# Simple Mathematical Facts for Lecture 2

# Cumulative Probability Functions (CDF) and Probability Density Functions (PDF)

- We say a random number $X$ is a **continuous random variable** if there exists a nonnegative function $f$ such that

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du.$$

- Example:
  - Normal distribution
  - Uniform distribution

# Expectation and Variance

- The **expectation** of a random variable characterizes its average.

  For continuously distributed random variable,

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x)\, dx.$$

- The **variance** is a reflection of the degree of its value uncertainty.

$$Var[X] = E[(X - E(X))^2] = E[X^2] - (E[X])^2.$$

# Independence between Random Variables

- The random variables $X$ and $Y$ are said to be **independent** if

$$P(X \le a, Y \le b) = P(X \le a)P(Y \le b)$$

for all $a$ and $b$. In terms of the joint distribution function of $X$ and $Y$, we have that

$$F(a,b) = F_X(a)F_Y(b).$$

- For two independent random variables,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)].$$

# Covariance

- The **covariance** of any two random variables is defined by

$$Cov(X,Y) = E[(X-E(X))(Y-E(Y))] = E[XY]-E[X]E[Y].$$

  - Positive vs. negative covariance
  - Zero-covariance and independence
    - Independence implies zero covariance, but the converse is not true.
    - Counterexample:

Let $X$ be a standard normal random variable, and $Y$ be an independent random variable with

$$P(Y=1) = P(Y=-1) = 1/2.$$

Let $Z = XY$. $X$ and $Z$ have zero covariance, but they are not independent.

# Sum of Two Independent Random Variables

- If $X$ and $Y$ are independent continuously distributed random variables, the PDF of $X+Y$ is given by

$$f_{X+Y}(u) = \int_{-\infty}^{+\infty} f(u-y)g(y)\,dy$$

where $f$ and $g$ are the PDFs of $X$ and $Y$, respectively.