified by the market liquidity to a great threat to the systemic stability. In this sense, this numerical example

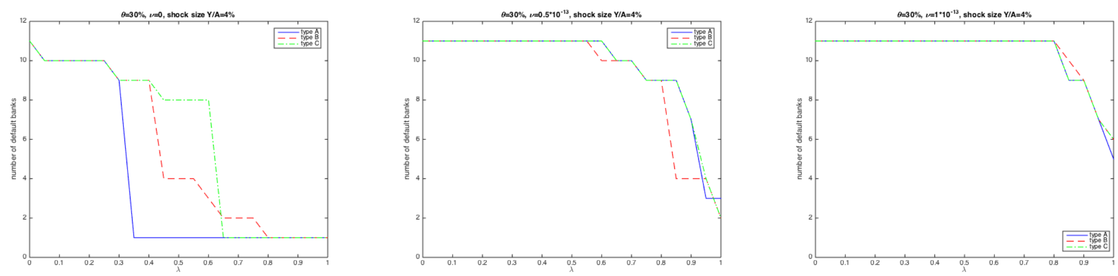

Figure 7: The number of defaults under different bankruptcy cost λ. The vertical axis is the default number in the repayment equilibrium and the horizontal axis is the value of λ. Here we apply a shock with size 4% to the asset of DE017. Assume that the illiquid asset ratio in the system is $\theta = 30\%$. The price impacts used in the plots, from left to right, are $\nu = 0$, 0.5×10^{-13}, and 1×10^{-13}, respectively.

underscores the importance of orderly resolution of failing banks in avoiding costly bankruptcy procedures and preventing systemic risks, especially during a financial crisis when the market is in turmoil.

6 Conclusions

This paper develops an equilibrium-constrained optimization approach to model the systemic risk in a banking system. In the literature of social networks and epidemiology, contagion of rumors, deceases, and so on, typically follows a diffusion process through the local neighborhood structure of a network; refer to, e.g., [22]. Hence, the network effect there is the predominant force. A distinct feature of the financial system is that two banks may not have any counter-party relationship at all, but they are still connected through a global channel, the market. Our formulation can incorporate both two important channels, the network and market liquidity, for the transmission of financial systemic risk. We present a partition algorithm to solve the equilibrium, by which we unify and extend the fixed-point-based approaches proposed in some major studies about financial networks. The numerical experiments in the paper reveal that, as the on-going de-leveraging practice in financial institutions has already significantly shrunk their mutual liability exposure, the market effect may overtake the former to become a dominant force to trigger large-scale financial contagion.

We illustrate the network and liquidity effects with data on the European banking system. In spite of the crude estimates we use to recover the system from the limited information disclosed by the test dataset, the message conveyed from the experiments is unequivocal: the market liquidity have a potential to play a dominating role in the development of the systemic risk. This echoes the concern of European Banking Authority chair Andrea Enria in his opening statement of the publication of the stress test results, in which he warned that a possible further deterioration in investors' risk appetite for sovereign debts during the ongoing EU sovereign crisis might create a liquidity problem for such assets and thus impair the net worth of the banks with sizeable exposures to them.

Several directions are of special interest for us to pursue in the future. First, we assume that the entire liability network, captured by the matrix P, is known for the purpose of solving for the ultimate equilibrium. However, the data observable from the market typically contain at best incomplete information about it. This calls for a need to develop some methodologies to handle systemic risk modeling with uncertain data. In this aspect, our optimization formulation provides a very appealing platform, because it can be easily extended to accommodate data uncertainty with the help of a rapidly burgeoning literature of robust optimization (see, e.g., [10], [12] and the references therein).

The second research direction is to endogenize the decision of network formation and illiquid asset holding in the banking system toward building a dynamic model, as opposed to the static model presented in the paper. Such types of models would shed more insights on the problem of how to monitor the accumulation of systemic risk within the system.

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Appendices

A Proof of Main Results

A.1 Proofs for the Results in Section 2

Recall the definition of H and the sequence of $\{(z^i, t^i, p^i), i \geq 1\}$ generated by it in Section 2.3. Endow a weak order $\succcurlyeq$ on $\mathcal{R}$ in the sense that $(z, t, p) \succcurlyeq (\tilde{z}, \tilde{t}, \tilde{p})$ if $z \geq \tilde{z}$, $t \leq \tilde{t}$, and $p \geq \tilde{p}$. We need several technical lemmas to establish the main result, Theorem 4.

Lemma 10. *For any two real numbers a and b, if $a \leq b$, then $(a - (a^+ \wedge d))^+ \leq (b - (b^+ \wedge d))^+$ for any real d.*

Proof. Proof. We have

$$(a - (a^+ \wedge d))^+ = [(a - a^+) \vee (a - d)]^+ = [(-a^-) \vee (a - d)]^+,$$

where a^- is defined to be $\max\{-a, 0\}$. Since $a \leq b$, $-a^- \leq -b^-$ and $a - d \leq b - d$. Then, the right hand side of the above equality should be less than

$$[(-b^-) \vee (b - d)]^+ = (b - (b^+ \wedge d))^+ . \square$$

Lemma 11. *(i) Function H is increasing on $\mathcal{R}$ relative the weak order $\succcurlyeq$.*
(ii) The sequence $\{(z^i, t^i, p^i), i \geq 1\}$ is decreasing in the sense of the weak order $\succcurlyeq$. Therefore, the limits $x_{\mathcal{D}} := \lim_{i \to +\infty} z^i$, $s_{\mathcal{N}} := \lim_{i \to +\infty} t^i$, and $q := \lim_{i \to +\infty} p^i$ exist.
(iii) For any other fixed-point (z, t, p) of function H, we have $(x_{\mathcal{D}}, s_{\mathcal{N}}, q) \succcurlyeq (z, t, p)$.

Proof. Proof. (i) Suppose that we have $(z, t, p), (\tilde{z}, \tilde{t}, \tilde{p}) \in \mathcal{R}$, $(z, t, p) \succcurlyeq (\tilde{z}, \tilde{t}, \tilde{p})$. Because $z \geq \tilde{z}$, $zP_{\mathcal{D}} \geq \tilde{z}P_{\mathcal{D}}$ and $zP_{\mathcal{D},\mathcal{N}} \geq \tilde{z}P_{\mathcal{D},\mathcal{N}}$ due to the non-negativeness of the matrices $P_{\mathcal{D}}$ and $P_{\mathcal{D},\mathcal{N}}$. Hence,

$$z' = (\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \bar{s}_{\mathcal{D}} p + zP_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}) \wedge \ell_{\mathcal{D}} \geq (\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \bar{s}_{\mathcal{D}} \tilde{p} + \tilde{z}P_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}) \wedge \ell_{\mathcal{D}} = \tilde{z}'$$

and

$$w = \bar{y}_{\mathcal{N}} \wedge [\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}} + zP_{\mathcal{D},\mathcal{N}})]^+ \leq \bar{y}_{\mathcal{N}} \wedge [\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}} + \tilde{z}P_{\mathcal{D},\mathcal{N}})]^+ = \tilde{w}.$$

Lemma 10 and the assumption $p \geq \tilde{p}$ implies that

$$t' = \bar{s}_{\mathcal{N}} \wedge \frac{[\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}} + zP_{\mathcal{D},\mathcal{N}}) - w]^+}{p} \leq \bar{s}_{\mathcal{N}} \wedge \frac{[\ell_{\mathcal{N}} - (\beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}} + \tilde{z}P_{\mathcal{D},\mathcal{N}}) - \tilde{w}]^+}{\tilde{p}} = \tilde{t}'.$$

Furthermore, we can obtain

$$p' = Q(|\bar{s}_{\mathcal{D}}| + |t|) \geq Q(|\bar{s}_{\mathcal{D}}| + |\tilde{t}|) = \tilde{p}'$$

from the monotonicity of Q and $t \leq \tilde{t}$. So far, we have shown $H(z, t, p) \succcurlyeq H(\tilde{z}, \tilde{t}, \tilde{p})$.

(ii) Notice that $(z^0, t^0, p^0) = (\ell_{\mathcal{D}}, \mathbf{0}_{\mathcal{N}}, 1)$. According to the definition of H, $(z^0, t^0, p^0) \succcurlyeq (z^1, t^1, p^1)$. By the monotonicity of H established in Part (i), we have $(z^1, t^1, p^1) = H(z^0, t^0, p^0) \succcurlyeq H(z^1, t^1, p^1) = (z^2, t^2, p^2)$. Repeatedly applying H, we can show that the sequence $\{(z^i, t^i, p^i), i \geq 1\}$ must be decreasing.
(iii) Consider any fixed point of H denoted by (z, t, p). We know that $(z^0, t^0, p^0) \succcurlyeq (z, t, p)$. Invoking the same arguments used in Part (ii), $(z^i, t^i, p^i) \succcurlyeq (z, t, p)$ for all $i \geq 1$. As the limits of the sequence, we know that $(x_{\mathcal{D}}, s_{\mathcal{N}}, q) \succcurlyeq (z, t, p)$. □

Lemma 11 establishes that $(x_{\mathcal{D}}, s_{\mathcal{N}}, q)$ must be a maximal fixed point of H. To justify that this triplet also solves the equation system (13-16), we have to show

Lemma 12. *For any partition generated by the algorithm, we have $x_{\mathcal{D}} < \ell_{\mathcal{D}}$.*

Proof. Proof. Use induction. The statement of the lemma is trivially true for the initial partition the algorithm starts with because $\mathcal{D} = \emptyset$. To complete the inductive arguments, we assume that $x_{\mathcal{D}} < \ell_{\mathcal{D}}$ for some intermediate partition $(\mathcal{D}, \mathcal{N})$ and its related fixed point $(x_{\mathcal{D}}, s_{\mathcal{N}}, q)$. Denote $(\mathcal{D}', \mathcal{N}')$ to be the next partition we obtain after the feasibility check in Step 2. Note that $\mathcal{D}'$ includes new banks for which the surplus constraint (cf. the second one in (9)) is violated under $(x_{\mathcal{D}}, y_{\mathcal{N}}, s_{\mathcal{N}}, q)$, namely,

$$\mathcal{D}' = \mathcal{D} \cup \left\{ i : i \in \mathcal{N}, \ell_i > \beta_i + y_i + s_i q + \sum_{j \in \mathcal{D}} x_j p_{ji} + \sum_{j \in \mathcal{N}} \ell_j p_{ji} \right\}.$$

Certainly we have $\mathcal{D} \subseteq \mathcal{D}'$, $\mathcal{N} \supseteq \mathcal{N}'$, and $\mathcal{N} = \mathcal{N}' \cup (\mathcal{D}' \backslash \mathcal{D})$. Denote $(x'_{\mathcal{D}'}, s'_{\mathcal{N}'}, q')$ to be the greatest fixed point under this new partition obtained as Part (ii) of Lemma 11 instructs.

We show first that $x'_{\mathcal{D}} < \ell_{\mathcal{D}}$. To this end, consider the following parameterized function $\tilde{H}$ on $\tilde{\mathcal{R}} := \prod_{i \in \mathcal{D}} [0, \ell_i] \otimes \prod_{i \in \mathcal{N}} [0, \bar{s}_i] \otimes [0, 1]$, where $(z', t', p') = \tilde{H}(z, t, p)$ such that

$$z' = (\theta_1 + \bar{s}_{\mathcal{D}} p + z P_{\mathcal{D}}) \wedge \ell_{\mathcal{D}}, \quad t' = \bar{s}_{\mathcal{N}} \wedge \frac{[\ell_{\mathcal{N}} - (\theta_2 + z P_{\mathcal{D},\mathcal{N}}) - w]^+}{p}, \quad p' = Q(|\bar{s}_{\mathcal{D}}| + |t|),$$

with

$$w = \bar{y}_{\mathcal{N}} \wedge [\ell_{\mathcal{N}} - (\theta_2 + z P_{\mathcal{D},\mathcal{N}})]^+$$

for some parameters θ_1 and θ_2. Notice that $(x_{\mathcal{D}}, s_{\mathcal{N}}, q)$ and $(x'_{\mathcal{D}}, s'_{\mathcal{N}}, q')$ are the maximal fixed points of $\tilde{H}$ with the parameter sets being

$$(\theta_1, \theta_2) = (\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}}, \quad \beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}}) \tag{30}$$

and

$$(\theta'_1, \theta'_2) = (\beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + x'_{\mathcal{D}' \backslash \mathcal{D}} P_{\mathcal{D}' \backslash \mathcal{D}, \mathcal{D}} + \ell_{\mathcal{N}'} P_{\mathcal{N}', \mathcal{D}}, \quad \beta_{\mathcal{N}} + x'_{\mathcal{D}' \backslash \mathcal{D}} P_{\mathcal{D}' \backslash \mathcal{D}, \mathcal{N}} + \ell_{\mathcal{N}'} P_{\mathcal{N}', \mathcal{N}}). \tag{31}$$

respectively.

Lemma 10 implies that the function $\tilde{H}$ is increasing in the parameter θ_1 and θ_2, i.e., for any $\theta_1 \geq \theta'_1$ and $\theta_2 \geq \theta'_2$, we have $\tilde{H}(z, t, p; \theta_1, \theta_2) \succcurlyeq \tilde{H}(z, t, p; \theta'_1, \theta'_2)$ for any $(z, t, p) \in \mathcal{R}$. By the celebrated Tarski's theorem (see, e.g., Corollary 2.5.2 of [7]), we know that the maximal fixed point of $\tilde{H}$ should be increasing in (θ_1, θ_2). On the other hand, (θ_1, θ_2) and (θ_1', θ_2') defined as above satisfy

$$\begin{aligned}\theta_1 &= \beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}} = \beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + \ell_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D},\mathcal{D}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}} \\ &\geq \beta_{\mathcal{D}} + \bar{y}_{\mathcal{D}} + x'_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D},\mathcal{D}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}} = \theta_1'\end{aligned}$$

and

$$\begin{aligned}\theta_2 &= \beta_{\mathcal{N}} + \ell_{\mathcal{N}} P_{\mathcal{N}} = \beta_{\mathcal{N}} + \ell_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D},\mathcal{N}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{N}} \\ &\geq \beta_{\mathcal{N}} + x'_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D},\mathcal{N}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{N}} = \theta_2',\end{aligned}$$

where we use the fact that $\ell_{\mathcal{D}'\backslash\mathcal{D}} \geq x'_{\mathcal{D}'\backslash\mathcal{D}}$. Therefore, $(x_{\mathcal{D}}, s_{\mathcal{N}}, q) \succcurlyeq (x'_{\mathcal{D}}, s'_{\mathcal{N}}, q')$. From this, we have $x'_{\mathcal{D}} \leq x_{\mathcal{D}} < \ell_{\mathcal{D}}$.

Turn to prove $x_i' < \ell_i$ for $i \in \mathcal{D}'\backslash\mathcal{D}$. From the definition of set $\mathcal{D}'$, we know that

$$\ell_i > \beta_i + y_i + s_i q + \sum_{j \in \mathcal{D}} x_j p_{ji} + \sum_{j \in \mathcal{N}} \ell_j p_{ji} \tag{32}$$

for such i. It implies that, for $i \in \mathcal{D}'\backslash\mathcal{D}$,

$$[\ell_i - (\beta_i + \sum_{j \in \mathcal{N}} \ell_j p_{ji} + \sum_{j \in \mathcal{D}} x_j p_{ji}) - y_i]^+ > s_i q \geq 0;$$

hence, $s_i = \bar{s}_i$ and $y_i = \bar{y}_i$. Consequently, by (32),

$$\begin{aligned}\ell_{\mathcal{D}'\backslash\mathcal{D}} &> \beta_{\mathcal{D}'\backslash\mathcal{D}} + \bar{y}_{\mathcal{D}'\backslash\mathcal{D}} + \bar{s}_{\mathcal{D}'\backslash\mathcal{D}} q + \ell_{\mathcal{N}} P_{\mathcal{N},\mathcal{D}'\backslash\mathcal{D}} + x_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}'\backslash\mathcal{D}} \\ &= \beta_{\mathcal{D}'\backslash\mathcal{D}} + \bar{y}_{\mathcal{D}'\backslash\mathcal{D}} + \bar{s}_{\mathcal{D}'\backslash\mathcal{D}} q + \ell_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}'\backslash\mathcal{D}} + x_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}'\backslash\mathcal{D}},\end{aligned} \tag{33}$$

where we split the index set $\mathcal{N}$ into a union of $\mathcal{D}'\backslash\mathcal{D}$ and $\mathcal{N}'$ in the second equality. Using the facts that $x_{\mathcal{D}} \geq x'_{\mathcal{D}}$, $q \geq q'$, the right hand side of (33) is greater than

$$\beta_{\mathcal{D}'\backslash\mathcal{D}} + \bar{y}_{\mathcal{D}'\backslash\mathcal{D}} + \bar{s}_{\mathcal{D}'\backslash\mathcal{D}} q' + \ell_{\mathcal{D}'\backslash\mathcal{D}} P_{\mathcal{D}'\backslash\mathcal{D}} + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}'\backslash\mathcal{D}} + x'_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}'\backslash\mathcal{D}}.$$

Then,

$$\ell_{\mathcal{D}'\backslash\mathcal{D}}(I_{\mathcal{D}'\backslash\mathcal{D}} - P_{\mathcal{D}'\backslash\mathcal{D}}) > \beta_{\mathcal{D}'\backslash\mathcal{D}} + \bar{y}_{\mathcal{D}'\backslash\mathcal{D}} + \bar{s}_{\mathcal{D}'\backslash\mathcal{D}} q' + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}'\backslash\mathcal{D}} + x'_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}'\backslash\mathcal{D}}.$$

Multiplying a nonnegative matrix $(I_{\mathcal{D}'\backslash\mathcal{D}} - P_{\mathcal{D}'\backslash\mathcal{D}})^{-1}$ on both sides of the above inequality will yield that

$$\ell_{\mathcal{D}'\backslash\mathcal{D}} > (\beta_{\mathcal{D}'\backslash\mathcal{D}} + \bar{y}_{\mathcal{D}'\backslash\mathcal{D}} + \bar{s}_{\mathcal{D}'\backslash\mathcal{D}} q' + \ell_{\mathcal{N}'} P_{\mathcal{N}',\mathcal{D}'\backslash\mathcal{D}} + x'_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}'\backslash\mathcal{D}})(I_{\mathcal{D}'\backslash\mathcal{D}} - P_{\mathcal{D}'\backslash\mathcal{D}})^{-1}.$$

Note that the right hand side of the above inequality equals to $x'_{\mathcal{D}'\backslash\mathcal{D}}$. That implies $x'_{\mathcal{D}'\backslash\mathcal{D}} < \ell_{\mathcal{D}'\backslash\mathcal{D}}$. In summary, we have $x'_{\mathcal{D}'} < \ell_{\mathcal{D}'}$. □

Proof. Proof of Theorem 4. In light of Lemmas 11 and 12, to complete the proof, what we need to establish is that, for any partition $(\mathcal{D}, \mathcal{N})$ generated sequentially from the algorithm, the corresponding equilibrium

(x, y, s, q) associated with it is greater than any optimal solution to the problem (8). Therefore, when the algorithm terminates at a primal feasible partition, this partition should be optimal.

Use induction again to show the above claim. Denote $(\tilde{x}, \tilde{y}, \tilde{s}, \tilde{q})$ to be any market-clearing repayment equilibrium satisfying (8). It defines a partition for $\{1, 2, \cdots, n\}$ as follows: $\tilde{\mathcal{D}} := \{i : \tilde{x}_i < l_i\}$ and $\tilde{\mathcal{N}} := \{i : \tilde{x}_i = l_i\}$. The algorithm starts with a partition such that $\mathcal{D} = \emptyset$ and $\mathcal{N} = \{\infty, \in, \cdots, \backslash\}$. The corresponding $x = x_{\mathcal{N}} = \ell$ obviously dominates $\tilde{x}$, i.e., $x \geq \tilde{x}$. Take the notations in the proof of Lemma 12. Suppose that for an intermediate partition $(\mathcal{D}, \mathcal{N})$, the corresponding $(x, s, q) \succcurlyeq (\tilde{x}, \tilde{s}, \tilde{q})$ with respect to the partial order $\succcurlyeq$ such that $(x, s, q) \succcurlyeq (\tilde{x}, \tilde{s}, \tilde{q})$ if and only if $x \geq \tilde{x}$, $s \leq \tilde{s}$ and $q \geq \tilde{q}$ for all $1 \leq i \leq n$. Following the algorithm instructions, we identify some new defaults and augment the default set from $\mathcal{D}$ to $\mathcal{D}'$. To accomplish the inductive step, we need to show that (x', s', q'), the equilibrium corresponding to the new partition $(\mathcal{D}', \mathcal{N}')$ dominates $(\tilde{x}, \tilde{s}, \tilde{q})$ in the sense of the weak order $\succcurlyeq$.

From the inductive assumption that $x_{\mathcal{D}} \geq \tilde{x}_{\mathcal{D}}$, we know that $\mathcal{D} \subseteq \tilde{\mathcal{D}}$; hence $\mathcal{N} = (\tilde{\mathcal{D}} \backslash \mathcal{D}) \cup \tilde{\mathcal{N}}$. For any $i \in \mathcal{D}' \backslash \mathcal{D}$, the inequality (32) in the Proof of Lemma 12holds. Therefore,

$$\begin{aligned} \ell_i &> \beta_i + \bar{y}_i + \bar{s}_i q + \sum_{j \in \mathcal{N}} \ell_j p_{ji} + \sum_{j \in \mathcal{D}} x_j p_{ji} \\ &= \beta_i + \bar{y}_i + \bar{s}_i q + \sum_{j \in \tilde{\mathcal{N}}} \ell_j p_{ji} + \sum_{j \in \tilde{\mathcal{D}} \backslash \mathcal{D}} \ell_j p_{ji} + \sum_{j \in \mathcal{D}} x_j p_{ji}, \end{aligned} \tag{34}$$

where we split the sum across the set $\mathcal{N}$ into two using the observation that $\mathcal{N} = (\tilde{\mathcal{D}} \backslash \mathcal{D}) \cup \tilde{\mathcal{N}}$. Because $\ell_{\tilde{\mathcal{D}} \backslash \mathcal{D}} \geq \tilde{x}_{\tilde{\mathcal{D}} \backslash \mathcal{D}}$, $x_{\mathcal{D}} \geq \tilde{x}_{\mathcal{D}}$, and $q \geq \tilde{q}$, the right hand side of (34) will be greater than

$$\beta_i + \bar{y}_i + \bar{s}_i \tilde{q} + \sum_{j \in \tilde{\mathcal{N}}} \ell_j p_{ji} + \sum_{j \in \tilde{\mathcal{D}}} \tilde{x}_j p_{ji}.$$

The above quantity in turn is larger than $\tilde{x}_i$ according to the first constraint in (8) which $\tilde{x}$ satisfies. Consequently, $\mathcal{D}' \backslash \mathcal{D} \subseteq \tilde{\mathcal{D}}$ and we have $\mathcal{D}' \subseteq \tilde{\mathcal{D}}$ in conjunction with $\mathcal{D} \subseteq \tilde{\mathcal{D}}$. As the complement set of $\mathcal{D}'$, $\mathcal{N}' \supseteq \tilde{\mathcal{N}}$.

With this relationship in mind, we can use the Tarski's fixed point theorem again to compare $(x'_{\mathcal{D}'}, s'_{\mathcal{N}'}, q')$ and $(\tilde{x}_{\mathcal{D}'}, \tilde{s}_{\mathcal{N}'}, \tilde{q})$. More specifically, consider H' on $\mathcal{R}' := \prod_{\rangle \in \mathcal{D}'}[\prime, \ell_\rangle] \otimes \prod_{\rangle \in \mathcal{N}'}[\prime, \bar{\int}_\rangle] \otimes [\prime, \infty]$, where $(z', t', p') = H'(z, t, p)$ such that

$$z' = (\theta_1 + \bar{s}_{\mathcal{D}'} p + z P_{\mathcal{D}'}) \wedge \ell_{\mathcal{D}'}, \quad t' = \bar{s}_{\mathcal{N}'} \wedge \frac{[\ell_{\mathcal{N}'} - (\theta_2 + z P_{\mathcal{D}', \mathcal{N}'}) - w]^+}{p}, \quad p' = Q(|\bar{s}_{\mathcal{D}'}| + |t|)$$

with

$$w = \bar{y}_{\mathcal{N}'} \wedge [\ell_{\mathcal{N}'} - (\theta_2 + z P_{\mathcal{D}', \mathcal{N}'})]^+$$

for some parameters θ_1 and θ_2. $(x'_{\mathcal{D}'}, s'_{\mathcal{N}'}, q')$ is its fixed point with the parameter

$$(\theta'_1, \theta'_2) = (\beta_{\mathcal{D}'} + \bar{y}_{\mathcal{D}'} + \ell_{\mathcal{N}'} P_{\mathcal{N}', \mathcal{D}'}, \quad \beta_{\mathcal{N}'} + \ell_{\tilde{\mathcal{N}}} P_{\tilde{\mathcal{N}}, \mathcal{N}'} + \ell_{\tilde{\mathcal{D}} \backslash \mathcal{D}'} P_{\tilde{\mathcal{D}} \backslash \mathcal{D}', \mathcal{N}'}),$$

whereas $(\tilde{x}_{\mathcal{D}'}, \tilde{s}_{\mathcal{N}'}, \tilde{q})$ is a fixed point of H' with the parameters

$$(\tilde{\theta}_1, \tilde{\theta}_2) = (\beta_{\mathcal{D}'} + \bar{y}_{\mathcal{D}'} + \ell_{\tilde{\mathcal{N}}} P_{\tilde{\mathcal{N}}, \mathcal{D}'} + \tilde{x}_{\tilde{\mathcal{D}} \backslash \mathcal{D}'} P_{\tilde{\mathcal{D}} \backslash \mathcal{D}', \mathcal{D}'}, \quad \beta_{\mathcal{N}'} + \ell_{\tilde{\mathcal{N}}} P_{\tilde{\mathcal{N}}, \mathcal{N}'} + \tilde{x}_{\tilde{\mathcal{D}} \backslash \mathcal{D}'} P_{\tilde{\mathcal{D}} \backslash \mathcal{D}', \mathcal{N}'}).$$

From $\theta_i' \geq \tilde{\theta}_i$, $i = 1, 2$, we know that $(x'_{\mathcal{D}'}, s'_{\mathcal{N}'}, q')$, as the greatest fixed point of H' with parameter (θ_1', θ_2'), dominates the greatest fixed point of H' with parameter $(\tilde{\theta}_1, \tilde{\theta}_2)$. Therefore, $(x'_{\mathcal{D}'}, s'_{\mathcal{N}'}, q') \succcurlyeq (\tilde{x}_{\mathcal{D}'}, \tilde{s}_{\mathcal{N}'}, \tilde{q})$. From the definition, we also know that $y'_{\tilde{\mathcal{N}}} \leq \tilde{y}_{\tilde{\mathcal{N}}}$. Furthermore, for $i \in \mathcal{N}'$, $x_i' = \ell_i \geq \tilde{x}_i$; for $i \in \tilde{\mathcal{D}}$, $\tilde{y}_i = \bar{y}_i \geq y_i'$ and $\tilde{s}_i = \bar{s}_i \geq s_i'$. Hence, we have finished the inductive step. The theorem is proved. □

A.2 Proofs for the Results in Section 3

Indeed, we can establish a more general result about the contagion estimates than Theorem 7. Assume that a multiple of failures are already caused by the shock on bank 1 in the banking system. Denote $\mathcal{D}$ to be a collection of all those defaulting banks. $1 \notin \mathcal{D}$. Let $e_i^{\mathcal{D}}$ be the equity value of bank i, after all the banks in $\mathcal{D} \cup \{\infty\}$ sell out the illiquid holdings, i.e.,

$$e_i^{\mathcal{D}} = \left(\beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i\right) \vee 0.$$

Let $Z^{\mathcal{D}} = (z_{ij})_{i,j \in \mathcal{D}^c} = (I_{\mathcal{D}^c} - P_{\mathcal{D}^c})^{-1}$. We have

Theorem 13. *For any $j \notin \mathcal{D} \cup \{\infty\}$,*

$$\mathbb{P}(\textit{Bank } j \textit{ defaults} \,|\, \mathcal{D} \cup \{\infty\} \textit{ default}) \geq \mathbb{P}\left(Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right).$$

Moreover,

$$\mathbb{E}(\# \textit{ of default banks} \,|\, \mathcal{D} \cup \{\infty\} \textit{ default}) \geq \sum_{j \notin \mathcal{D}} \mathbb{P}\left(Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right).$$

It is easy to see that Theorem 7 is a special case of the above theorem by taking $\mathcal{D} = \emptyset$.

Proof. Proof of Theorem 13. Given that the banks in $\mathcal{D} \cup \{\infty\}$ default in the ultimate equilibrium, their liquidation amounts of the illiquid security should be $s_i = \bar{s}_i$ for $i \in \mathcal{D} \cup \{\infty\}$. Consider any $k \neq 1$, the equilibrium repayment of bank k when bank 1 receives a shock of size Y_1, satisfies

$$\begin{aligned} x_k &\leq \beta_k + y_k + s_k Q(|s_{\mathcal{D}}| + s_1 + |s_{\mathcal{D}^c \setminus \{1\}}|) + \sum_{j \in \mathcal{D}} x_j p_{jk} + \sum_{j \in \mathcal{D}\rfloor} x_j p_{jk} \\ &\leq \beta_k + \bar{y}_k + \bar{s}_k Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \sum_{j \in \mathcal{D}} \ell_j p_{jk} + \sum_{j \in \mathcal{D}\rfloor} x_j p_{jk}, \end{aligned} \tag{35}$$

where the first inequality is due to the limited liability condition in the definition for market clearing equilibrium, and the second inequality uses the facts that $s_i = \bar{s}_i$ for $i \in \mathcal{D}$, $y_k \leq \bar{y}_k$, $s_k \leq \bar{s}_k$, and $Q(\cdot)$ is decreasing. Meanwhile, the repayment of bank 1

$$x_1 \leq \beta_1 - Y_1 + \bar{y}_1 + \bar{s}_1 Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \sum_{j \in \mathcal{D}} \ell_j p_{j1} + \sum_{j \in \mathcal{D}\rfloor} x_j p_{j1}. \tag{36}$$

Rewriting (35) and (36) in a matrix form, we have

$$x_{\mathcal{D}^{\rfloor}} \leq \beta_{\mathcal{D}^{\rfloor}} - Y_1\mathbf{e}_1 + \bar{y}_{\mathcal{D}^{\rfloor}} + \bar{s}_{\mathcal{D}^{\rfloor}} Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \ell_{\mathcal{D}} P_{\mathcal{D},\mathcal{D}^{\rfloor}} + x_{\mathcal{D}^{\rfloor}} P_{\mathcal{D}^{\rfloor},\mathcal{D}^{\rfloor}}. \tag{37}$$

From the definition of $e^{\mathcal{D}}$,

$$e_i^{\mathcal{D}} = \left(\beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i\right) \vee 0 \geq \beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_{\mathcal{D}}| + \bar{s}_1) + \sum_{j=1}^{n} \ell_j p_{ji} - \ell_i,$$

which implies, for any $i \in \mathcal{D}^{\rfloor}$,

$$\beta_i + \bar{y}_i + \bar{s}_i Q(|\bar{s}_{\mathcal{D}}| + s_1) \leq e_i^{\mathcal{D}} + \ell_i - \sum_{j \in \mathcal{D}} \ell_j p_{ji} - \sum_{j \in \mathcal{D}^{\rfloor}} \ell_j p_{ji}.$$

Substituting the above inequality into (37), we have

$$x_{\mathcal{D}^{\rfloor}} \leq e_{\mathcal{D}^{\rfloor}}^{\mathcal{D}} - Y_1\mathbf{e}_1 + \ell_{\mathcal{D}^{\rfloor}}(I_{\mathcal{D}^{\rfloor}} - P_{\mathcal{D}^{\rfloor}}) + x_{\mathcal{D}^{\rfloor}} P_{\mathcal{D}^{\rfloor}}. \tag{38}$$

In junction of the non-negativeness of the matrix $Z^{\mathcal{D}} = (I_{\mathcal{D}^{\rfloor}} - P_{\mathcal{D}^{\rfloor}})^{-1}$, the inequality (38) implies that

$$x_{\mathcal{D}^{\rfloor}} \leq (e_{\mathcal{D}^{\rfloor}}^{\mathcal{D}} - Y_1\mathbf{e}_1) Z^{\mathcal{D}} + \ell_{\mathcal{D}^{\rfloor}}.$$

In particular, for any $j \notin \mathcal{D} \cup \{\infty\}$,

$$x_j \leq e_1^{\mathcal{D}} z_{1j} + \sum_{i \in \mathcal{D}^{\rfloor}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij} - Y_1 z_{1j} + \ell_j. \tag{39}$$

On the other hand, note that the condition

$$Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}$$

implies that

$$(Y_1 - e_1^{\mathcal{D}}) z_{1j} > \sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}. \tag{40}$$

Combining (39) and (40) will lead to $x_j < \ell_j$, in other words, bank j defaults. So far we have established that

$$\{\text{Banks in } \mathcal{D} \cup \{\infty\} \text{ default}\} \cap \left\{ Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}} \right\} \subseteq \{\text{Bank } j \text{ defaults}\}.$$

Therefore,

$$\mathbb{P}(\text{Bank } j \text{ defaults}) \geq \mathbb{P}\left(\text{Banks in } \mathcal{D} \cup \{\infty\} \text{ default}, \quad Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right). \tag{41}$$

It is easy to show that the events,

$$\{\text{Banks in } \mathcal{D} \cup \{\infty\} \text{ default}\} \quad \text{and} \quad \left\{ Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i \notin \mathcal{D}, \rangle \neq \infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}} \right\}$$

are positively correlated in the sense that

$$
\begin{aligned}
& \mathbb{P}\left(\text{Banks in } \mathcal{D}\cup\{\infty\} \text{ default}, \quad Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i\notin\mathcal{D},\rangle\neq\infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right) \\
\geq \quad & \mathbb{P}(\text{Banks in } \mathcal{D}\cup\{\infty\} \text{ default})\mathbb{P}\left(Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i\notin\mathcal{D},\rangle\neq\infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right). \qquad (42)
\end{aligned}
$$

In fact, recall that we established in the proof of Theorem 4 that the equilibrium repayment x is increasing with respect to the value of β. Therefore, for two shocks Y_1 and Y_1', $Y_1 < Y_1'$, we have $x \geq x'$, where x and x' are the corresponding equilibrium repayments under the two shocks respectively. If the banks in $\mathcal{D}\cup\{\infty\}$ default under shock Y_1, i.e., $x_i < \ell_i$ for all $i \in \mathcal{D}\cup\{\infty\}$, then $x_i' < \ell_i$, meaning that this bank will also fail under a larger shock. In this sense, the indicator function $1_{\{\text{Banks in } \mathcal{D}\cup\{\infty\} \text{ default}\}}$ is an increasing function of Y_1. Meanwhile, $1_{\{Y_1 > a\}}$ is obviously an increasing function in Y_1. Invoking Proposition 7.2.1 of [6], we know that the inequality (42) must be true. From (41),

$$
\mathbb{P}(\text{Bank } j \text{ defaults} \,|\, \mathcal{D}\cup\{\infty\} \text{ default}) \geq \mathbb{P}\left(Y_1 - e_1^{\mathcal{D}} > \frac{\sum_{i\notin\mathcal{D},\rangle\neq\infty} e_i^{\mathcal{D}} z_{ij}}{z_{1j}}\right). \square
$$

A.3 Proofs for the Results in Section 4

Proof. Proof of Theorem 8.Recall that in the equilibrium the banking system is divided into three subgroups: $\mathcal{D}^*$, $\mathcal{L}^*$, and $\mathcal{N}^*\backslash\mathcal{L}^*$. The amounts of asset liquidation from $\mathcal{D}^*$ and $\mathcal{L}^*$ are $\bar{s}_{\mathcal{D}^*}$ and $s^*_{\mathcal{L}^*}$, respectively, whereas the banks in $\mathcal{N}^*\backslash\mathcal{L}^*$ do not need to sell any assets to raise funds. We have

$$
q^* = Q(|s^*|) = Q(|s^*_{\mathcal{L}^*}| + |\bar{s}_{\mathcal{D}^*}|).
$$

Therefore, for $i \in \mathcal{D}^*$,

$$
\frac{\partial q^*}{\partial \beta_i} = {q^*}' \sum_{k\in\mathcal{L}^*} \frac{\partial s_k^*}{\partial \beta_i} = {q^*}' \frac{\partial s^*_{\mathcal{L}^*}}{\partial \beta_i}\mathbf{1}. \qquad (43)
$$

Furthermore, since s_k^* satisfies $s_k^* = d_k^*/q^*$ for $k \in \mathcal{L}^*$, we know that

$$
q^* s_k^* = d_k^* = \ell_k - (\beta_k + \bar{y}_k + \sum_{h\in\mathcal{D}^*} x_h^* p_{hk} + \sum_{j\in\mathcal{N}^*} \ell_j p_{jk}). \qquad (44)
$$

Taking derivatives with respect to β_i on both sides,

$$
q^* \frac{\partial s_k^*}{\partial \beta_i} + s_k^* \frac{\partial q^*}{\partial \beta_i} = -\sum_{h\in\mathcal{D}^*} \frac{\partial x_h^*}{\partial \beta_i} p_{hk} = -\frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} P_{\mathcal{D}^*,k}.
$$

Sum the above equalities over $k \in \mathcal{L}^*$. We obtain

$$
q^* \frac{\partial s^*_{\mathcal{L}^*}}{\partial \beta_i}\mathbf{1} + \frac{\partial q^*}{\partial \beta_i}|s^*_{\mathcal{L}^*}| = -\frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}. \qquad (45)
$$

From (43) and (45), we can solve

$$
\frac{\partial q^*}{\partial \beta_i} = \Big(-\frac{q^*}{{q^*}'} - |s^*_{\mathcal{L}^*}| \Big)^{-1} \frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}. \qquad (46)
$$

On the other hand, the repayment vector $x^*_{\mathcal{D}^*}$ in the largest equilibrium admits the following representation:

$$x^*_{\mathcal{D}^*} = (\beta_{\mathcal{D}^*} + \bar{y}_{\mathcal{D}^*} + \bar{s}_{\mathcal{D}^*} q^* + x^*_{\mathcal{N}^*} P_{\mathcal{N}^*,\mathcal{D}^*})(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}. \tag{47}$$

Taking partial derivative with respect to β_i on the expression of $x^*_{\mathcal{D}^*}$, we have

$$\frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} = \Big(\mathbf{e}_i + \frac{\partial q^*}{\partial \beta_i} \bar{s}_{\mathcal{D}^*}\Big)(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}. \tag{48}$$

Finally, substituting (48) into (46) and recollecting terms will lead us to the first equality in the theorem statement.

Next we proceed to derive $\partial q^*/\partial \beta_i$ for $i \in \mathcal{L}^*$. Similarly, we start from (43). Taking derivatives with respect to β_i on both sides of (44) for $i \in \mathcal{L}^*$ will lead to

$$q^* \frac{\partial s^*_k}{\partial \beta_i} + s^*_k \frac{\partial q^*}{\partial \beta_i} = -\mathbf{e}_i - \frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} P_{\mathcal{D}^*,k}, \quad \forall k \in \mathcal{L}^*.$$

From it, we have

$$q^* \frac{\partial s^*_{\mathcal{L}^*}}{\partial \beta_i}\mathbf{1} + \frac{\partial q^*}{\partial \beta_i} |s^*_{\mathcal{L}^*}| = -1 - \frac{\partial x^*_{\mathcal{D}^*}}{\partial \beta_i} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}. \tag{49}$$

Note the difference between (49) and (45). Following similar arguments as the proof of the first half from now on, we can show the second equality in the theorem.

Finally, we can easily get the last two equality in the theorem statement through taking partial derivative with respect to β_i on the expression of $x^*_{\mathcal{D}^*}$ which is shown in equation (47). □

Proof. Proof of Theorem 9. Notice that

$$PE^{I}_{\text{DAP, Market}} = \frac{\partial q^*}{\partial \bar{y}_i} - \frac{1}{q^*}\frac{\partial q^*}{\partial \bar{s}_i}, \qquad PE^{II}_{\text{DAP, Market}} = \frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{y}_i} - \frac{1}{q^*}\frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{s}_i};$$

$$PE^{I}_{\text{Capital}} = \frac{\partial q^*}{\partial \bar{y}_i}, \qquad PE^{II}_{\text{Capital}} = \frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{y}_i};$$

and

$$PE^{I}_{\text{DAP, Market}} = \frac{\partial q^*}{\partial \bar{y}_i} - \frac{\partial q^*}{\partial \bar{s}_i}, \qquad PE^{II}_{\text{DAP, Market}} = \frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{y}_i} - \frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{s}_i}.$$

Therefore it suffices to derive $\partial q^*/\partial \bar{y}_i$, $\partial x^*_{\mathcal{D}^*}/\partial \bar{y}_i$, $\partial q^*/\partial \bar{s}_i$, and $\partial x^*_{\mathcal{D}^*}/\partial \bar{s}_i$. Since the proof is highly similar as Theorem 8, we only present the calculation related to the first two sensitivities here for the interest of space.

On one hand, we can obtain

$$\frac{\partial q^*}{\partial \bar{y}_i} = \Big(-\frac{q^*}{q^{*\prime}} - |s^*_{\mathcal{L}^*}|\Big)^{-1} \frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{y}_i} P_{\mathcal{D}^*,\mathcal{L}^*}\mathbf{1}, \tag{50}$$

invoking the same arguments leading to (46). On the other hand,

$$\frac{\partial x^*_{\mathcal{D}^*}}{\partial \bar{y}_i} = \Big(\mathbf{e}_i + \frac{\partial q^*}{\partial \bar{y}_i} \bar{s}_{\mathcal{D}^*}\Big)(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1} \tag{51}$$

from the expression of $x^*_{\mathcal{D}^*}$. Then we can solve for $\partial q^*/\partial \bar{y}_i$ and $\partial x^*_{\mathcal{D}^*}/\partial \bar{y}_i$ by substituting (51) into (50). □

B Capital Adequacy Requirement and Contagion

It is long known in the literature that the capital adequacy requirement has a potentially destabilizing effect. Under a wide range of market conditions, prudential regulations such as liquidity or capital requirements enhance the resilience of a financial system against external shocks. However, they may have an unexpected effect of forcing financial institutions to sell assets at times of market turbulence. Such forced sales press down the asset price further, resulting in adverse impacts to the capital bases of other institutions when their assets are marked to market at the new price. With a weaker capital level, more institutions are induced to sell assets to meet the requirement of the externally imposed regulations. In this way, the combination of market liquidity and capital adequacy constraint forms another endogenous channel that amplifies the initial shock.

In this section, we intend to demonstrate that the optimization formulation and the related partition algorithm provide an appropriate tool to capture analytically this undesirable spillover effect. To fix the idea, we assume that every solvent bank in the ultimate equilibrium must satisfy the following capital adequacy constraint

$$\frac{\beta_i + \bar{y}_i + \bar{s}_i q + \sum_{j \neq i} x_j p_{ji} - x_i}{\beta_i + (\bar{y}_i - y_i) + (\bar{s}_i - s_i) q + \sum_{j \neq i} x_j p_{ji}} \geq R \tag{52}$$

for some pre-specified ratio R.

The numerator in (52) is the value of the bank's equity after it makes its repayments and marks the illiquid assets to the equilibrium price q, whereas the denominator reflects the marked-to-market value of its assets after the bank sells y_i units of liquid assets and s_i units of illiquid assets. Intuitively, when a bank receives a negative shock on its equity value, causing a decline in the value of the numerator in (52), the bank may be forced to liquidate its assets to meet the minimum capital ratio R. The underlying assumption is that the assets are sold for cash, and that cash does not attract any capital requirement.

Add the constraint (52) to the original optimization formulation (8). The partition algorithm is still applicable to solve this extended problem. Indeed, the inequality (52) can easily be transformed into an linear constraint, which will not bring about too much structural change to the problem. More specifically, it is easy to see that the total amount of liquid assets that the bank needs to sell now is given by $y_i = \bar{y}_i \wedge d_i^1$, where

$$d_i^1 = [\ell_i - (\beta_i + \sum_j x_j p_{ji})]^+ \vee \left[\frac{\ell_i - (1 - R)(\beta_i + \bar{y}_i + \bar{s}_i q + \sum_j x_j p_{ji})}{R} \right]^+ . \tag{53}$$

After it sells amount y_i of liquid assets, the amount of illiquid asset sales should be $s_i = \bar{s}_i \wedge (d_i^2 / q)$, with

$$d_i^2 = [\ell_i - (\beta_i + \sum_j x_j p_{ji}) - y_i]^+ \vee \left[\frac{\ell_i - (1 - R)(\beta_i + \bar{y}_i + \bar{s}_i q + \sum_j x_j p_{ji})}{R} - y_i \right]^+ . \tag{54}$$

The differences between the original definitions of d^1 and d^2 and (53) and (54) reflect additional liquidation needs caused by the capital adequacy requirement. With these changed d^1 and d^2, we can run the same partition algorithm to obtain the maximal equilibrium.

Recall from the optimization theory that for a general optimization problem, an additional constraint will introduce an additional positive Lagrange multiplier, making the solution more sensitive. Performing sensitivity analysis on this extended problem, we have

$$\frac{\partial q^*}{\partial \beta_i} = \frac{\gamma}{1-\gamma\big(|s^*_{\mathcal{L}_1^*}| + |s^*_{\mathcal{L}_2^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}(P_{\mathcal{D}^*,\mathcal{L}_1^*}\mathbf{1}_{\mathcal{L}_1^*} + \frac{1-R}{R}P_{\mathcal{D}^*,\mathcal{L}_2^*}\mathbf{1}_{\mathcal{L}_2^*}) + \frac{1-R}{R}|\bar{s}_{\mathcal{L}_2^*}|\big)}, \quad \text{for } i \in \mathcal{L}_1^*,$$

with

$$\mathcal{L}_1^* = \left\{ i \in \mathcal{N}^* : s_i^* > 0,\ \frac{\beta_i + \bar{y}_i + \bar{s}_i q^* + \sum_j x_j^* p_{ji} - \ell_i}{\beta_i + \bar{y}_i - y_i^* + (\bar{s}_i - s_i^*)q^* + \sum_j x_j^* p_{ji}} > R \right\}$$

and

$$\mathcal{L}_2^* = \left\{ i \in \mathcal{N}^* : s_i^* > 0, \frac{\beta_i + \bar{y}_i + \bar{s}_i q^* + \sum_j x_j^* p_{ji} - \ell_i}{\beta_i + \bar{y}_i - y_i^* + (\bar{s}_i - s_i^*)q^* + \sum_j x_j^* p_{ji}} = R \right\}.$$

Compared with the result in Theorem 8, several additional terms appear in the denominator, causing this sensitivity to be larger than the one without the constraint (52).

We attribute this greater sensitivity to a new amplification channel opened by the capital requirement policy. To see this, like what we did in explaining Theorem 8, consider the impact of a \$1 negative shock in β_i to the equilibrium price q^*. This shock causes bank i to sell more, depressing the price by a factor of γ. As observed in Section 4, the price decline will create a loss to the banks in $\mathcal{L}_1^*$ amounting to

$$\gamma(|s^*_{\mathcal{L}_1^*}| + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}_1^*}\mathbf{1}_{\mathcal{L}_1^*}), \tag{55}$$

and hence they are obliged to sell part of their assets to compensate the loss for the purpose of meeting the liability repayments. On the top of such sales, the banks in $\mathcal{L}_2^*$ are also forced to liquidate their illiquid holdings. From the numerator part of the ratio, the price decline creates a loss of

$$\gamma(\bar{s}_{\mathcal{L}_2^*} + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}_2^*}\mathbf{1}_{\mathcal{L}_2^*})$$

on their equity values. Note that the capital ratio constraint is binding for those banks in the equilibrium. Therefore, it will trigger them to reduce the asset sizes, calculated as the denominator indicates, by an amount of

$$\frac{1}{R}\cdot\gamma(\bar{s}_{\mathcal{L}_2^*} + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}_2^*}\mathbf{1}_{\mathcal{L}_2^*})$$

to maintain the capital ratio at the level R, or equivalently, to liquidate additionally

$$\frac{1-R}{R}\cdot\gamma(\bar{s}_{\mathcal{L}_2^*} + \bar{s}_{\mathcal{D}^*}(I_{\mathcal{D}^*} - P_{\mathcal{D}^*})^{-1}P_{\mathcal{D}^*,\mathcal{L}_2^*}\mathbf{1}_{\mathcal{L}_2^*}) + \gamma|s^*_{\mathcal{L}_2^*}| \tag{56}$$

worth of the illiquid asset. In total, the extra sale amount induced by the price decline in this round will be the sum of (55) and (56). Continuing the argument through all the rounds of ripple effects, we will reach down to the expression of $\partial q^*/\partial \beta_i$.

C Supplement for Numerical Experiments

C.1 Details of Network Reconstruction

To create a network of structure type B, we specify a sparse configuration for Y. The idea is to concentrate the liability exposures of one bank to one of its neighboring bank, as long as it does not exceed the minimum of the total amounts of the interbank liabilities and assets of both banks. This requires us to force the column/row sums of Y to its super- and sub-diagonal entries as much as possible. More formally, we use a greedy algorithm as follows to define Y: let $y_{ii} = 0$ for all i and

```
for i = 1 to n do
    for j = 1 to n do
        if (j ≠ i) then
            y_ij ← min{l_i, a_j};
            l_i ← l_i − y_ij  and  a_j ← a_j − y_ij .
```

The algorithm can be viewed as a variant of the northwest corner rule in the literature of transportation problem or Monge problem. Under some regularity conditions, people use the rule to couple two random variables with known marginal distributions so as to produce largest covariance between them; see Chapter 8 of [4] for a more comprehensive discussion. Once the matrix Y is obtained, we substitute it into the optimization program (28) to find a feasible liability matrix L.

Finally, we proceed to present how we construct a network with a core-periphery structure, which may resemble the market reality more closely. [8] find strong evidences from the balance sheets data at the end of 1998 that the German interbank deposit market was organized in two tiers. The lower tier consisted of saving and cooperative banks, and the upper tier consists of the head institutions of two giro systems (Landesbanken and cooperative central banks) and commercial banks. The banks in the lower tier had very few direct linkages with banks in the same tier, whereas the upper tier banks maintained wide lending relationship with a variety of other banks including banks in other categories.

The information about the types of the counterparty banks in interbank lending transactions is available in their dataset so that they can identify the upper tier banks from their estimation more precisely. In contrast, our dataset provides very limited information about the banks identification. We group the 11 banks into 2 classes simply according to the sizes of their interbank EAD: DE019, DE020 and DE021 as the core and the remaining 8 as the periphery. The above specification of the core and periphery banks surely is not an accurate reflection of the real market situation. The underlying assumption is that the large EAD values of the specified core banks should be resulted by their wide bilateral exposures to the other banks. We also try different specifications in the experiments and find that they do not have qualitative impacts on our conclusion.

After the classification of core and periphery banks is fixed, we let $y_{ij} = 0$ if banks i and j both belong to the periphery class and $y_{ij} = l_i a_j$ otherwise. The program (28) is invoked again to solve for a feasible liability matrix L. Tables 7 - 9 show the reconstruction outcomes under all the three structure types.

	DE017	DE018	DE019	DE020	DE021	DE022	DE023	DE024	DE025	DE027	DE028
DE017	0	4857	9971	11306	6753	5413	712	2213	413	2573	2891
DE018	4857	0	10625	12047	7196	5768	758	2358	440	2741	3081
DE019	9971	10625	0	24734	14774	11841	1557	4841	904	5628	6325
DE020	11306	12047	24734	0	16751	13427	1766	5489	1025	6382	7172
DE021	6753	7196	14774	16751	0	8020	1055	3279	612	3812	4284
DE022	5413	5768	11841	13427	8020	0	845	2628	491	3055	3434
DE023	712	758	1557	1766	1055	845	0	346	65	402	452
DE024	2213	2358	4841	5489	3279	2628	346	0	201	1249	1404
DE025	413	440	904	1025	612	491	65	201	0	233	262
DE027	2573	2741	5628	6382	3812	3055	402	1249	233	0	1632
DE028	2891	3081	6325	7172	4284	3434	452	1404	262	1632	0

Table 7: The Liability matrix of the complete network.

	DE017	DE018	DE019	DE020	DE021	DE022	DE023	DE024	DE025	DE027	DE028
DE017	0	31855	0	0	0	0	0	0	0	0	15247
DE018	31855	0	18016	0	0	0	0	0	0	0	0
DE019	0	18016	0	73184	0	0	0	0	0	0	0
DE020	0	0	73185	0	26915	0	0	0	0	0	0
DE021	0	0	0	26915	0	39620	0	0	0	0	0
DE022	0	0	0	0	39620	0	7956	7345	0	0	0
DE023	0	0	0	0	0	7956	0	0	0	0	0
DE024	0	0	0	0	0	7345	0	0	4645	12017	0
DE025	0	0	0	0	0	0	0	4645	0	0	0
DE027	0	0	0	0	0	0	0	12017	0	0	15690
DE028	15247	0	0	0	0	0	0	0	0	15690	0

Table 8: The Liability matrix of the ring-link network.

	DE017	DE018	DE019	DE020	DE021	DE022	DE023	DE024	DE025	DE027	DE028
DE017	0	0	16673	18340	12089	0	0	0	0	0	0
DE018	0	0	17653	19418	12800	0	0	0	0	0	0
DE019	16673	17653	0	2242	1478	19440	2816	8498	1644	9807	10951
DE020	18340	19418	2242	0	1625	21385	3098	9348	1809	10788	12046
DE021	12089	12800	1478	1625	0	14096	2042	6162	1192	7111	7940
DE022	0	0	19440	21385	14096	0	0	0	0	0	0
DE023	0	0	2816	3098	2042	0	0	0	0	0	0
DE024	0	0	8498	9348	6162	0	0	0	0	0	0
DE025	0	0	1644	1809	1192	0	0	0	0	0	0
DE027	0	0	9807	10788	7111	0	0	0	0	0	0
DE028	0	0	10951	12046	7940	0	0	0	0	0	0

Table 9: The Liability matrix of the core-periphery network.

C.2 Contagion via Market Liquidity using an Exponential Demand Function

As a robust check, we use an exponential demand function in this appendix to assess the impact of functional forms of Q to our results. Still assume that the illiquid holding counts for 30% of the total asset of each bank in the system, i.e., $\theta = 30\%$. Let $Q(s) = \exp(-\gamma s)$, an exponential demand function that is widely used in the literature of systemic risk (see, e.g., [1], [3], [5]). We consider three cases in the following examples: $\gamma = 0.52 \times 10^{-13}$, 1.08×10^{-13}, and 3.94×10^{-13}. For comparison purposes, we choose such values that they can result in comparable price impacts as what we used in Section 5. Note that the total amount of the illiquid assets owned by the banking system is 1,456,349.7 million euros under the assumption of $\theta = 30\%$ (30% of the sum of the first column of Table 4). With $\gamma = 0.52 \times 10^{-13}$, the market price of the illiquid asset will decline to \$0.927 from its face value \$1 when every bank in the system sells out all its illiquid holdings; or equivalently, 5 basis points of price change per 10 billion euro asset sales. Similarly, $\gamma = 1.08 \times 10^{-13}$ and 3.94×10^{-13} corresponds price changes of 10 and 30 basis points per 10 billion euro asset sales, respectively. Figure 8 illustrates the number of defaults in equilibrium when different sizes of external shocks are applied

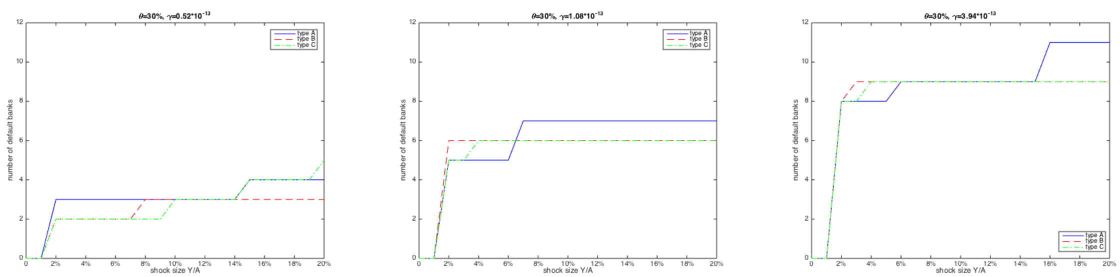

Figure 8: The number of defaults under different shock sizes with a demand function $Q(s) = \exp(-\gamma s)$. The vertical axis is the default number in the repayment equilibrium. The percentages in the horizontal axis are the relative size of an external shock Y to the total asset of Bank DE017. In the first, second, and third rows, we specify the illiquid asset ratios as $\theta = 10\%$, 30%, and 60%, respectively, while in the first, second, and third columns, we use the market depth as $\gamma = 0.52 \times 10^{-13}$, 1.08×10^{-13}, and 3.94×10^{-13}.

on the external projects of Bank DE017. Comparing it with the central row of Figure 5, we can see that results are qualitatively similar.

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