Stochastic Models
Lecture 2 Part I
Poisson Processes

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Outline

1. Exponential Distributions
2. Poisson Process: Definition and Distribution
3. Further Properties of Poisson Processes

2.1 The Exponential Distribution

Definition

- A continuous random variable X is said to have an **exponential distribution** with parameter λ, if its probability density function is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, x\ge 0;\\ 0, x<0.\end{cases}$$

CDF, Mean and Variance

- The CDF of an exponentially distributed random variable is given by

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \begin{cases} 1 - e^{-\lambda x}, x \geq 0; \\ 0, x < 0. \end{cases}$$

- The mean and variance of an exponentially distributed random variable are
  - Mean: $E[X] = \frac{1}{\lambda}.$
  - Variance:

$$Var[X] = \frac{1}{\lambda^2}.$$

The Memory-less Property

- We can show that

$$P(X > s+t \mid X > t) = P(X > s)$$

  for a random variable with exponential distribution.

- If we think of X as being the lifetime of some instrument, then this property states that the instrument does not remember that it has already been in use for a time t.

Example I: Service Time in a Bank

- Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes, that is, $\lambda = 0.1$.
  - What is the probability that a customer will spend more than 15 minutes in the bank?
  - What is the probability that a customer will spend more than 15 minutes in the bank, given that she is still in the bank after 10 minutes?

Example II: Post Office

- Consider a post office that is run by two clerks. Suppose that when Mr. Smith enters the post office he discovers that Mr. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Jones or Brown leaves.

Example II: Post Office (Continued)

- If the amount of service time for a clerk is exponentially distributed, what is the probability that, of the three customers, Mr. Smith is the last one to leave?

Several Further Properties of Exponential Distribution

- Suppose that X_1 and X_2 are independent exponential random variables with respect means $1/\lambda_1$ and $1/\lambda_2$.
- The probability

$$P(X_1 < X_2) = \int_0^{+\infty} P(X_1 < X_2 \mid X_1 = x)\lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^{+\infty} P(x < X_2)\lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^{+\infty} e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}\, dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Several Further Properties (Continued)

- Suppose that $\{X_1, X_2, \ldots, X_n\}$ are independent exponential random variables, with X_i having rate μ_i, $i = 1, 2, \ldots, n$.
- The distribution of

$$Y = \min\{X_1, X_2, \ldots, X_n\}$$

follows an exponential distribution with a rate equal to the sum of the μ_i. That is,

$$P(Y > x) = \exp\left(\left(-\sum_{i=1}^{n} \mu_i\right) x\right).$$

Several Further Properties (Continued)

- Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables having mean $1/\lambda$. The sum

$$Z = X_1 + X_2 + \ldots + X_n$$

has a gamma distribution; that is, its PDF is given by

$$f_Z(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

Example III: Post Office Again

- Suppose that you arrive at a post office having two clerks at a moment when both are busy but there is no one else waiting in line. You will enter service when either clerk becomes free.
- If service times for clerk i are exponential with rate λ_i , $i = 1,2,$ find $E[T]$, where T is the amount of time that you spend in the post office.

2.2 The Poisson Process

Construction of a Poisson Process

- To construct a Poisson process, we begin with a sequence $\tau_1, \tau_2, \ldots$ of independent exponential random variables, all with the same mean $1/\lambda$.
- We build a model in which "events" occur from time to time. In particular,
  - the first event occurs at time τ_1;
  - the second event occurs τ_2 time units after the first event;
  - …

Construction of a Poisson Process (Continued)

- The τ_k random variables are called the **inter-arrival times**.
- The **waiting times** of events are

$$S_n = \sum_{k=1}^{n} \tau_k .$$

- A process $\{N_t, t \geq 0\}$ is said to be a **Poisson process** if it counts the number of events in the above model that occur at or before time t.

Distribution of Poisson Processes

- It is easy to know that the waiting time until the nth event follows a gamma distribution.
- A simple observation:

$$N_t \geq n \quad \text{if and only if} \quad S_n \leq t.$$

From it, we can show

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0,1,2,\ldots$$

i.e., N_t follows a Poisson distribution.

Increments of Poisson Process

- **Theorem:**

  Let $\{N_t, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, and let $0 = t_0 < t_1 < \ldots < t_n$ be given. Then, the increments

$$N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$$

  are independent and stationary, and

$$P(N_t - N_s = k) = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \qquad k = 0, 1, 2, \ldots$$

Mean and Variance of Poisson Increments

- Mean:

$$E[N_t - N_s] = \lambda(t - s).$$

- Variance:

$$Var[N_t - N_s] = \lambda(t - s).$$

Poisson Process as a Counting Process

- In summary, the Poisson process must satisfy:
  - $N_t \geq 0$;
  - N_t is integer valued;
  - $N_t \geq N_s$ for $t \geq s$;
  - $N_t - N_s$, the number of events that occur in the interval $(s,t]$, follows a Poisson distribution.
- The Poisson process is a special case of **counting processes**.

Example IV: Immigration

- Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.
- What is the expected time until the 10^{th} immigrant arrives?
- What is the probability that the elapsed time between the 10^{th} and the 11^{th} immigrants exceeds 2 days?

2.3 Further Properties of Poisson Processes

Decomposition of a Poisson Process

- Consider a Poisson process $\{N_t, t \geq 0\}$ having rate $\lambda > 0$, and suppose that each time an event occurs it is randomly classified as either a type I or a type II. Suppose further that each event is classified as a type I event with probability p or a type II event with probability $1-p$, independently of all other events.

Decomposition of a Poisson Process (Continued)

- **Theorem:**

  Let N_t^1 and N_t^2 denote respectively the number of type I and II events occurring in $[0,t]$. Then, $\{N_t^1, t \geq 0\}$ and $\{N_t^2, t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$. They are independent.

Superposition of Poisson Processes

- **Theorem:**

  Let $\{N_t^1, t \geq 0\}$ and $\{N_t^2, t \geq 0\}$ be two independent Poisson processes having intensities λ_1 and λ_2, respectively. Then, $\{N_t^1 + N_t^2, t \geq 0\}$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

Example V: Immigration Again

- If immigrants to area A arrive at a Poisson rate of 10 per week, and if each immigrant is of English descent with probability 1/12, then what is the probability that no people of English descent will immigrate to area A in one month?

Example VI: Selling an Apartment

- Suppose that you want to sell your apartment. Potential buyers arrive according to a Poisson process with rate λ. Assume that each buyer offer you a random price with continuous PDF $f(x)$. Once the offer is presented to you, you must either accept it or reject it and wait for the next offer.

Example VI: Selling an Apartment (Continued)

- Suppose that you employ the policy of accepting the first offer that is greater than some pre-specified price y. Meanwhile, assume that a cost of rate c will be incurred per unit time until the apartment is sold.
- If your objective is to maximize the expected return, the difference between the amount received and the total cost incurred, what is the best price you should set?

Example VI: Selling an Apartment (Continued)

- For simplicity, consider the case that the offer distribution f is exponential.

Conditional Distribution of Arrival Times

- Suppose that we are told that exactly one event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred.
- $$P(S_1 < s \mid N_t = 1) = \frac{P(S_1 < s, N_t = 1)}{P(N_t = 1)}$$

$$= \frac{P(N_s = 1, N_t - N_s = 0)}{P(N_t = 1)} = \frac{\lambda s e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}}$$

$$= \frac{s}{t}.$$

Conditional Distribution of Arrival Times (Continued)

- In other words, the time of the event should be uniformly distributed over $[0,t]$.
- This result may be generalized, but before doing so we need to introduce the concept of order statistics.

Order Statistics

- Let $Y_1, Y_2, \ldots, Y_n$ be n random variables. We say that $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ are the corresponding **order statistics** if $Y_{(k)}$ is the k th smallest value among $Y_1, Y_2, \ldots, Y_n$.
- For instance,

$$(Y_1, Y_2, Y_3) = (4, 5, 1)$$

The corresponding order statistics are

$$(Y_{(1)}, Y_{(2)}, Y_{(3)}) = (1, 4, 5).$$

Order Statistics (Continued)

- If the $Y_i,\ i = 1, \ldots, n,$ are independent identically distributed continuous random variables with PDF $f,$ then the joint density of the order statistics $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is given by

$$g(y_1, y_2, \ldots, y_n) = n! \prod_{i=1}^{n} f(y_i),$$

for $y_1 < y_2 < \ldots < y_n.$

Order Statistics (Continued)

- In particular, if the $Y_i, \ \ i = 1,\ldots,n,$ are uniformly distributed over $[0,t],$ then

$$g(y_1, y_2, \ldots, y_n) = \frac{n!}{t^n},$$

for $0 < y_1 < y_2 < \ldots < y_n < t.$

Conditional Distribution of Arrival Times

- **Theorem:**

  Given that $N_t = n,$ the n arrival times $S_1, S_2, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Example VII: Insurance Claims

- Insurance claims are made at times distributed according to a Poisson process with rate λ; each claim amount is the same as $\$a$.
- Compute the discounted value of all claims made up to time t.

Homework Assignments

- Read Ross Chapter 5.1, 5.2, and 5.3.
- Answer Questions:
  - Exercises 1, 2, 4 (Page 338, Ross)
  - Exercises 14 (Page 339, Ross)
  - Exercises 34 (Page 343, Ross)
  - Exercise 42 (Page 344, Ross)
  - Exercise 68 (Page 349, Ross)
  - Exercise 44, 49 (page 345-346, Ross).

  Due on Oct 6, Fri.